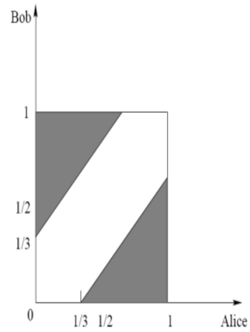
Solution

1. (a) $A \cup B \cup C$
   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
   (d) $A \cap B \cap C$
   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
   (f) $A \cap B \cap C^c$
   (g) $A \cup (A^c \cap B^c)$

2.

P(A)

If Alice picked 0, Bob would have to pick 1/3 or higher for the difference to be greater than 1/3, and vice versa. Following this logic, we obtain the following figure where the axes represent Alice and Bob's choices and the shaded areas represent points where Alice's and Bob's choices differ by at least 1/3.



By the uniform law, the probability of the event is proportional to the area of the event in the sample space:

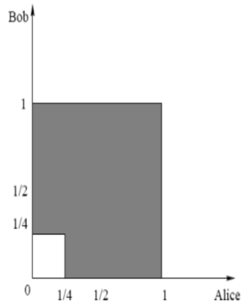
Total Area of the Sample Space = 1 * 1 = 1

$$\begin{aligned}\text{Area of the event} &= 2 * \frac{(2/3)^2}{2} \\ &= 4/9\end{aligned}$$

$$\begin{aligned}P(A) &= \frac{\text{Area of the event}}{\text{Total Area of the sample space}} \\ &= \frac{4/9}{1} \\ &= \boxed{4/9}\end{aligned}$$

P(B)

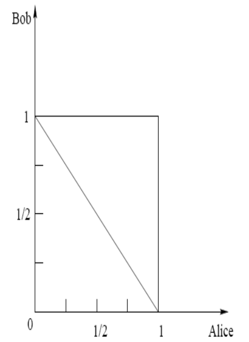
The shaded area in the following figure is the union of Alice's pick being greater than 1/4 and Bob's pick being greater than 1/4.



$$\begin{aligned}P(B) &= 1 - P(\text{both numbers are smaller than } 1/4) \\ &= 1 - \frac{\text{Area of small square}}{\text{Total sample area}} \\ &= 1 - \frac{1/4 * 1/4}{1} \\ &= 1 - 1/16 \\ &= \boxed{15/16}\end{aligned}$$

P(C)

In the following figure, the diagonal of slope −1 represents the set of points where the sum of the two numbers is one.



The line has an area of 0. Thus,

$$\begin{aligned}P(C) &= \frac{\text{Area of line}}{\text{Total sample area}} \\ &= \frac{0}{1} \\ &= \boxed{0}\end{aligned}$$

3. (a) No. A and B are not independent. To see this, note that $A \subset B$, hence $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. This is equal to $\mathbf{P}(A) \cdot \mathbf{P}(B)$ only when $\mathbf{P}(B) = 1$ or $\mathbf{P}(A) = 0$. But in our example, clearly $\mathbf{P}(B) < 1$ and $\mathbf{P}(A) > 0$. Hence $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$, and thus A and B are not independent.

(b) Yes. Conditioned on C, A will happen if and only if Imno meets 5 people during the second week. Hence $\mathbf{P}(A|C) = 1/5$.

If Imno made 5 friends in the first week, she is certain to make more than 5 friends in total. Hence $\mathbf{P}(B|C) = 1$.

If A happens, B will also happen, so clearly $\mathbf{P}(A \cap B|C) = \mathbf{P}(A|C) = \mathbf{P}(A|C) \cdot \mathbf{P}(B|C)$, therefore A and B are conditionally independent. Note that A and B were not independent prior to the conditioning.

4. The pdf of X and Y are: $f_x(x) = \begin{cases} 1/2a, & |x| \le a \\ 0 & others \end{cases}$, $f_y(y) = \begin{cases} 1/2a, & |y| \le a \\ 0 & others \end{cases}$

Using the equation: $f_z(z) = \int_{-\infty}^{+\infty} f_x(x) f_y(z-x)dx$

Consider X-Z region: G={(x,z): $|x| \le a$, $|z-x| \le a$ }, then: $f_x(x) f_y(y) = \begin{cases} 1/4a^2, & (x,z) \in G \\ 0, & others \end{cases}$.

Finally, $f_z(z) = \begin{cases} \dfrac{2a+z}{4a^2}, & -2a \le z < 0 \\ \dfrac{2a-z}{4a^2}, & 0 \le z \le 2a \\ 0, & others \end{cases}$.

5.

$$f_Y(x) = \begin{cases} 0.08x, & x \in [0,5], \\ 0, & \text{otherwise}, \end{cases} \qquad F_Y(x) = \begin{cases} 0, & x < 0, \\ 0.04x^2, & x \in [0,5], \\ 1, & \text{otherwise}, \end{cases}$$

$E[Y] = 10/3$, $\text{var}[X] = 25/18$.