Solution 5

1. Solution: $E[X(t)] = E[V] = 1/2$

   $R(t+\tau, t) = E[X(t+\tau)X(t)] = E[V^2] = 1/3$

   Therefore, $X(t)$ is stationary.

   However, $Average[X(t,s)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Vdt = V \neq E[X(t)]$

   $X(t)$ is not ergodic. Actually, when sample $X(t)$ is a constant, it cannot go through all the value between 0 and 1.

2. Solution:

   (1) The transition probability from m_1 to m_2: $\pi_{i_1 i_2}(m_1, m_2) = P(X(m_2) = i_2 \mid X(m_1) = i_1)$;

   The transition probability from m_1 to m_3: $\pi_{i_1 i_3}(m_1, m_3) = P(X(m_3) = i_3 \mid X(m_1) = i_1)$;

   The transition probability from m_2 to m_3: $\pi_{i_2 i_3}(m_1, m_2) = P(X(m_3) = i_3 \mid X(m_2) = i_2)$;

   (2)

$$
\begin{aligned}
\pi_{i_1 i_3}(m_1, m_3) &= P(X(m_3) = i_3 \mid X(m_1) = i_1) \\
&= \sum_{i_2} P(X(m_3) = i_3, X(m_2) = i_2 \mid X(m_1) = i_1) \\
&= \sum_{i_2} \left\{ P(X(m_3) = i_3 \mid X(m_1) = i_1, X(m_2) = i_2) P(X(m_2) = i_2 \mid X(m_1) = i_1) \right\} \\
&= \sum_{i_2} \left\{ P(X(m_3) = i_3 \mid X(m_2) = i_2) P(X(m_2) = i_2 \mid X(m_1) = i_1) \right\} \\
&= \sum_{i_2} \pi_{i_1 i_2}(m_1, m_2) \pi_{i_2 i_3}(m_2, m_3)
\end{aligned}
$$

3. Solution:

   (1)

$$
\begin{aligned}
P(Y(n) = k) &= P[\sum_{i=1}^{n} X(i) = k] = P[\text{there are k ones among } X(1), X(2)...X(n)] \\
&= C_n^k p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k}
\end{aligned}
$$

   (2)

$$
\begin{aligned}
\pi_{ij}(k, k+5) &= P[Y(k+5) = j \mid Y(k) = i] \\
&= P[Y(k) + \sum_{i=1}^{5} X(k+i) = j \mid Y(k) = i] \\
&= P[\sum_{i=1}^{5} X(k+i) = j - i \mid Y(k) = i] \\
&= P[\sum_{i=1}^{5} X(k+i) = j - i] \\
&= \begin{cases} C_5^{j-i} p^{i-i} q^{5-j-i}, & (j-i) \in [0,5] \\ 0 \quad , & \text{otherwise} \end{cases}
\end{aligned}
$$

4. Solution: According $\sum_j p_{ij} = \sum_j p(x_n = j \mid x_{n-1} = i) = 1$, $p_{12} = 0.3$.

$$
\pi(2) = \pi(0) \cdot P^2 = (0.5, 0.5) \begin{pmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{pmatrix}^2 = (0.725, 0.725)
$$

$$
\pi(3) = \pi(0) \cdot P^3 = (0.5, 0.5) \begin{pmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{pmatrix}^3 = (0.728, 0.272)
$$