Birth/Death Process

Consider a discrete-state, continuous-time Markov chain $\{N_t\}$ with state space

$$E = \{0,1,2,...,N\}$$

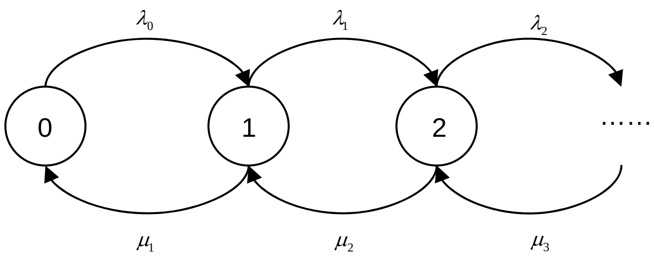
Then $\{N_t\}$ is a birth/death process iff

$$P(N_{t+\Delta t} = j | N_t = i) = \begin{cases} \lambda_i \Delta t, & j = i+1 \\ \mu_i \Delta t, & j = i-1 \\ 0, & |j-i| > 1 \end{cases}$$

Future of $\{N_t\}$ is regardless of its past history. The only possible transitions from any state i are $i \to i+1$ and $i \to i-1$.

$$\lambda_i = \text{birth coefficient (rate) in state i}$$

$$\mu_i = \text{death coefficient (rate) in state i}$$



Birth/death equations:
Denote the state probabilities at time t by: $p_i(t) = P(N_t = i)$, for $i = 0,1,...,N$. Since $\{N_t\}$ is Markov, we can compute the state probabilities $p_i(t)$ from knowledge of the initial distribution $\{p_i(0)\}$ plus the transition rates λ_i, μ_i.
So, $P(N_{t+\Delta t} = i) = P(N_t = i, N_{t+\Delta t} = i) + P(N_t = i-1, N_{t+\Delta t} = i) + P(N_t = i+1, N_{t+\Delta t} = i)$
That is, $p_i(t+\Delta t) = p_{i-1}(\lambda_{i-1}\Delta t) + p_{i+1}(t)(\mu_{i+1}\Delta t) + p_i(t)(1 - \lambda_i \Delta t - \mu_i \Delta t)$. Now subtract $p_i(t)$ from each side, divide throughout by Δt, and let $\Delta t \to 0$. The result is:

$$\frac{dp_i(t)}{dt} = \lambda_{i-1} p_{i-1}(t) - (\lambda_i + \mu_i) p_i(t) + \mu_{i+1} p_{i+1}(t)$$

Now suppose for each state i: $p_i(t) \xrightarrow{t} \text{constant probability}, \pi_i$, that is $\frac{dp_i(t)}{dt} = 0$.

The birth/death equations then become algebraic equations:

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0$$
$$\lambda_0 \pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2 \pi_2 = 0$$
$$\lambda_1 \pi_1 - (\lambda_2 + \mu_2)\pi_2 + \mu_3 \pi_3 = 0$$
$$\vdots$$

These equations are called the equilibrium equations. We have $\underline{\pi} \cdot \underline{Q} = 0$.

M/M/1 queuing system
(Poisson arrival stream/ Exponential service times/ 1 server)