Solution 7

1. Solution:

   Given L_{n-1}, the history of the process (i.e., $L_{n-2}, L_{n-3}, \ldots$) is irrelevant for determining the probability distribution of L_n, the number of remaining unlocked doors at time n. Therefore, L_n is Markov. More precisely,

   $$\mathbf{P}(L_n = j | L_{n-1} = i, L_{2-2} = k, \ldots, L_1 = q) = \mathbf{P}(L_n = j | L_{n-1} = i) = p_{ij}.$$

   Clearly, at one step the number of unlocked doors can only decrease by one or stay constant. So, for $1 \le i \le d$, if $j = i-1$, then $p_{ij} = \mathbf{P}(\text{selecting an unlocked door on day } n+1 | L_n = i) = \frac{i}{d}$. For $0 \le i \le d$, if $j = i$, then $p_{ij} = \mathbf{P}(\text{selecting an locked door on day } n+1 | L_n = i) = \frac{d-i}{d}$. Otherwise, $p_{ij} = 0$. To summarize, for $0 \le i, j \le d$, we have the following:

   $$p_{ij} = \begin{cases} \frac{d-i}{d} & j = i \\ \frac{i}{d} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

2. Solution:

   (a) Considering each component separately as a two state Markov chain and using their statistical independence;

   Here we have that for machine A the generator is

   $$G_A = \begin{bmatrix} -\alpha_A & \alpha_A \\ \beta_A & -\beta_A \end{bmatrix},$$

   and for machine B the generator is

   $$G_B = \begin{bmatrix} -\alpha_B & \alpha_B \\ \beta_B & -\beta_B \end{bmatrix}.$$

   Solving the equations $\pi^A G_A = 0$, and a similar one for B, gives us that the *long run distribution* of each system is given by $\pi_A = \frac{1}{\alpha_A + \beta_A}\{\beta_A, \alpha_A\}$, and similarly for B. Hence, the probability that the system is operating may be written as

   $$\begin{aligned} & P(\text{either } A \text{ or } B \text{ is working}) \\ = \; & 1 - P(\text{both } A \text{ and } B \text{ not working}) \\ = \; & 1 - \frac{\beta_A \beta_B}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}. \end{aligned}$$

   (b) Considering the system as a four state Markov chain.

   In this case we carefully right down the the generator for the four state Markov chain, with $S = \{00, 01, 10, 11\}$. Here, for example, 01 means that machine A is not working and machine B is working. The generator becomes

   $$G = \begin{bmatrix} -(\alpha_A + \alpha_B) & \alpha_B & \alpha_A & 0 \\ \beta_B & -(\alpha_A + \beta_B) & 0 & \alpha_A \\ \beta_A & 0 & -(\beta_A + \alpha_B) & \alpha_B \\ 0 & \beta_A & \beta_B & -(\beta_A + \beta_B) \end{bmatrix}.$$

   Now, we have to show that the answers from part (a) are the same as the answers here. That is, we should solve $\pi G = 0$. However, this could be a bit lengthy. However, we do know what the answer is from part (a). Hence, using Theorem 5.10 from the summary notes, once we have solution π we know it'll be unique. The answer π in this case is given by

   $$\pi = \frac{1}{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}\{\beta_A\beta_B,\ \beta_A\alpha_B,\ \alpha_A\beta_B,\ \alpha_A\alpha_B\}.$$

3. No. Try to calculate $P(Y_n = 0 | Y_{n-1} = 1)$ - you can't. The problem is that we've lost too much information by "putting together" states 1 and 2 of the original chain X_n.