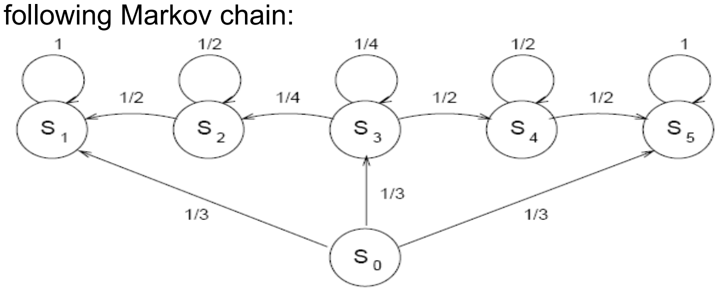
Tutorial 6

1. If "whether tomorrow is raining" is only related with the weather of today, and suppose the probability of "Both of today and tomorrow are raining" is 0.8; the probability of "Neither today nor tomorrow is raining" is 0.7. The state of raining or not on any given day is 1 and 0, respectively.
   (1) Is "whether it's raining on any given day" a Markov chain. If yes, find the matrix of transition probability.
   (2) Find the probability of "Day 3" is raining, given "Day 1" is raining.

2. Let $X_1, X_2, \ldots$ be independent, identical distributed random variables such that $P[X_i = j] = \alpha_j$, $j \geq 0$. We say that a record occurs at time n if $X_n > \max(X_1, \ldots, X_{n-1})$, where $X_0 = -\infty$, and if a record does occur at time n, let X_n be called the record value. Let R_i denote the ith record value.
   (1) Argue that $\{R_i, i \geq 1\}$ is a Markov chain.
   (2) Compute its transition probabilities.

3. Consider the following Markov chain:



   Given that the above process is in state S_0 just before the first trial, determine by inspection the probability that:
   (1) The process enters S_2 for the first time as the result of the $k-th$ trial
   (2) The process enters S_2 and then leaves S_2 on the next trial.
   (3) The process enters S_1 for the first time on the third trial.
   (4) The process is in state S_3 immediately after the nth trial.

Solution 6

1. Solution:

(1) We define the state "Whether it's raining on any given day" as $X(n)$, there are two states for $X(n)$: 0,1. n is the day sequence. $\{X(n), n \in Z\}$ is a random sequence. From the definition of Markov process and the given question, this is a Markov chain. The corresponding matrix of transition probability is:

$$(P_{ij}(1)) = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

(2) By the Markov property, we have $(P_{ij}(n)) = (P_{ij}(1))^n$ .Then,

$$(P_{ij}(3)) = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}^3 = \begin{pmatrix} 0.475 & 0.525 \\ 0.350 & 0.650 \end{pmatrix}$$

Because "Day 1" is raining, then $\pi_1(n) = 1$ and $\pi_0(n) = 0$.

The state vector for "Day 3" is:

$$\begin{aligned} &(\pi_0(n+3), \pi_1(n+3)) = (\pi_0(n), \pi_1(n))(P_{ij}(3)) \\ &= (0,1)\begin{pmatrix} 0.475 & 0.525 \\ 0.350 & 0.650 \end{pmatrix} \\ &= (0.350, 0.650) \end{aligned}$$

Therefore, the probability of "Day 3" is raining, given "Day 1" is raining, is 0.650.

2. Solution:

(1) Consider that the current record value is R_i for some $i \geq 1$. The value of the next record value R_{i+1} must be larger than R_i . So given R_i ,the value of R_{i+1} is completely characterized by the X_i's , which are i.i.d. Therefore, $\{R_i, i \geq 1\}$ is a Markov chain.

(2) Without loss of generality, let X_k be the element in the sequence $\{X_j, j \geq 1\}$ that corresponds to R_{i+1} . We have:

$$P(R_{i+1} = n | R_i = m) = P(X_k = n | X_k > m) = \frac{\alpha_n}{\sum_{j=m+1}^{\infty} \alpha_j}$$

Where $n > m$. So, considering the cases with $n \leq m$ leads to:

$$P(R_{i+1} = n | R_i = m) = \begin{cases} 0, & \text{for n} \leq \text{m} \\ \dfrac{\alpha_n}{\sum_{j=m+1}^{\infty} \alpha_j}, & \text{for n} > \text{m} \end{cases} \quad \text{for all m=0,1,2,...}$$

3. Solution:

(1) Let A_k be the event that the process enters S_2 for first time on trial k. The only way to enter state S_2 for the first time on the kth trial is to enter state S_3 on the first trial, remain in S_3 for the next k− 2 trials, and finally enter S_2 on the last trial. Thus,

$$P(A_k) = P_{03} \cdot P_{33}^{K-2} \cdot P_{32} = \frac{1}{3} \cdot \left(\frac{1}{4}\right)^{k-2} \cdot \frac{1}{4} = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1}, \quad \text{for k=2,3,...;}$$

(2)

$$\begin{aligned} &P(\{\text{Process enters } S_2 \text{ and then leaves } S_2 \text{ on next trial}\}) \\ &= P(\{\text{Process enters } S_2\})P(\{\text{leaves } S_2 \text{ on next trial}\} | \{\text{in } S_2\}) \\ &= [\sum_{k=2}^{\infty} P(A_k)] \cdot \frac{1}{2} = [\sum_{k=2}^{\infty} \frac{1}{3} (\frac{1}{4})^{k-1}] \cdot \frac{1}{2} = \frac{1}{6} \cdot \frac{1/4}{1-1/4} = \frac{1}{18} \end{aligned}$$

(3) This event can only happen if the sequence of state transitions is as follows:

$$S_0 \to S_3 \to S_2 \to S_1$$

Thus, $P(\{\text{Process enters } S_1 \text{ for first time on third trial}\}) = P_{03}P_{32}P_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$.

(4)

$$\begin{aligned} &P(\{\text{Process in } S_3 \text{ immediately after the Nth trial}\}) \\ &= P(\{\text{moves to } S_3 \text{ in first trial and stays in } S_3 \text{ for next N-1 trials}\}) \\ &= \frac{1}{3} \cdot (\frac{1}{4})^{n-1}, \quad \text{for n=1,2,3,...} \end{aligned}$$