Solution 4

1. Solution:

$$R_Y(t_1,t_2)=E[[X(t_1)+A(t_1)][X(t_2)+A(t_2)]^*]$$
$$=R_X(t_1,t_2)+R_A(t_1,t_2)$$

Furthermore, if $X(t)$ and $A(t)$ are stationary, then $Y(t)$ is stationary as well. We have
$$R_Y(\tau)=R_X(\tau)+R_A(\tau)$$

2. Solution:

$$R_x(0)=E[x_{k+1}x_{k+1}^T]=E[(Ax_k+be_k)(Ax_k+be_k)^T]=E[(Ax_k+be_k)(x_k^T\cdot A^T+e_k^Tb^T)]$$
$$=E[Ax_kx_k^TA^T+Ax_ke_k^Tb^T+be_kx_k^TA^T+be_ke_k^Tb^T]$$
$$=A\cdot R_x(0)\cdot A^T+b\cdot\sigma^2\cdot b^T=A\cdot R_x(0)\cdot A^T+bb^T$$
$$R_y(0)=E[c^Tx_kx_k^Tc]=c^T\cdot R_x(0)\cdot c$$

3. Solution:

$$\underset{a,b}{\text{Minimize}}\ E[(X_k-aX_{k-1}-bX_{k-2})^2]$$

Take the derivative of the mean square error w.r.t. a and set the result to zero. we get
$$E[(X_k-aX_{k-1}-bX_{k-2})X_{k-1}]=0$$
$$\Rightarrow\quad E[X_kX_{k-1}]-aE[X_{k-1}X_{k-1}]-bE[X_{k-2}X_{k-1}]=0 \quad ①$$

Similarly, differentiation w.r.t. b yields
$$E[(X_k-aX_{k-1}-bX_{k-2})X_{k-2}]=0$$
$$\Rightarrow\quad E[X_kX_{k-2}]-aE[X_{k-1}X_{k-2}]-bE[X_{k-2}^2]=0 \quad ②$$

Solving ① and ② yields
$$a=\frac{var(X_{k-2})\,cov(X_k,X_{k-1})-cov(X_k,X_{k-2})\,cov(X_{k-2},X_{k-1})}{var(X_{k-1})\,var(X_{k-2})-cov(X_{k-1},X_{k-2})^2}$$
$$b=\frac{var(X_{k-1})\,cov(X_k,X_{k-2})-cov(X_k,X_{k-1})\,cov(X_{k-1},X_{k-2})}{var(X_{k-2})\cdot var(X_{k-1})-cov(X_{k-2},X_{k-1})^2}$$

As $var(X_k)=\sigma^2$ for $\forall k$,
$$a=\frac{\sigma^2cov(X_k,X_{k-1})-cov(X_k,X_{k-2})\,cov(X_{k-1},X_{k-2})}{\sigma^4-cov(X_{k-1},X_{k-2})^2}$$
$$b=\frac{\sigma^2cov(X_k,X_{k-2})-cov(X_k,X_{k-1})\,cov(X_{k-1},X_{k-2})}{\sigma^4-cov(X_{k-1},X_{k-2})^2}$$

4. Solution:

Subs. $cov(X_i, X_j) = \rho_{|i-j|}\sigma^2$ into the expressions for a and b in part b:

$$a = \frac{\sigma^2 \rho_1 \sigma^2 - \rho_2 \sigma^2 \rho_1 \sigma^2}{\sigma^4 - \rho_1^2 \sigma^4} = \frac{\rho_1 (1 - \rho_2)}{1 - \rho_1^2}$$

$$b = \frac{\sigma^2 \rho_2 \sigma^2 - \rho_1 \sigma^2 \rho_1 \sigma^2}{\sigma^4 - \rho_1^2 \sigma^4} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$