Tutorial 7

1. Out of the d doors of my house, suppose that in the beginning k> 0 are unlocked and $d-k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.
   Show that the number of unlocked doors at the end of day n, L_n ,evolves as the state in a Markov process for n ≥ 1. Write down the transition probability p_{ij}

2. A certain type of component has two states: ON (1) and OFF (0). In state 0, the process remains there for an exponential(α) length of time, and then moves to state 1. The time in state 1 is also exponentially distributed with parameter β. The system has two components, A and B, with distinct parameters: α_A, β_A and α_B, β_B respectively. In order for the system to operate, *at least one* of the components A of B must be operating (i.e. the system is parallel). Assume that the two components in the system are independent of one another. Determine the long run probability that the system is operating by

   (a) Considering each component separately as a two state Markov chain and using their statistical independence;

   (b) Considering the system as a four state Markov chain.

3. Consider the Markov chain $\{X_n\}_{n\geq 0}$, with $S = \{0, 1, 2\}$, whose transition matrix is

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   Let $f(0) = 0, f(1) = f(2) = 1$. Now define $Y_n = f(X_n)$. Is $\{Y_n\}_{n\geq 0}$ a Markov chain?

Solution 7

1. Solution:

Given L_{n-1}, the history of the process (i.e., $L_{n-2}, L_{n-3}, \ldots$) is irrelevant for determining the probability distribution of L_n, the number of remaining unlocked doors at time n. Therefore, L_n is Markov. More precisely,

$$\mathbf{P}(L_n = j | L_{n-1} = i, L_{2-2} = k, \ldots, L_1 = q) = \mathbf{P}(L_n = j | L_{n-1} = i) = p_{ij}.$$

Clearly, at one step the number of unlocked doors can only decrease by one or stay constant. So, for $1 \leq i \leq d$, if $j = i-1$, then $p_{ij} = \mathbf{P}$(selecting an unlocked door on day $n+1 | L_n = i) = \frac{i}{d}$. For $0 \leq i \leq d$, if $j = i$, then $p_{ij} = \mathbf{P}$(selecting an locked door on day $n+1 | L_n = i) = \frac{d-i}{d}$. Otherwise, $p_{ij} = 0$. To summarize, for $0 \leq i, j \leq d$, we have the following:

$$p_{ij} = \begin{cases} \frac{d-i}{d} & j = i \\ \frac{i}{d} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

2. Solution:

(a) Considering each component separately as a two state Markov chain and using their statistical independence;

Here we have that for machine A the generator is

$$G_A = \begin{bmatrix} -\alpha_A & \alpha_A \\ \beta_A & -\beta_A \end{bmatrix},$$

and for machine B the generator is

$$G_B = \begin{bmatrix} -\alpha_B & \alpha_B \\ \beta_B & -\beta_B \end{bmatrix}.$$

Solving the equations $\pi^A G_A = 0$, and a similar one for B, gives us that the *long run distribution* of each system is given by $\pi_A = \frac{1}{\alpha_A+\beta_A}\{\beta_A, \alpha_A\}$, and similarly for B. Hence, the probability that the system is operating may be written as

$$\begin{aligned} & P(\text{either } A \text{ or } B \text{ is working}) \\ = \ & 1 - P(\text{both } A \text{ and } B \text{ not working}) \\ = \ & 1 - \frac{\beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}. \end{aligned}$$

(b) Considering the system as a four state Markov chain.

In this case we carefully right down the the generator for the four state Markov chain, with $S = \{00, 01, 10, 11\}$. Here, for example, 01 means that machine A is not working and machine B is working. The generator becomes

$$G = \begin{bmatrix} -(\alpha_A + \alpha_B) & \alpha_B & \alpha_A & 0 \\ \beta_B & -(\alpha_A + \beta_B) & 0 & \alpha_A \\ \beta_A & 0 & -(\beta_A + \alpha_B) & \alpha_B \\ 0 & \beta_A & \beta_B & -(\beta_A + \beta_B) \end{bmatrix}.$$

Now, we have to show that the answers from part (a) are the same as the answers here. That is, we should solve $\pi G = 0$. However, this could be a bit lengthy. However, we do know what the answer is from part (a). Hence, using Theorem 5.10 from the summary notes, once we have solution π we know it'll be unique. The answer π in this case is given by

$$\pi = \frac{1}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)} \{\beta_A\beta_B,\ \beta_A\alpha_B,\ \alpha_A\beta_B,\ \alpha_A\alpha_B\}.$$

3. No. Try to calculate $P(Y_n = 0 | Y_{n-1} = 1)$ - you can't. The problem is that we've lost too much information by "putting together" states 1 and 2 of the original chain X_n.