Solution

1. Solution:

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 2 & -2 \\ 2 & \lambda+2 & -4 \\ -2 & -4 & \lambda+2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda-1 & 2 & -2 \\ 2 & \lambda+2 & -4 \\ 0 & \lambda-2 & \lambda-2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda-1 & 4 & -2 \\ 2 & \lambda+6 & -4 \\ 0 & 0 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-1 & 4 \\ 2 & \lambda+6 \end{vmatrix}$$

$$= (\lambda-2)^2(\lambda+7) = 0$$

Therefore, the eigenvalue are $\lambda_1 = 2$ (double), $\lambda_2 = 7$

For $\lambda_1 = 2$, the coefficient matrix of $(\lambda_1 I - A)X = 0$ is:

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have $\begin{cases} x_1 = -2x_2 + 2x_3 \\ x_2 = \quad x_2 \\ x_3 = \qquad x_3 \end{cases}$, the corresponding eigenvectors are $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -7$, the coefficient matrix of $(\lambda_1 I - A)X = 0$ is:

$$\begin{pmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have $\begin{cases} x_1 = -0.5x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$, the corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

2. Solution:

$$(A, I) = \left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 + r_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 + r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-2r_3 + r_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{2} \cdot r_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & -1.5 & -2.5 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \qquad \Rightarrow \quad A^{-1} = \begin{pmatrix} -0.5 & -1.5 & -2.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \end{pmatrix}$$

3. Solution:
Y(t) is wide-sense stationary.

$$
\begin{aligned}
E[Y(t)] &= E[X(t)]\cdot E[\cos(wt+\theta)] \\
&= E[X(t)]\cdot \int_{-\pi}^{+\pi} \cos(wt+\theta)\frac{1}{2\pi}d\theta \\
&= 0 \quad \text{(constant)}
\end{aligned}
$$

$$
\begin{aligned}
E[Y(t+\tau)Y(t)] &= E[X(t+\tau)X(t)]\cdot E[\cos(wt+w\tau+\theta)\cos(wt+\theta)] \\
&= R_x(\tau)\cdot\frac{1}{2}E[\cos(wt)+\cos(2wt+2\theta+w\tau)] \\
&= \frac{1}{2}R_x(\tau)\cos(w\tau)
\end{aligned}
$$

4. Solution:

$\operatorname{cov}(X,Y+Z) = E[X(Y+Z)] - E(X)E[Y+Z]$

a). $= E[XY]+E[XZ]-E(X)E(Y)-E(X)E(Z)$

$\Rightarrow \text{cov(X,Y)+cov(X,Z)}$

b). $\operatorname{cov}(X,E(Y|X)) = E(X\cdot E(Y|X)) - E(X)E(E(Y|X))$

$\Rightarrow \operatorname{cov}(X,E(Y|X)) = E(XY)-E(X)E(Y) = \operatorname{cov}(X,Y)$

c). Given $E[Y|X] = \alpha+\beta X$

$\operatorname{cov}(X,Y) = \operatorname{cov}(X,E(Y|X)) \quad \text{from part b}$

$=\operatorname{cov}(X,\alpha+\beta X) = E[X(\alpha+\beta X)] - E(X)E(\alpha+\beta X)$

$\Rightarrow \text{cov(X,Y)}=\alpha\text{E(X)}+\beta\text{E}[\text{X}^2]\text{-}\alpha\text{E(X)-}\beta\text{E(X)E(X)}=\beta[\text{E(X}^2\text{)-}\{\text{E(x)}\}^2]$

$\Rightarrow \text{cov(X,Y)}=\beta\text{var(X)} \Rightarrow \beta=\dfrac{\text{cov(X,Y)}}{\text{var(X)}}$

5. Solution:

$$
\begin{aligned}
E[x_k x_k^T] &= E[(Ax_k+be_k)(Ax_k+be_k)^T] = E[(Ax_k+be_k)(x_k^T\cdot A^T+e_k^T b^T)] \\
&= E[Ax_k x_k^T A^T + Ax_k e_k^T b^T + be_k x_k^T A^T + be_k e_k^T b^T] \\
&= A\cdot R_x(0)\cdot A^T + b\cdot\sigma^2\cdot b^T = A\cdot R_x(0)\cdot A^T + \sigma^2 bb^T
\end{aligned}
$$