Tutorial 8

1. (a). If random variable X and Y are identically distributed, not necessarily independent, show that:
$$\operatorname{cov}(X+Y, X-Y)=0$$
(b). Now consider random variables X, Y and Z. The conditional covariance of X and Y, given Z, is defined by
$$\operatorname{cov}(X,Y\,|\,Z)=E[(X-E[X\,|\,Z])(Y-E[Y\,|\,Z])\,|\,Z]$$
   i. Show that
   $$\operatorname{cov}(X,Y\,|\,Z)=E[XY\,|\,Z]-E[X\,|\,Z]E[Y\,|\,Z]$$
   ii. Show that
   $$\operatorname{cov}(X,Y)=E[\operatorname{cov}(X,Y\,|\,Z)]+\operatorname{cov}(E[X\,|\,Z],E[Y\,|\,Z])$$
   (Hint: take the expectation of both sides of the result in part i)
   iii. Set X=Y in part ii and obtain a formula for var(X) in terms of conditional expectations.

2. (a). Let X, Y and Z be zero-mean random variables. Determine the linear squares estimate $\hat{Z}=\alpha X+\beta Y$ of Z given X and Y, i.e., find α and β to minimize the mean square error. You should express the optimal values of α and β in terms of variances and covariances of the random variables.

   (b). A stationary, second order, stochastic process $\{y_k\}$ is given by the following difference equation:
   $$y_k+ay_{k-1}+by_{k-2}=e_k$$
   where a and b are constants and $\{e_k\}$ is a sequence of zero mean, uncorrelated random variables with unit variance.
   i. Find the first three values of the covariance function $r_y(k)$,k=0,1 and 2 for the sequence $\{y_k\}$.
   ii. Using results in part a, determine the linear least squares estimate of y_k given y_{k-1} and y_{k-2} .

Solution 8

1. Solution:

$$Cov(X+Y, X-Y) = cov(X,X) + cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$$

$$= var(X) - cov(X,Y) + Cov(Y,X) - var(Y)$$

$$= var(X) - var(Y)$$

$$= 0 \qquad \because X \text{ and } Y \text{ are identically distributed.}$$

$Cov(X,Y|Z)$

$$= E\{ XY - E[X|Z]Y - XE[Y|Z] + E[X|Z]E[Y|Z] \,|\, Z \}$$

$$= E[XY|Z] - E[X|Z]E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]\cdot E[Y|Z] \qquad \because E[\cdot] \text{ is a constant}$$

$$= E[XY|Z] - E[X|Z]E[Y|Z]$$

Take the expectation on both sides of result in part i:

$$E[Cov(X,Y|Z)] = E\{E[XY|Z] - E[X|Z]E[Y|Z]\}$$

$$E[Cov(X,Y|Z)] = E[XY] - E\{E[X|Z]E[Y|Z]\} \quad \text{—— ①}$$

consider $cov(E[X|Z], E[Y|Z])$.

$$cov(E[X|Z], E[Y|Z]) = E\{E[X|Z]E[Y|Z]\} - E\{E[X|Z]\}\cdot E\{E[Y|Z]\}$$

$$cov(E[X|Z], E[Y|Z]) = E\{E[X|Z]E[Y|Z]\} - E[X]E[Y] \quad \text{②}$$

Combine ① and ②:

$$E[cov(X,Y|Z)] + cov(E[X|Z], E[Y|Z])$$

$$= E[XY] - E(X)E(Y)$$

$$= Cov(X,Y) \qquad \text{Q.E.D.}$$

iii) $cov(X,Y|Z) = Var(X|Z)$ when $Y = X$.

The LHS of result in part ii:

$$cov(X,Y) = var(X)$$

The RHS: $E[cov(X,X|Z)] + cov(E(X|Z), E(\ |Z))$

$$= E[var(X|Z)] + var(E[X|Z])$$

$$\Rightarrow \quad var(X) = E[var(X|Z)] + var(E[X|Z])$$

2. Solution:

$$r_y(0) = \frac{1}{1-b^2}$$

$$r_y(1) = \frac{-a}{1+b}\cdot\frac{1}{1-b^2}$$

$$r_y(2) = \left(-b + \frac{a^2}{1+b}\right)\cdot\frac{1}{1-b^2}$$