Discrete-Time Fourier Transform
Discrete Fourier Transform
z-Transform

Tania Stathaki

811b

t.stathaki@imperial.ac.uk

Joseph Fourier (1768-1830)

Discrete-Time Fourier Transform

- Definition - The **Discrete-Time Fourier Transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + j\, X_{\text{im}}(e^{j\omega})$$

Discrete-Time Fourier Transform

- $X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω
- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

Discrete-Time Fourier Transform

- $|X(e^{j\omega})|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

Discrete-Time Fourier Transform

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega})$$

$$X_{\mathrm{re}}(e^{j\omega}) = \left|X(e^{j\omega})\right|\cos\theta(\omega)$$

$$X_{\mathrm{im}}(e^{j\omega}) = \left|X(e^{j\omega})\right|\sin\theta(\omega)$$

$$\left|X(e^{j\omega})\right|^2 = X_{\mathrm{re}}^2(e^{j\omega}) + X_{\mathrm{im}}^2(e^{j\omega})$$

$$\tan\theta(\omega) = \frac{X_{\mathrm{im}}(e^{j\omega})}{X_{\mathrm{re}}(e^{j\omega})}$$

Discrete-Time Fourier Transform

- For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω **(Prove using previous slide relationships)**
- Note:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega+2\pi k)}$$

$$= \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

for any integer k

- The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the **principal value**

Discrete-Time Fourier Transform

- The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses
- An alternate type of phase function that is a continuous function of ω is often used
- It is derived from the original phase function by removing the discontinuities of 2π

Discrete-Time Fourier Transform

- The process of removing the discontinuities is called "**unwrapping**"
- The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$
- In some cases, discontinuities of π may be present after unwrapping

Discrete-Time Fourier Transform

- Example - The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\Delta(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

- Example - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1, \mu[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Discrete-Time Fourier Transform

- Its DTFT is given by

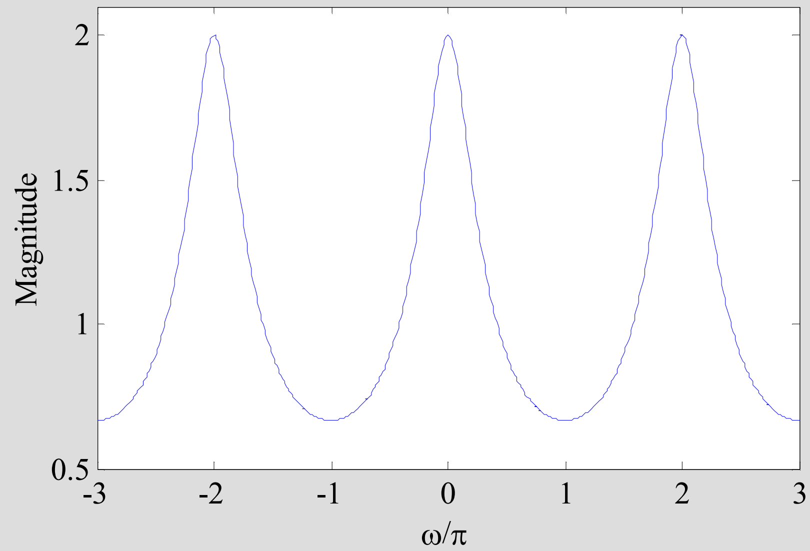
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

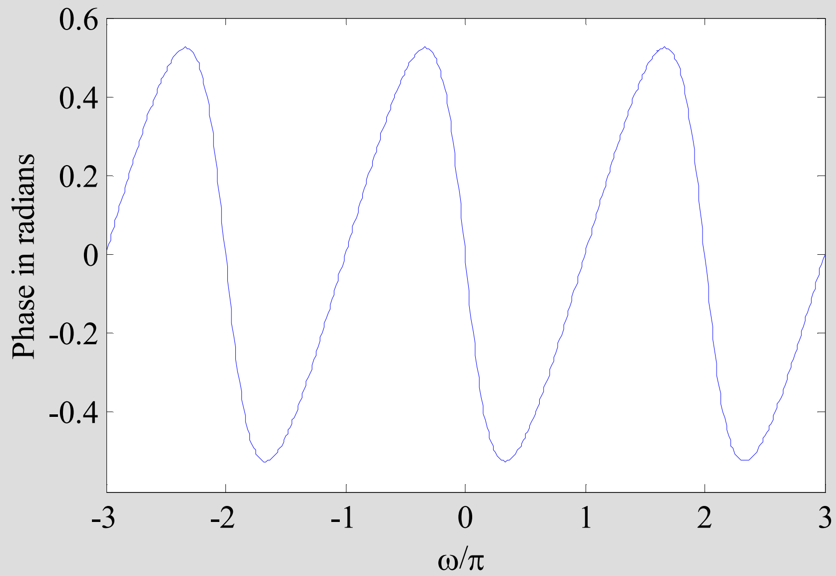
$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

Discrete-Time Fourier Transform

- The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1 - 0.5\,e^{-j\omega})$ are shown below





Discrete-Time Fourier Transform

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π:

$$X(e^{j(\omega_o+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega_o+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_o n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_o n} = X(e^{j\omega_o})$$

Discrete-Time Fourier Transform

- **Inverse Discrete-Time Fourier Transform**:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Proof:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega \ell} \right) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e., $X(e^{j\omega})$ exists
- Then $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty}x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$

$$= \sum_{\ell=-\infty}^{\infty}x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty}x[\ell]\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}$$

Discrete-Time Fourier Transform

- Now

$$\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases}$$

$$= \delta[n-\ell]$$

- Hence

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] = x[n]$$

Discrete-Time Fourier Transform

- **Convergence Condition -** An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  may or may not converge

- Consider the following approximation

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Then for uniform convergence of $X(e^{j\omega})$,

$$\lim_{K\to\infty} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right| = 0$$

- If $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

for all values of ω

- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT

Discrete-Time Fourier Transform

- Example - The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and therefore its DTFT $X(e^{j\omega})$ converges to $1/(1-\alpha\, e^{-j\omega})$ uniformly

Discrete-Time Fourier Transform

- Since

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 \le \left(\sum_{n=-\infty}^{\infty} \left| x[n] \right| \right)^2,$$

an **absolutely summable sequence** has always a **finite energy**

- However, a finite-energy sequence is not necessarily absolutely summable

Discrete-Time Fourier Transform

- Example - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathsf{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

- However, $x[n]$ is not absolutely summable since the summation

$$\sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

Discrete-Time Fourier Transform

- To represent a finite energy sequence that is not absolutely summable by a DTFT, it is necessary to consider a **mean-square convergence** of $X(e^{j\omega})$

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

must approach zero at each value of ω as K goes to ∞

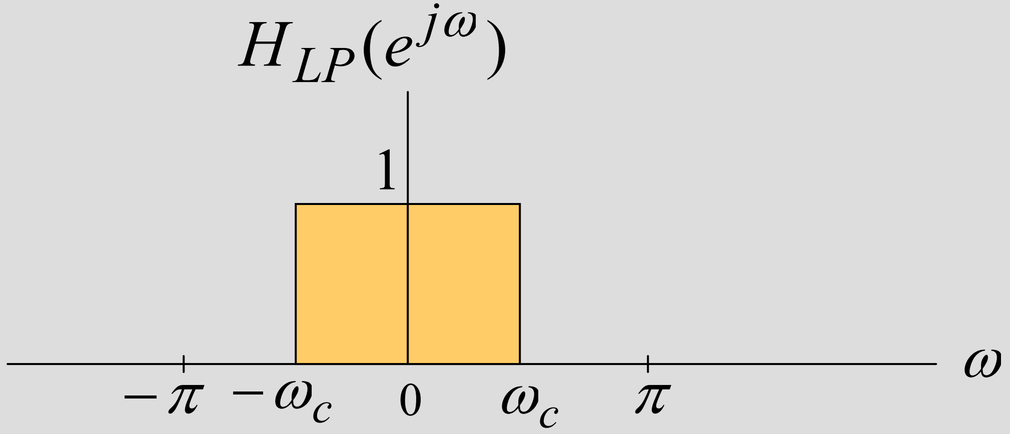
- In such a case, the absolute value of the error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ may not go to zero as K goes to ∞ and the DTFT is no longer bounded

Discrete-Time Fourier Transform

- Example - Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

shown below



Discrete-Time Fourier Transform

- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by ω_c / π

(See slide 46 for proof. Parseval's Theorem stated in slide 37 is used).

- $\Longrightarrow$ $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable

Discrete-Time Fourier Transform

- As a result

$$\sum_{n=-K}^{K} h_{LP}[n] e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense

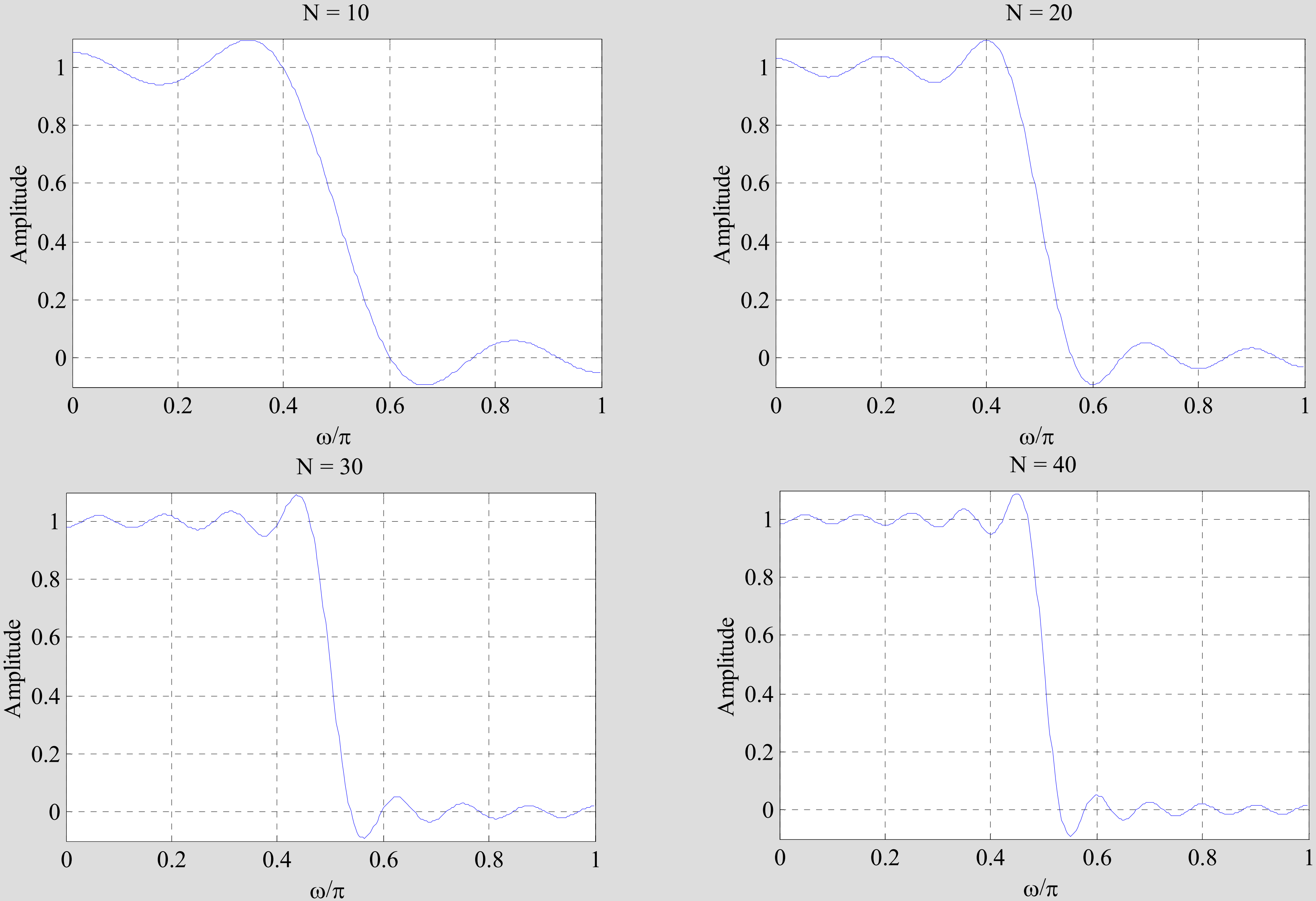
Discrete-Time Fourier Transform

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next

Discrete-Time Fourier Transform



Discrete-Time Fourier Transform

- As can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K

Discrete-Time Fourier Transform

- As K goes to infinity, the condition

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega}) \right|^2 d\omega = 0$$

  holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$
- The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**

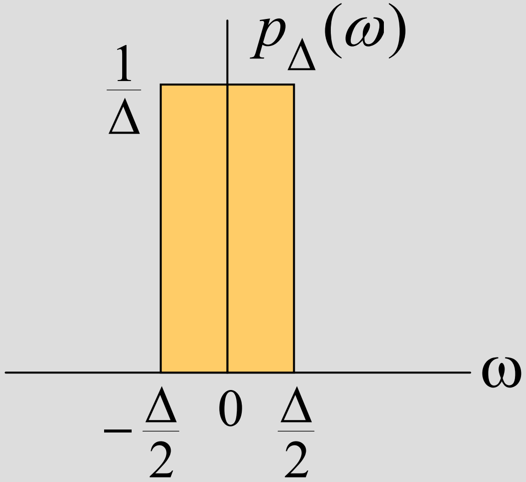
Discrete-Time Fourier Transform

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_o n + \phi)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function** $\delta(\omega)$

Discrete-Time Fourier Transform

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero, satisfying

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_\Delta(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$



Discrete-Time Fourier Transform

- Example - Consider the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \le \omega_o \le \pi$$

Discrete-Time Fourier Transform

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

  is a periodic function of ω with a period 2π and is called a **periodic impulse train**

- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_o n}$ we compute the inverse DTFT of $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Thus

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_o) e^{j\omega n} d\omega = e^{j\omega_o n}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Sequence		**DTFT**
$\delta[n]$	$\leftrightarrow$	1
1	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\mu[n], (\lvert\alpha\rvert < 1)$	$\leftrightarrow$	$\dfrac{1}{1 - \alpha e^{-j\omega}}$

DTFT Properties

- There are a number of important properties of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table: General Properties of DTFT

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\, d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\, d\omega$	

Table: Symmetry relations of the DTFT of a complex sequence

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\text{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

$x[n]$: A complex sequence

Table: Symmetry relations of the DTFT of a real sequence

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

$x[n]$: A real sequence

DTFT Properties

- Example - Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \quad |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], |\alpha| < 1$
- We can therefore write

$$y[n] = n\,x[n] + x[n]$$

- From Tables above, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha\, e^{-j\omega}}$$

DTFT Properties

- Using the differentiation property of the DTFT given in Table above, we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next using the linearity property of the DTFT given in Table above we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

DTFT Properties

- Example - Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- The DTFT of $\delta[n]$ is 1
- Using the time-shifting property of the DTFT given in Table above we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Properties

- Using the linearity property of we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- From Parseval's relation given above we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = \left| G(e^{j\omega}) \right|^2$$

is called the **energy density spectrum**

- Therefore, the area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence

Energy Density Spectrum

- Example - Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) \right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Energy Density Spectrum

- Therefore

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}[n]$ is a finite-energy sequence

DTFT Computation Using MATLAB

- The function **`freqz`** can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + ... + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + ... + d_N e^{-j\omega N}}$$

  at a prescribed set of discrete frequency points $\omega = \omega_\ell$

DTFT Computation Using MATLAB

- For example, the statement

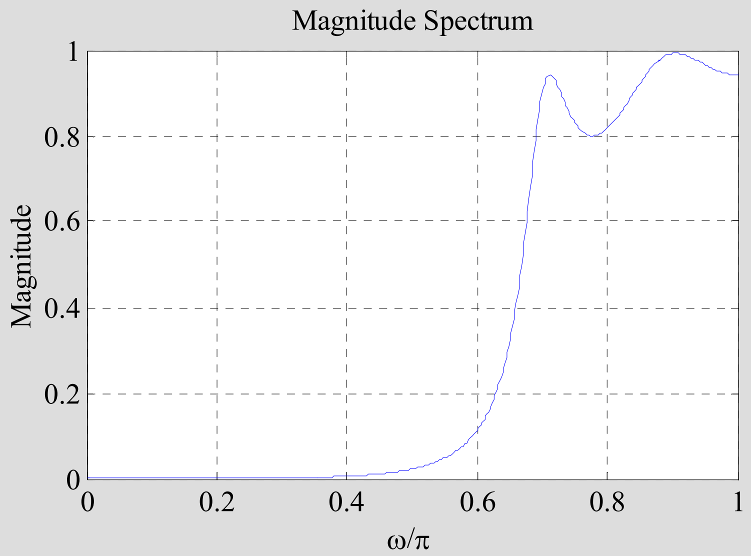
  ```
  H = freqz(num,den,w)
  ```

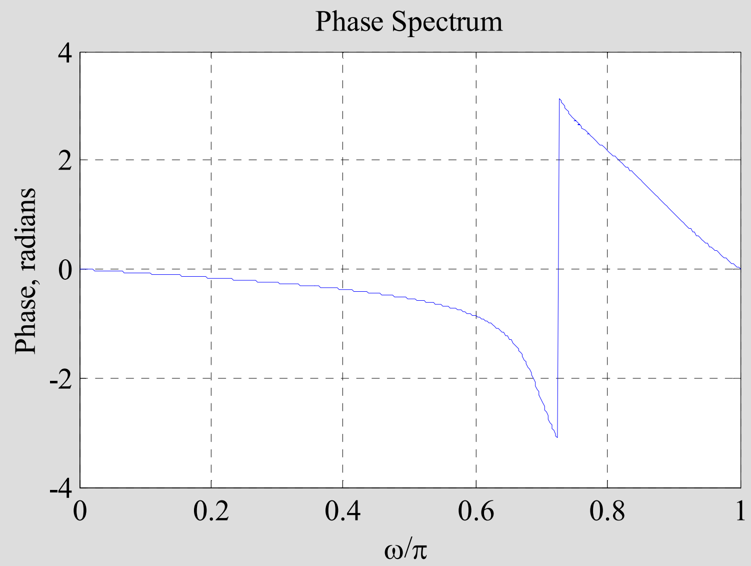
  returns the frequency response values as a vector **`H`** of a DTFT defined in terms of the vectors **`num`** and **`den`** containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector **`w`**
- There are several other forms of the function **`freqz`**

DTFT Computation Using MATLAB

- Example – We illustrate the magnitude and phase of the following DTFT

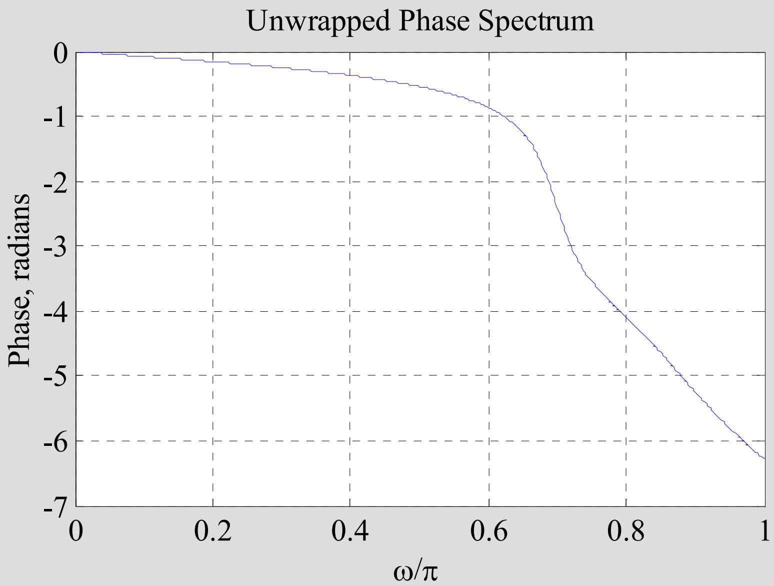
$$X(e^{j\omega}) = \frac{0.008 - 0.033\,e^{-j\omega} + 0.05\,e^{-j2\omega} - 0.033\,e^{-j3\omega} + 0.008\,e^{-j4\omega}}{1 + 2.37\,e^{-j\omega} + 2.7\,e^{-j2\omega} + 1.6\,e^{-j3\omega} + 0.41\,e^{-j4\omega}}$$





DTFT Computation Using MATLAB

- <u>Note</u>: The phase spectrum displays a discontinuity of 2π at $\omega = 0.72$
- This discontinuity can be removed using the function **`unwrap`** as indicated below



Linear Convolution Using DTFT

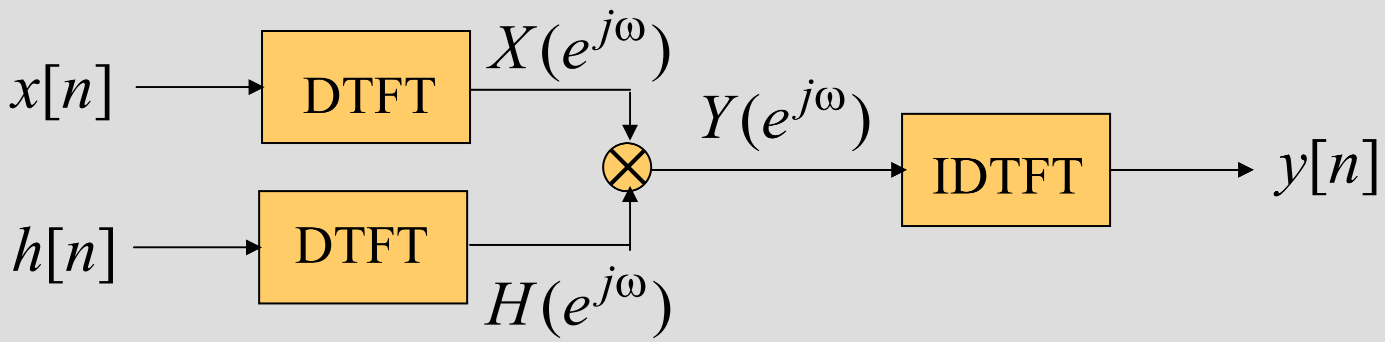
- An important property of the DTFT is given by the convolution theorem
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Linear Convolution Using DTFT

- 1) Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
- 2) Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



Discrete Fourier Transform

- Definition - For a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$ only N samples of its DTFT are required, which are obtained by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \leq \omega \leq 2\pi$ at $\omega_k = 2\pi k/N$, $0 \leq k \leq N-1$

- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{k=0}^{N-1} x[n] e^{-j2\pi k/N},$$

$0 \leq k \leq N-1$

Discrete Fourier Transform

- Note: $X[k]$ is also a length-N sequence in the frequency domain
- The sequence $X[k]$ is called the **Discrete Fourier Transform** (DFT) of the sequence $x[n]$
- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \ 0 \leq k \leq N-1$$

Discrete Fourier Transform

- The **Inverse Discrete Fourier Transform** (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

- To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n = 0$ to $n = N-1$

Discrete Fourier Transform

resulting in

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right) W_N^{\ell n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X[k] W_N^{-(k-\ell)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} X[k] W_N^{-(k-\ell)n}$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, & \text{for } k-\ell = rN, r \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

  we observe that the RHS of the last equation is equal to $X[\ell]$

- Hence

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = X[\ell]$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \le n \le m-1,\ m+1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence defined for $0 \le n \le N-1$

$$g[n] = \cos(2\pi rn/N), \quad 0 \le r \le N-1$$

- Using a trigonometric identity we can write

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

Discrete Fourier Transform

- The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \leq k, r \leq N-1$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, \ r \text{ an integer} \\ 0, \quad \text{otherwise} \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N - r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k, r \le N-1$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \dots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \dots & x[N-1] \end{bmatrix}^T$$

Matrix Relations

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

Matrix Relations

- Likewise, the IDFT relation given by

$$x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

Matrix Relations

where

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

- Note:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

DFT Computation Using MATLAB

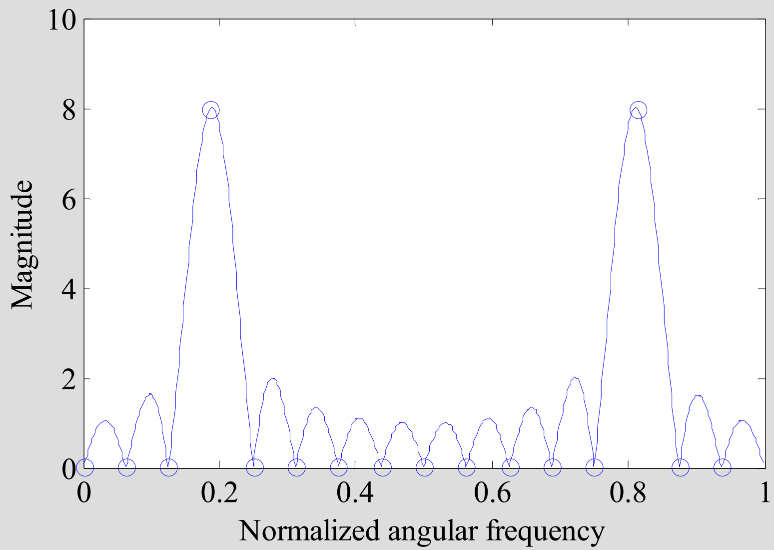
- The functions to compute the DFT and the IDFT are `fft` and `ifft`
- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation

DFT Computation Using MATLAB

- Example - The DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

are shown below



○ indicates DFT samples

DTFT from DFT by Interpolation

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points

$$\omega = \omega_k = 2\pi k / N, \quad 0 \le k \le N-1$$

- Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$!

DTFT from DFT by Interpolation

- Thus

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n}}_{S}$$

DTFT from DFT by Interpolation

- To develop a compact expression for the sum S, let $r = e^{-j(\omega - 2\pi k/N)}$

$$= \sum_{n=1}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

- Then $S = \sum_{n=0}^{N-1} r^n$
- From the above

$$rS = \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N-1} r^n + r^N - 1$$

$$= \sum_{n=1}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

DTFT from DFT by Interpolation

- Or, equivalently,

$$S - rS = (1-r)S = 1 - r^N$$

- Hence

$$S = \frac{1-r^N}{1-r} = \frac{1-e^{-j(\omega N - 2\pi k)}}{1-e^{-j[\omega-(2\pi k/N)]}}$$

$$= \frac{\sin\left(\dfrac{\omega N - 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

DTFT from DFT by Interpolation

- Therefore

$$X(e^{j\omega})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\dfrac{\omega N - 2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

Sampling the DTFT

- Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k / N,\ 0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$

Sampling the DTFT

- Now $X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$
- Thus $Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\ell/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}$$

- An IDFT of $Y[k]$ yields

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}$$

Sampling the DTFT

- i.e. $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-k(n-r)} = \begin{cases} 1, & \text{for } r = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Sampling the DTFT

we arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n + mN], \;\; 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

Sampling the DTFT

- To apply

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \leq n \leq N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros

- Thus if $x[n]$ is a length-M sequence with $M \leq N$, then $y[n] = x[n]$ for $0 \leq n \leq N-1$

Sampling the DTFT

- If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$
- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

Sampling the DTFT

$$y[n] = x[n] + x[n+4] + x[n-4]\ ,\ 0 \le n \le 3$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

⇒ $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

Numerical Computation of the DTFT Using the DFT

- A practical approach to the numerical computation of the DTFT of a finite-length sequence
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x[n]$
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k / M$, $0 \le k \le M - 1$, where $M >> N$:

Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x[n]e^{-j2\pi kn/M}$$

Numerical Computation of the DTFT Using the DFT

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$
- The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2
- The function `freqz` employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$

DFT Properties

- Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications
- Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different
- A summary of the DFT properties are given in Tables in the following slides

Table: General Properties of DFT

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k] H[k]$
Modulation	$g[n] h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Table: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\text{Re}\{x[n]\}$	$X_{\text{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\text{Im}\{x[n]\}$	$X_{\text{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\text{pcs}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{pca}}[n]$	$j\,\text{Im}\{X[k]\}$

Note: $x_{\text{pcs}}[n]$ and $x_{\text{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\text{pcs}}[k]$ and $X_{\text{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

Table: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{\text{pe}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{po}}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$
	$\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$
	$\text{Im}\,X[k] = -\,\text{Im}\,X[\langle -k\rangle_N]$
	$\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$
	$\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\text{pe}}[n]$ and $x_{\text{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

Circular Shift of a Sequence

- This property is analogous to the time-shifting property of the DTFT, but with a subtle difference
- Consider length-N sequences defined for

$$0 \leq n \leq N-1$$

- Sample values of such sequences are equal to zero for values of $n < 0$ and $n \geq N$

Circular Shift of a Sequence

- If $x[n]$ is such a sequence, then for any arbitrary integer n_o, the shifted sequence

$$x_1[n] = x[n - n_o]$$

  is no longer defined for the range $0 \le n \le N-1$
- We thus need to define another type of a shift that will always keep the shifted sequence in the range $0 \le n \le N-1$

Circular Shift of a Sequence

- The desired shift, called the **circular shift**, is defined using a modulo operation:

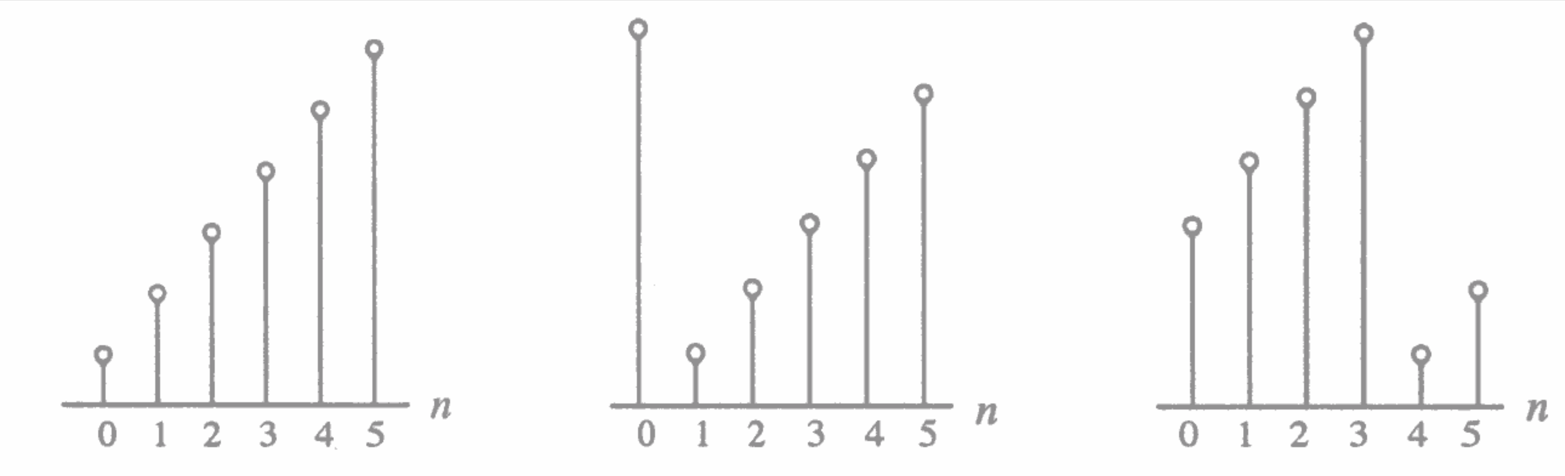
$$x_c[n] = x[\langle n - n_o \rangle_N]$$

- For $n_o > 0$ (right circular shift), the above equation implies

$$x_c[n] = \begin{cases} x[n - n_o], & \text{for } n_o \le n \le N-1 \\ x[N - n_o + n], & \text{for } 0 \le n < n_o \end{cases}$$

Circular Shift of a Sequence

- Illustration of the concept of a circular shift



$x[n]$ $\quad$ $x[\langle n-1\rangle_6]$ $\quad$ $x[\langle n-4\rangle_6]$

$= x[\langle n+5\rangle_6]$ $\quad$ $= x[\langle n+2\rangle_6]$

Circular Shift of a Sequence

- As can be seen from the previous figure, a right circular shift by n_o is equivalent to a left circular shift by $N - n_o$ sample periods
- A circular shift by an integer number n_o greater than N is equivalent to a circular shift by $\langle n_o \rangle_N$

Circular Convolution

- This operation is analogous to linear convolution, but with a subtle difference
- Consider two length-N sequences, $g[n]$ and $h[n]$, respectively
- Their linear convolution results in a length-$(2N-1)$ sequence $y_L[n]$ given by

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \quad 0 \le n \le 2N-2$$

Circular Convolution

- In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to $2N-1$
- The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right
- The first nonzero value of $y_L[n]$ is $y_L[0] = g[0]h[0]$, and the last nonzero value is $y_L[2N-2] = g[N-1]h[N-1]$

Circular Convolution

- To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift
- Resulting operation, called a **circular convolution**, is defined by

$$y_C[n] = \sum_{m=0}^{N-1} g[m] h[\langle n-m \rangle_N], \quad 0 \le n \le N-1$$

Circular Convolution

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y[n] = g[n] \circledN h[n]$$

- The circular convolution is commutative, i.e.

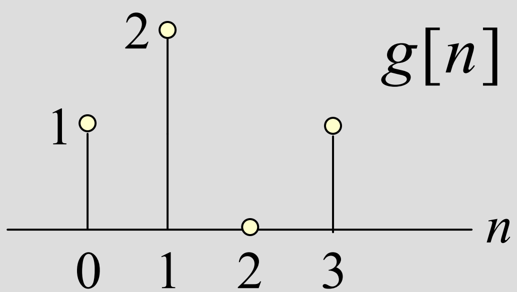
$$g[n] \circledN h[n] = h[n] \circledN g[n]$$

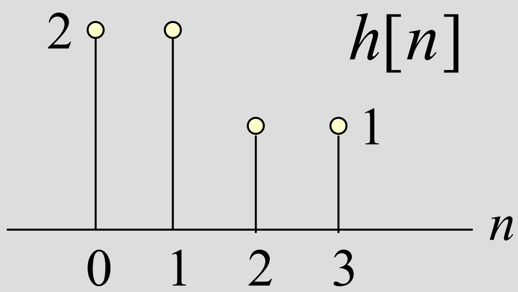
Circular Convolution

- Example - Determine the 4-point circular convolution of the two length-4 sequences:

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

as sketched below





Circular Convolution

- The result is a length-4 sequence $y_C[n]$ given by

$$y_C[n] = g[n] \,④\, h[n] = \sum_{m=0}^{3} g[m]h[\langle n-m\rangle_4], \quad 0 \le n \le 3$$

- From the above we observe

$$y_C[0] = \sum_{m=0}^{3} g[m]h[\langle -m\rangle_4]$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]g[1]$$

$$= (1\times 2) + (2\times 1) + (0\times 1) + (1\times 2) = 6$$

Circular Convolution

- Likewise $y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m \rangle_4]$

$$= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= (1\times 2) + (2\times 2) + (0\times 1) + (1\times 1) = 7$$

$$y_C[2] = \sum_{m=0}^{3} g[m]h[\langle 2-m \rangle_4]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) + (1\times 1) = 6$$

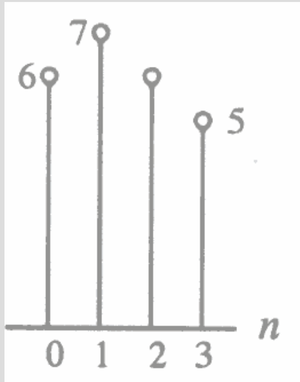
Circular Convolution

and

$$y_C[3] = \sum_{m=0}^{3} g[m]h[\langle 3-m\rangle_4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

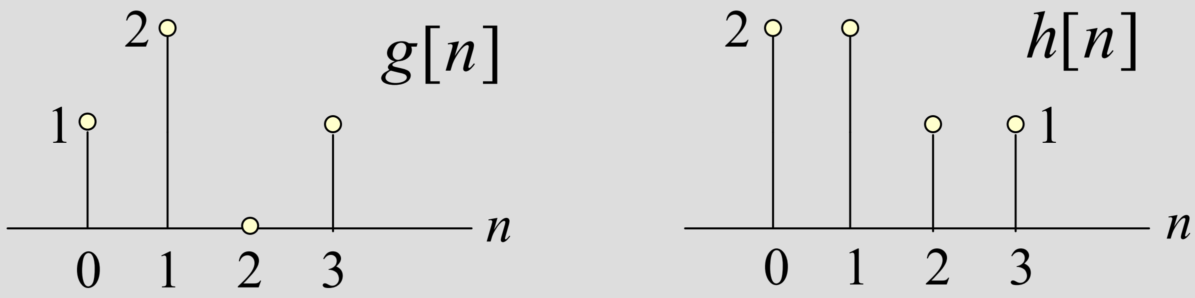


$y_C[n]$

- The circular convolution can also be computed using a DFT-based approach as indicated in previous Table

Circular Convolution

- Example - Consider the two length-4 sequences repeated below for convenience:



- The 4-point DFT $G[k]$ of $g[n]$ is given by

$$G[k] = g[0] + g[1]e^{-j2\pi k/4} + g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4}$$

$$= 1 + 2e^{-j\pi k/2} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

Circular Convolution

- Therefore $G[0]=1+2+1=4,$

$$G[1]=1-j2+j=1-j,$$
$$G[2]=1-2-1=-2,$$
$$G[3]=1+j2-j=1+j$$

- Likewise,

$$H[k]=h[0]+h[1]e^{-j2\pi k/4}$$
$$+h[2]e^{-j4\pi k/4}+h[3]e^{-j6\pi k/4}$$
$$=2+2e^{-j\pi k/2}+e^{-j\pi k}+e^{-j3\pi k/2},\ \ 0\leq k\leq 3$$

Circular Convolution

- Hence, $H[0] = 2 + 2 + 1 + 1 = 6,$

$$H[1] = 2 - j2 - 1 + j = 1 - j,$$

$$H[2] = 2 - 2 + 1 - 1 = 0,$$

$$H[3] = 2 + j2 - 1 - j = 1 + j$$

- The two 4-point DFTs can also be computed using the matrix relation given earlier

Circular Convolution

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$\mathbf{D}_4$ is the 4-point DFT matrix

Circular Convolution

- If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then from Table above we observe

$$Y_C[k] = G[k]H[k], \quad 0 \le k \le 3$$

- Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Circular Convolution

- A 4-point IDFT of $Y_C[k]$ yields

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\mathbf{D}_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

Circular Convolution

- Example - Now let us extended the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.

$$g_e[n] = \begin{cases} g[n], & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 6 \end{cases}$$

Circular Convolution

- We next determine the 7-point circular convolution of $g_e[n]$ and $h_e[n]$:

$$y[n] = \sum_{m=0}^{6} g_e[m] h_e[\langle n-m \rangle_7], \quad 0 \le n \le 6$$

- From the above $y[0] = g_e[0]h_e[0] + g_e[1]h_e[6]$

$$+ g_e[3]h_e[4] + g_e[4]h_e[3] + g_e[5]h_e[2] + g_e[6]h_e[1]$$

$$= g[0]h[0] = 1 \times 2 = 2$$

Circular Convolution

- Continuing the process we arrive at

$$y[1] = g[0]h[1] + g[1]h[0] = (1\times 2) + (2\times 2) = 6,$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) = 5,$$

$$y[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5,$$

$$y[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$

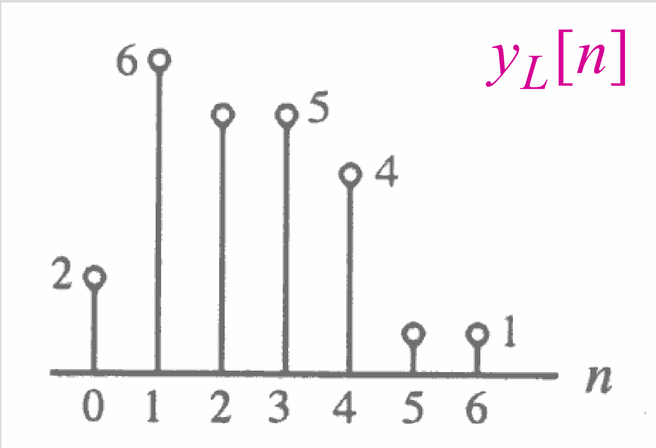
$$= (2\times 1) + (0\times 1) + (1\times 2) = 4,$$

Circular Convolution

$$y[5] = g[2]h[3] + g[3]h[2] = (0\times 1) + (1\times 1) = 1,$$

$$y[6] = g[3]h[3] = (1\times 1) = 1$$

- As can be seen from the above that $y[n]$ is precisely the sequence $y_L[n]$ obtained by a linear convolution of $g[n]$ and $h[n]$



Circular Convolution

- The N-point circular convolution can be written in matrix form as

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

- Note: The elements of each diagonal of the $N \times N$ matrix are equal
- Such a matrix is called a **circulant matrix**

Computation of the DFT of Real Sequences

- In most practical applications, sequences of interest are real
- In such cases, the symmetry properties of the DFT can be exploited to make the DFT computations more efficient

N-Point DFTs of Two Length-N Real Sequences

- Let $g[n]$ and $h[n]$ be two length-N real sequences with $G[k]$ and $H[k]$ denoting their respective N-point DFTs
- These two N-point DFTs can be computed efficiently using a single N-point DFT
- Define a complex length-N sequence

$$x[n] = g[n] + j\,h[n]$$

- Hence, $g[n] = \text{Re}\{x[n]\}$ and $h[n] = \text{Im}\{x[n]\}$

N-Point DFTs of Two Length-*N* Real Sequences

- Let $X[k]$ denote the N-point DFT of $x[n]$
- Then, DFT properties we arrive at

$$G[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k \rangle_N]\}$$

- Note that

$$X^*[\langle -k \rangle_N] = X^*[\langle N - k \rangle_N]$$

N-Point DFTs of Two Length-N Real Sequences

- Example - We compute the 4-point DFTs of the two real sequences $g[n]$ and $h[n]$ given below

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

- Then $\{x[n]\} = \{g[n]\} + j\{h[n]\}$ is given by

$$\{x[n]\} = \{\underset{\uparrow}{1+j2} \quad 2+j2 \quad j \quad 1+j\}$$

N-Point DFTs of Two Length-N Real Sequences

- Its DFT $X[k]$ is

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

- From the above

$$X^*[k] = [4-j6 \quad 2 \quad -2 \quad -j2]$$

- Hence

$$X^*[\langle 4-k \rangle_4] = [4-j6 \quad -j2 \quad -2 \quad 2]$$

N-Point DFTs of Two Length-*N* Real Sequences

- Therefore

$$\{G[k]\} = \{4 \quad 1-j \quad -2 \quad 1+j\}$$

$$\{H[k]\} = \{6 \quad 1-j \quad 0 \quad 1+j\}$$

verifying the results derived earlier

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

- Let $v[n]$ be a length-$2N$ real sequence with an $2N$-point DFT $V[k]$
- Define two length-N real sequences $g[n]$ and $h[n]$ as follows:

$$g[n] = v[2n], \quad h[n] = v[2n+1], \quad 0 \le n \le N$$

- Let $G[k]$ and $H[k]$ denote their respective N-point DFTs

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

- Define a length-N complex sequence

$$\{x[n]\} = \{g[n]\} + j\{h[n]\}$$

with an N-point DFT $X[k]$

- Then as shown earlier

$$G[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k \rangle_N]\}$$

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

- Now $V[k] = \sum_{n=0}^{2N-1} v[n] W_{2N}^{nk}$

$$= \sum_{n=0}^{N-1} v[2n] W_{2N}^{2nk} + \sum_{n=0}^{N-1} v[2n+1] W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + \sum_{n=0}^{N-1} h[n] W_{N}^{nk} W_{2N}^{k}$$

$$= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} h[n] W_{N}^{nk}, \; 0 \le k \le 2N-1$$

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

- i.e.,

$$V[k] = G[\langle k \rangle_N] + W_{2N}^{k} H[\langle k \rangle_N], \quad 0 \le k \le 2N-1$$

- Example - Let us determine the 8-point DFT V[k] of the length-8 real sequence

$$\{v[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 2 \quad 2 \quad 0 \quad 1 \quad 1 \quad 1\}$$

- We form two length-4 real sequences as follows

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

$$\{g[n]\} = \{v[2n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}$$

$$\{h[n]\} = \{v[2n+1]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

- Now

$$V[k] = G[\langle k \rangle_4] + W_8^k H[\langle k \rangle_4], \;\; 0 \le k \le 7$$

- Substituting the values of the 4-point DFTs $G[k]$ and $H[k]$ computed earlier we get

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

$$V[0] = G[0] + H[0] = 4 + 6 = 10$$

$$V[1] = G[1] + W_8^1 H[1]$$

$$= (1 - j) + e^{-j\pi/4}(1 - j) = 1 - j2.4142$$

$$V[2] = G[2] + W_8^2 H[2] = -2 + e^{-j\pi/2} \cdot 0 = -2$$

$$V[3] = G[3] + W_8^3 H[3]$$

$$= (1 + j) + e^{-j3\pi/4}(1 + j) = 1 - j0.4142$$

$$V[4] = G[0] + W_8^4 H[0] = 4 + e^{-j\pi} \cdot 6 = -2$$

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

$$V[5] = G[1] + W_8^5 H[1]$$

$$= (1 - j) + e^{-j5\pi/4}(1 - j) = 1 + j0.4142$$

$$V[6] = G[2] + W_8^6 H[2] = -2 + e^{-j3\pi/2} \cdot 0 = -2$$

$$V[7] = G[3] + W_8^7 H[3]$$

$$= (1 + j) + e^{-j7\pi/4}(1 + j) = 1 + j2.4142$$

Linear Convolution Using the DFT

- Linear convolution is a key operation in many signal processing applications
- Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT

Linear Convolution of Two Finite-Length Sequences

- Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively
- Denote $L = N + M - 1$
- Define two length-L sequences

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

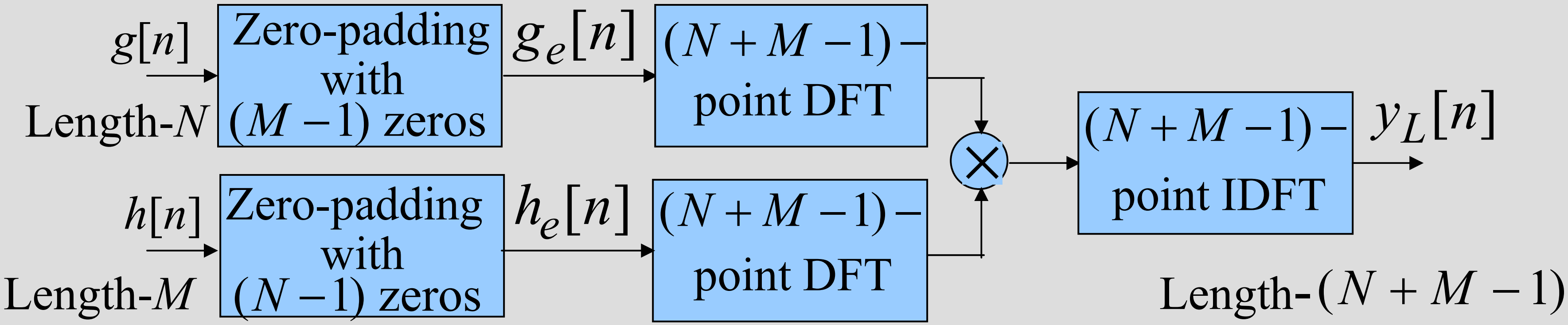
$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

Linear Convolution of Two Finite-Length Sequences

- Then

$$y_L[n] = g[n] \circledast h[n] = y_C[n] = g[n] \, \textcircled{L} \, h[n]$$

- The corresponding implementation scheme is illustrated below



Linear Convolution of a Finite-Length Sequence with an Infinite-Length Sequence

- We next consider the DFT-based implementation of

$$y[n] = \sum_{\ell=0}^{M-1} h[\ell]x[n-\ell] = h[n] \circledast x[n]$$

where $h[n]$ is a finite-length sequence of length M and $x[n]$ is an infinite length (or a finite length sequence of length much greater than M)

Overlap-Add Method

- We first segment $x[n]$, assumed to be a causal sequence here without any loss of generality, into a set of contiguous finite-length subsequences $x_m[n]$ of length N each:

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mN]$$

where

$$x_m[n] = \begin{cases} x[n + mN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Overlap-Add Method

- Thus we can write

$$y[n] = h[n] \circledast x[n] = \sum_{m=0}^{\infty} y_m[n - mN]$$

where

$$y_m[n] = h[n] \circledast x_m[n]$$

- Since $h[n]$ is of length M and $x_m[n]$ is of length N, the linear convolution $h[n] \circledast x_m[n]$ is of length $N + M - 1$

Overlap-Add Method

- As a result, the desired linear convolution $y[n] = h[n] \circledast x[n]$ has been broken up into a sum of infinite number of short-length linear convolutions of length $N + M - 1$ each: $y_m[n] = x_m[n] \text{ Ⓛ } h[n]$
- Each of these short convolutions can be implemented using the DFT-based method discussed earlier, where now the DFTs (and the IDFT) are computed on the basis of $(N + M - 1)$ points

Overlap-Add Method

- There is one more subtlety to take care of before we can implement

$$y[n] = \sum_{m=0}^{\infty} y_m[n - mN]$$

using the DFT-based approach

- Now the first convolution in the above sum, $y_0[n] = h[n] \circledast x_0[n]$, is of length $N + M - 1$ and is defined for $0 \le n \le N + M - 2$

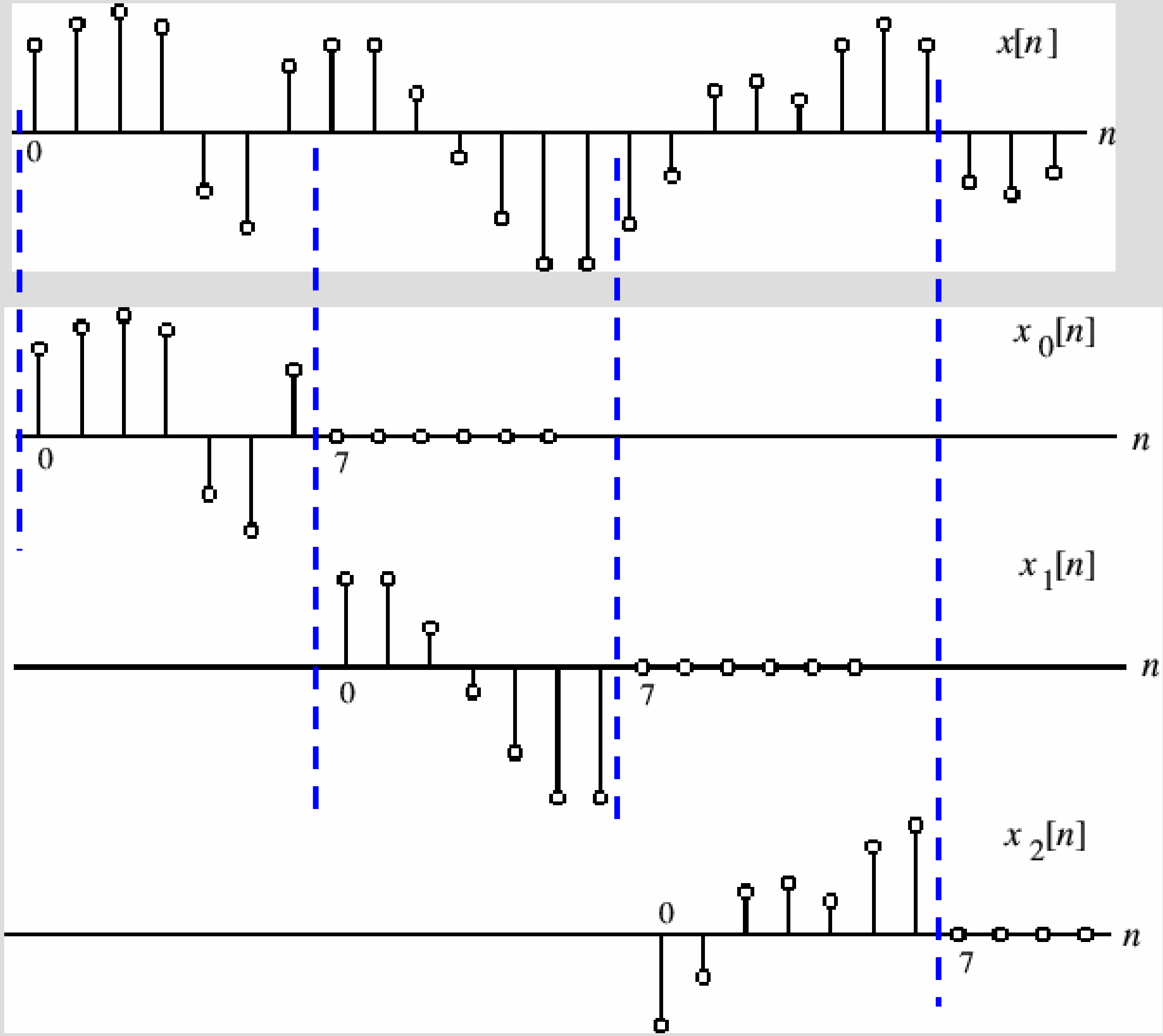
Overlap-Add Method

- The second short convolution $y_1[n] = h[n] \circledast x_1[n]$, is also of length $N + M - 1$ but is defined for $N \leq n \leq 2N + M - 2$
- $\Rightarrow$ There is an overlap of $M - 1$ samples between these two short linear convolutions
- Likewise, the third short convolution $y_2[n] = h[n] \circledast x_2[n]$, is also of length $N + M - 1$ but is defined for $0 \leq n \leq N + M - 2$

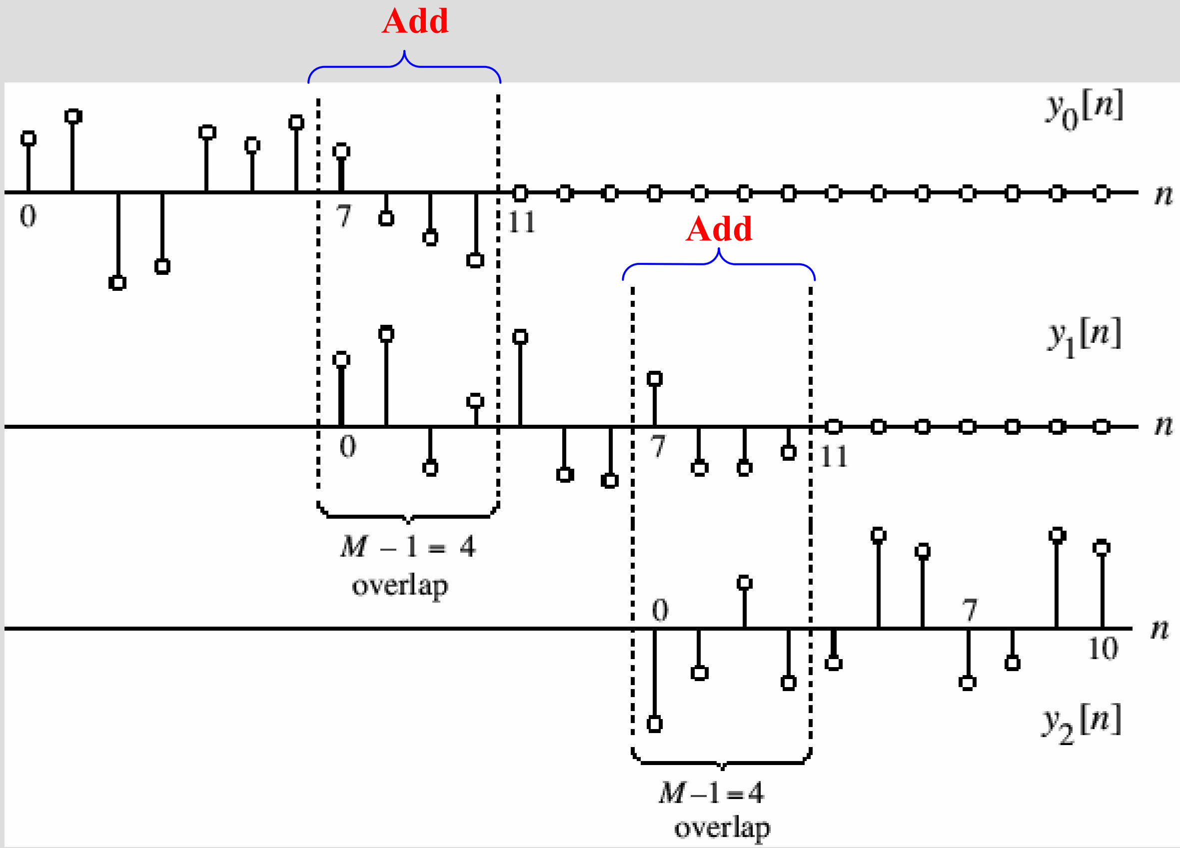
Overlap-Add Method

- Thus there is an overlap of $M-1$ samples between $h[n] \circledast x_1[n]$ and $h[n] \circledast x_2[n]$
- In general, there will be an overlap of $M-1$ samples between the samples of the short convolutions $h[n] \circledast x_{r-1}[n]$ and $h[n] \circledast x_r[n]$ for
- This process is illustrated in the figure on the next slide for $M = 5$ and $N = 7$

Overlap-Add Method



Overlap-Add Method



Overlap-Add Method

- Therefore, $y[n]$ obtained by a linear convolution of $x[n]$ and $h[n]$ is given by

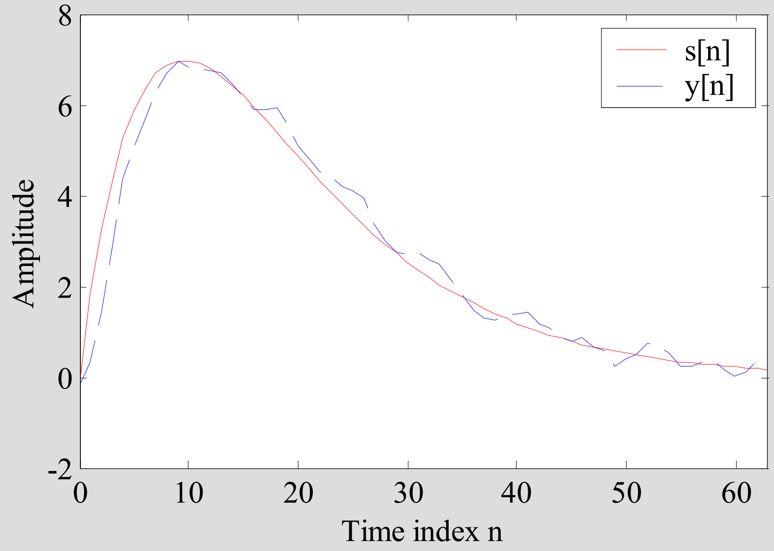
$$
\begin{aligned}
y[n] &= y_0[n], & 0 \le n \le 6 \\
y[n] &= y_0[n] + y_1[n-7], & 7 \le n \le 10 \\
y[n] &= y_1[n-7], & 11 \le n \le 13 \\
y[n] &= y_1[n-7] + y_2[n-14], & 14 \le n \le 17 \\
y[n] &= y_2[n-14], & 18 \le n \le 20 \\
&\vdots
\end{aligned}
$$

Overlap-Add Method

- The above procedure is called the **overlap-add method** since the results of the short linear convolutions overlap and the overlapped portions are added to get the correct final result
- The function **`fftfilt`** can be used to implement the above method

Overlap-Add Method

- We have created a program which uses **`fftfilt`** for the filtering of a noise-corrupted signal y[n] using a length-3 moving average filter. The
- The plots generated by running this program is shown below



Overlap-Save Method

- In implementing the overlap-add method using the DFT, we need to compute two $(N+M-1)$-point DFTs and one $(N+M-1)$-point IDFT since the overall linear convolution was expressed as a sum of short-length linear convolutions of length $(N+M-1)$ each
- It is possible to implement the overall linear convolution by performing instead circular convolution of length shorter than $(N+M-1)$

Overlap-Save Method

- To this end, it is necessary to segment $x[n]$ into overlapping blocks $x_m[n]$, keep the terms of the circular convolution of $h[n]$ with $x_m[n]$ that corresponds to the terms obtained by a linear convolution of $h[n]$ and $x_m[n]$, and throw away the other parts of the circular convolution

Overlap-Save Method

- To understand the correspondence between the linear and circular convolutions, consider a length-4 sequence $x[n]$ and a length-3 sequence $h[n]$
- Let $y_L[n]$ denote the result of a linear convolution of $x[n]$ with $h[n]$
- The six samples of $y_L[n]$ are given by

Overlap-Save Method

$$y_L[0] = h[0]x[0]$$

$$y_L[1] = h[0]x[1] + h[1]x[0]$$

$$y_L[2] = h[0]x[2] + h[1]x[1] + h[2]x[0]$$

$$y_L[3] = h[0]x[3] + h[1]x[2] + h[2]x[1]$$

$$y_L[4] = h[1]x[3] + h[2]x[2]$$

$$y_L[5] = h[2]x[3]$$

Overlap-Save Method

- If we append $h[n]$ with a single zero-valued sample and convert it into a length-4 sequence $h_e[n]$, the 4-point circular convolution $y_C[n]$ of $h_e[n]$ and $x[n]$ is given by

$$y_C[0] = h[0]x[0] + h[1]x[3] + h[2]x[2]$$

$$y_C[1] = h[0]x[1] + h[1]x[0] + h[2]x[3]$$

$$y_C[2] = h[0]x[2] + h[1]x[1] + h[2]x[0]$$

$$y_C[3] = h[0]x[3] + h[1]x[2] + h[2]x[1]$$

Overlap-Save Method

- If we compare the expressions for the samples of $y_L[n]$ with the samples of $y_C[n]$, we observe that the first 2 terms of $y_C[n]$ do not correspond to the first 2 terms of $y_L[n]$, whereas the last 2 terms of $y_C[n]$ are precisely the same as the 3rd and 4th terms of $y_L[n]$, i.e.,

$$y_L[0] \neq y_C[0], \qquad y_L[1] \neq y_C[1]$$

$$y_L[2] = y_C[2], \qquad y_L[3] = y_C[3]$$

Overlap-Save Method

- General case: N-point circular convolution of a length-M sequence $h[n]$ with a length-N sequence $x[n]$ with $N > M$
- First $M-1$ samples of the circular convolution are incorrect and are rejected
- Remaining $N-M+1$ samples correspond to the correct samples of the linear convolution of $h[n]$ with $x[n]$

Overlap-Save Method

- Now, consider an infinitely long or very long sequence $x[n]$
- Break it up as a collection of smaller length (length-4) overlapping sequences $x_m[n]$ as

$$x_m[n] = x[n+2m], \qquad 0 \le n \le 3, \quad 0 \le m \le \infty$$

- Next, form

$$w_m[n] = h[n] \circledast_4 x_m[n]$$

Overlap-Save Method

- Or, equivalently,

$$w_m[0] = h[0]x_m[0] + h[1]x_m[3] + h[2]x_m[2]$$

$$w_m[1] = h[0]x_m[1] + h[1]x_m[0] + h[2]x_m[3]$$

$$w_m[2] = h[0]x_m[2] + h[1]x_m[1] + h[2]x_m[0]$$

$$w_m[3] = h[0]x_m[3] + h[1]x_m[2] + h[2]x_m[1]$$

- Computing the above for $m = 0, 1, 2, 3, \ldots,$ and substituting the values of $x_m[n]$ we arrive at

Overlap-Save Method

$$w_0[0] = h[0]x[0] + h[1]x[3] + h[2]x[2] \quad \leftarrow \text{Reject}$$

$$w_0[1] = h[0]x[1] + h[1]x[0] + h[2]x[3] \quad \leftarrow \text{Reject}$$

$$w_0[2] = h[0]x[2] + h[1]x[1] + h[2]x[0] = y[2] \quad \leftarrow \text{Save}$$

$$w_0[3] = h[0]x[3] + h[1]x[2] + h[2]x[1] = y[3] \quad \leftarrow \text{Save}$$

$$w_1[0] = h[0]x[2] + h[1]x[5] + h[2]x[4] \quad \leftarrow \text{Reject}$$

$$w_1[1] = h[0]x[3] + h[1]x[2] + h[2]x[5] \quad \leftarrow \text{Reject}$$

$$w_1[2] = h[0]x[4] + h[1]x[3] + h[2]x[2] = y[4] \quad \leftarrow \text{Save}$$

$$w_1[3] = h[0]x[5] + h[1]x[4] + h[2]x[3] = y[5] \quad \leftarrow \text{Save}$$

Overlap-Save Method

$$w_2[0] = h[0]x[4] + h[1]x[5] + h[2]x[6] \quad \leftarrow \text{Reject}$$

$$w_2[1] = h[0]x[5] + h[1]x[4] + h[2]x[7] \quad \leftarrow \text{Reject}$$

$$w_2[2] = h[0]x[6] + h[1]x[5] + h[2]x[4] = y[6] \quad \leftarrow \text{Save}$$

$$w_2[3] = h[0]x[7] + h[1]x[6] + h[2]x[5] = y[7] \quad \leftarrow \text{Save}$$

Overlap-Save Method

- It should be noted that to determine $y[0]$ and $y[1]$, we need to form $x_{-1}[n]$:

$$x_{-1}[0] = 0, \quad x_{-1}[1] = 0,$$

$$x_{-1}[2] = x[0], \quad x_{-1}[3] = x[1]$$

and compute $w_{-1}[n] = h[n] \circledast_4 x_{-1}[n]$ for $0 \le n \le 3$ reject $w_{-1}[0]$ and $w_{-1}[1]$, and save $w_{-1}[2] = y[0]$ and $w_{-1}[3] = y[1]$

Overlap-Save Method

- General Case: Let $h[n]$ be a length-N sequence
- Let $x_m[n]$ denote the m-th section of an infinitely long sequence $x[n]$ of length N and defined by

$$x_m[n] = x[n + m(N - m + 1)], \quad 0 \le n \le N - 1$$

with $M < N$

Overlap-Save Method

- Let $w_m[n] = h[n] \textcircled{N} x_m[n]$
- Then, we reject the first $M-1$ samples of $w_m[n]$ and "abut" the remaining $N-M+1$ samples of $w_m[n]$ to form $y_L[n]$, the linear convolution of $h[n]$ and $x[n]$
- If $y_m[n]$ denotes the saved portion of $w_m[n]$, i.e.

$$y_m[n] = \begin{cases} 0, & 0 \le n \le M-2 \\ w_m[n], & M-1 \le n \le N-2 \end{cases}$$

Overlap-Save Method

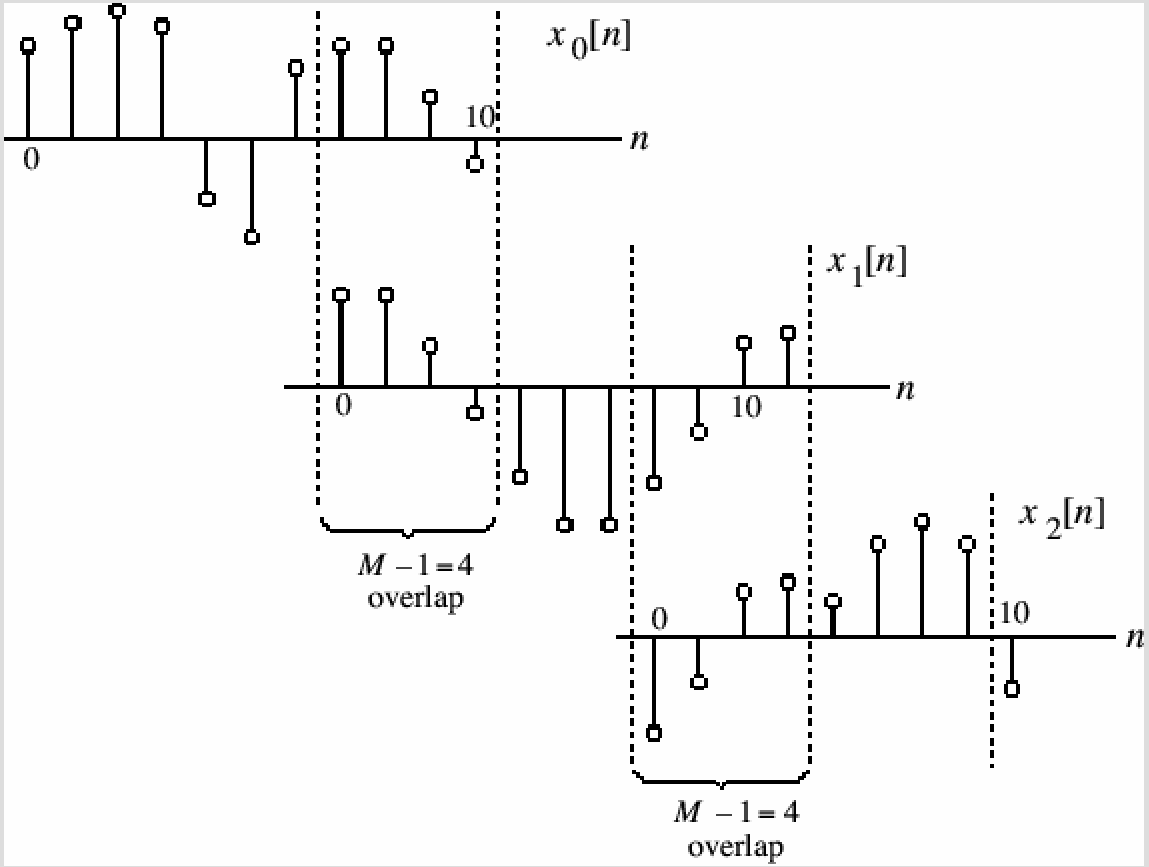
- Then

$$y_L[n+m(N-M+1)] = y_m[n], \qquad M-1 \le n \le N-1$$

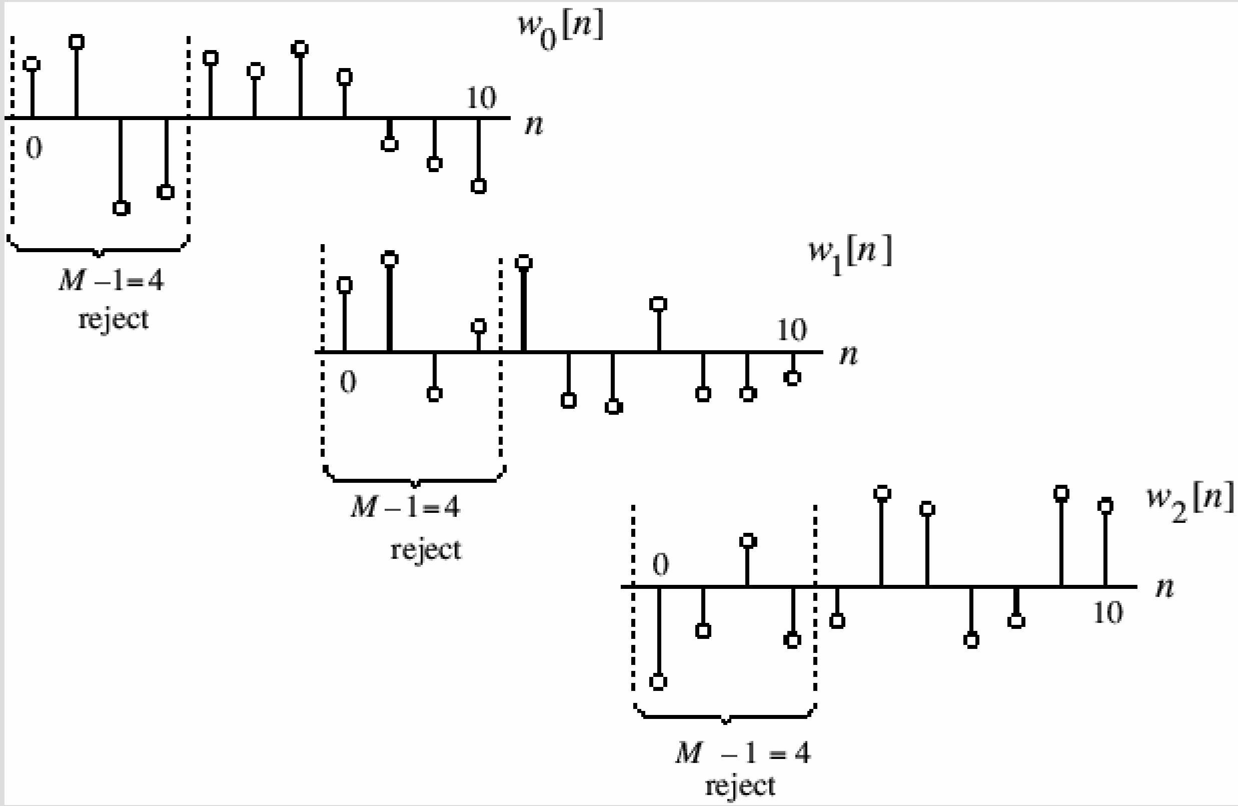
- The approach is called **overlap-save method** since the input is segmented into overlapping sections and parts of the results of the circular convolutions are saved and abutted to determine the linear convolution result

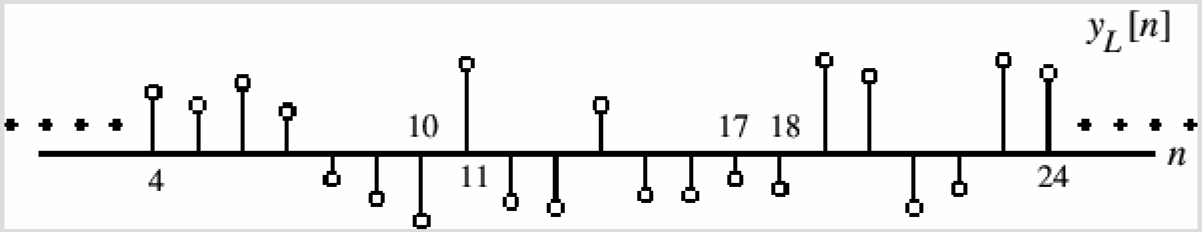
Overlap-Save Method

- Process is illustrated next



Overlap-Save Method





z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  leads to the *z*-transform

- *z*-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of *z*-transform techniques permits simple algebraic manipulations

z-Transform

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where $z = \text{Re}(z) + j\text{Im}(z)$ is a complex variable

z-Transform

- If we let $z = r\,e^{j\omega}$, then the z-transform reduces to

$$G(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

- The above can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z-transform reduces to its DTFT, provided the latter exists

z-Transform

- The contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the **unit circle**
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set R of values of z for which its z-transform converges is called the **region of convergence** (ROC)

z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}\,e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g[n]r^{-n} \right| < \infty$$

z-Transform

- In general, the ROC R of a *z*-transform of a sequence $g[n]$ is an annular region of the *z*-plane:

$$R_{g^-} < |z| < R_{g^+}$$

where $0 \leq R_{g^-} < R_{g^+} \leq \infty$

- Note: The *z*-transform is a form of a Laurent series and is an analytic function at every point in the ROC

z-Transform

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left| \alpha z^{-1} \right| < 1$$

- ROC is the annular region $|z| > |\alpha|$

z-Transform

- Example - The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1-z^{-1}}, \quad \text{for } |z^{-1}| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

z-Transform

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

z-Transform

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region $|z| < |\alpha|$

z-Transform

- Note: The z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

z-Transform

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the *z*-transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the *z*-transform

z-Transform

- Example - The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

which converges in the mean-square sense

z-Transform

- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r

- Some commonly used z-transform pairs are listed on the next slide

Table: Commonly Used *z*-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Rational *z*-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent *z*-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational *z*-Transforms

- The **degree** of the numerator polynomial $P(z)$ is M and the **degree** of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational *z*-transform is as a ratio of two polynomials in *z*:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational *z*-Transforms

- A rational z-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the **zeros** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the **poles** of $G(z)$

Rational *z*-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

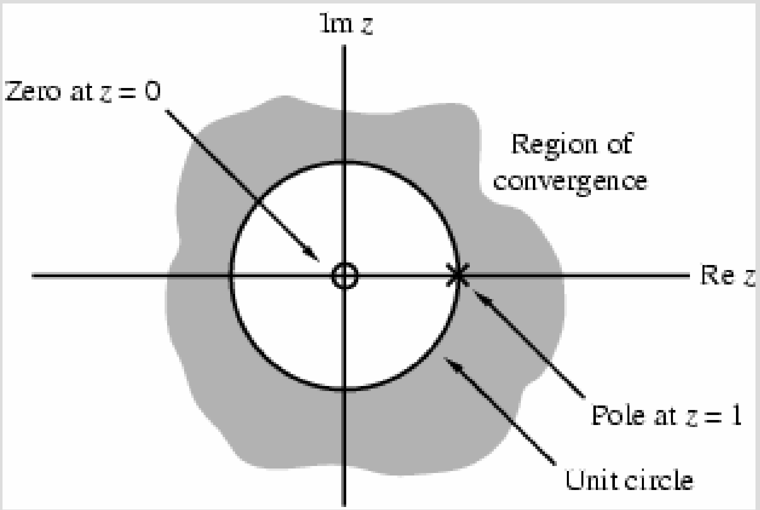
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational z-Transforms

- Example - The z-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

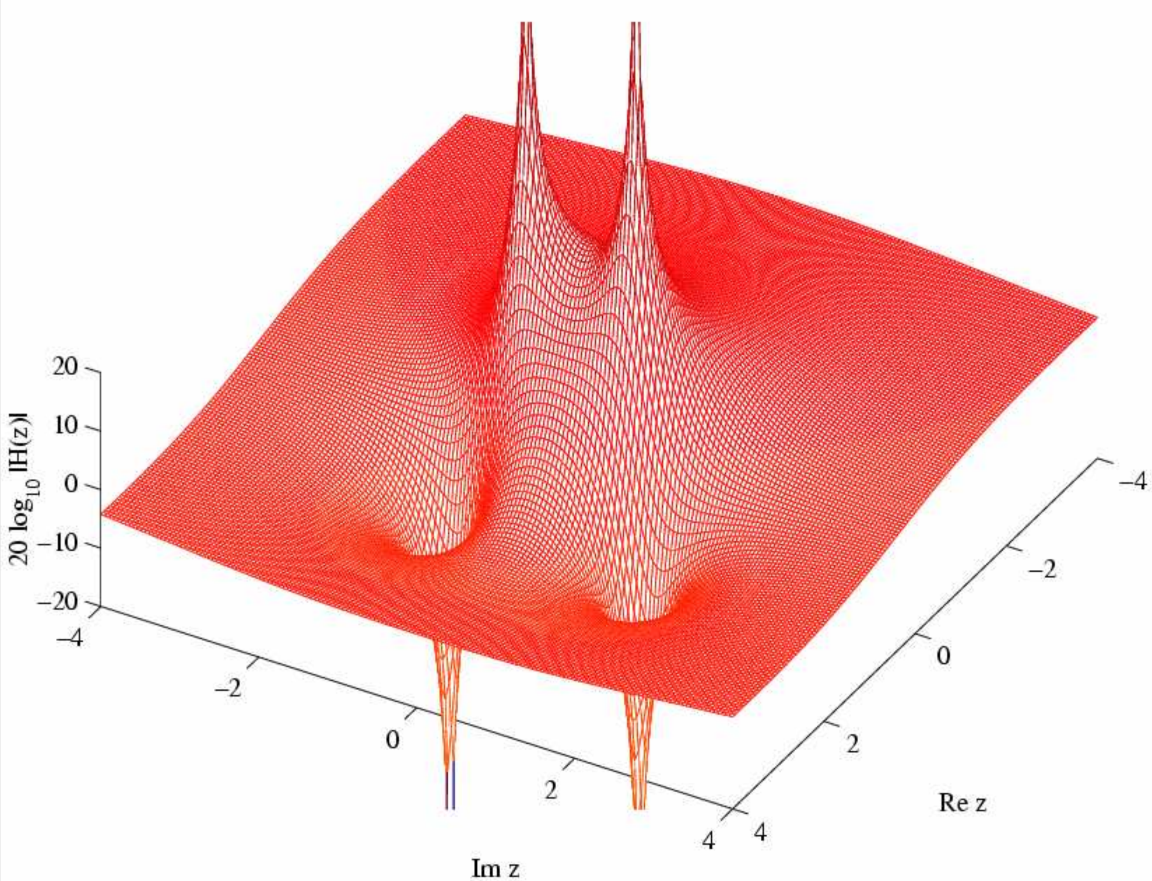


Rational *z*-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4\,z^{-1} + 2.88\,z^{-2}}{1 - 0.8\,z^{-1} + 0.64\,z^{-2}}$$

Rational *z*-Transforms



Rational *z*-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

ROC of a Rational *z*-Transform

- ROC of a *z*-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its *z*-transform
- Hence, the *z*-transform must always be specified with its ROC

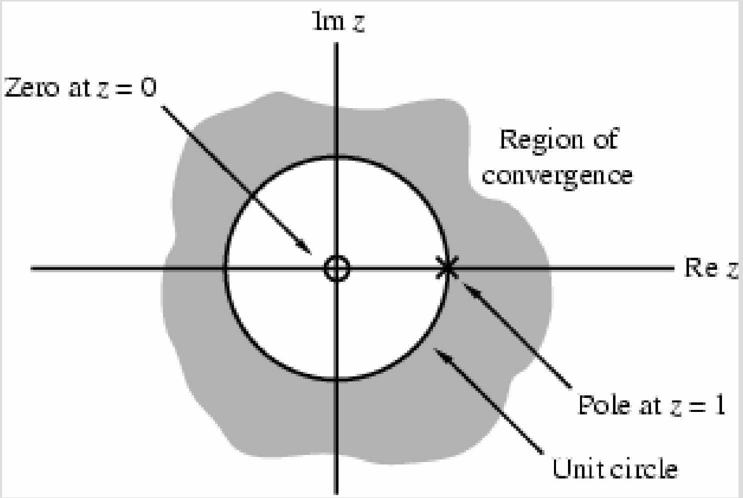
ROC of a Rational *z*-Transform

- Moreover, if the ROC of a *z*-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the *z*-transform on the unit circle
- There is a relationship between the ROC of the *z*-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$

ROC of a Rational z-Transform



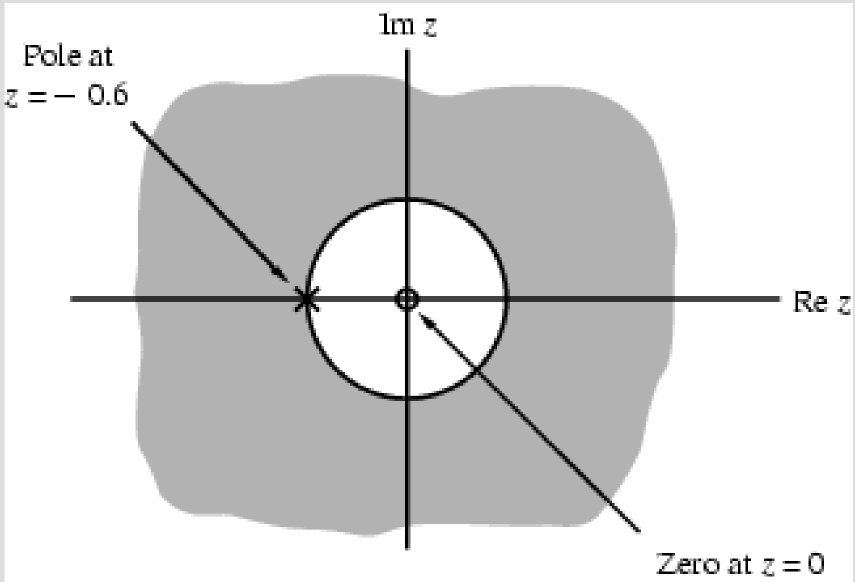
- In this plot, the ROC, shown as the shaded area, is the region of the z-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1 + 0.6\,z^{-1}},$$

$$|z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational *z*-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

ROC of a Rational z-Transform

- Example - Consider a finite-length sequence $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n] z^{-n} = \frac{\sum_{0}^{N+M} g[n-M] z^{N+M-n}}{z^{N}}$$

ROC of a Rational *z*-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$ (explain why)
- As can be seen from the expression for $G(z)$, the *z*-transform of a finite-length bounded sequence converges everywhere in the *z*-plane except possibly at $z = 0$ and/or at $z = \infty$

ROC of a Rational *z*-Transform

- <u>Example</u> - A **right-sided sequence** with nonzero sample values for $n \geq 0$ is sometimes called a **causal sequence**
- Consider a causal sequence $u_1[n]$
- Its *z*-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

ROC of a Rational *z*-Transform

- Example - A **left-sided sequence** with nonzero sample values for $n \leq 0$ is sometimes called a **anticausal sequence**
- Consider an anticausal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \le N$ with N nonnegative has a *z*-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

ROC of a Rational *z*-Transform

- Example - The *z*-transform of a two-sided sequence $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the *z*-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

ROC of a Rational z-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n] z^{-n}$, can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the z-transform does not exist

ROC of a Rational z-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

ROC of a Rational *z*-Transform

- There is no overlap between these two regions
- Hence, the *z*-transform of $u[n] = \alpha^n$ does not exist

ROC of a Rational z-Transform

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
- To show that the z-transform is bounded by the poles, assume that the z-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$
- Assume that the corresponding sequence $x[n]$ is a right-sided sequence

ROC of a Rational z-Transform

- Then $x[n]$ has the form

$$x[n] = \left(r_1 \alpha^n + r_2 \beta^n\right) \mu[n - N_o], \quad |\alpha| < |\beta|$$

where N_o is a positive or negative integer

- Now, the z-transform of the right-sided sequence $\gamma^n \mu[n - N_o]$ exists if

$$\sum_{n=N_o}^{\infty} \left|\gamma^n z^{-n}\right| < \infty$$

for some z

ROC of a Rational *z*-Transform

- The condition

$$\sum_{n=N_o}^{\infty} \left|\gamma^n z^{-n}\right| < \infty$$

holds for $|z| > |\gamma|$ but not for $|z| \le |\gamma|$

- Therefore, the *z*-transform of

$$x[n] = \left(r_1\alpha^n + r_2\beta^n\right)\mu[n - N_o], \quad |\alpha| < |\beta|$$

has an ROC defined by $|\beta| < |z| \le \infty$

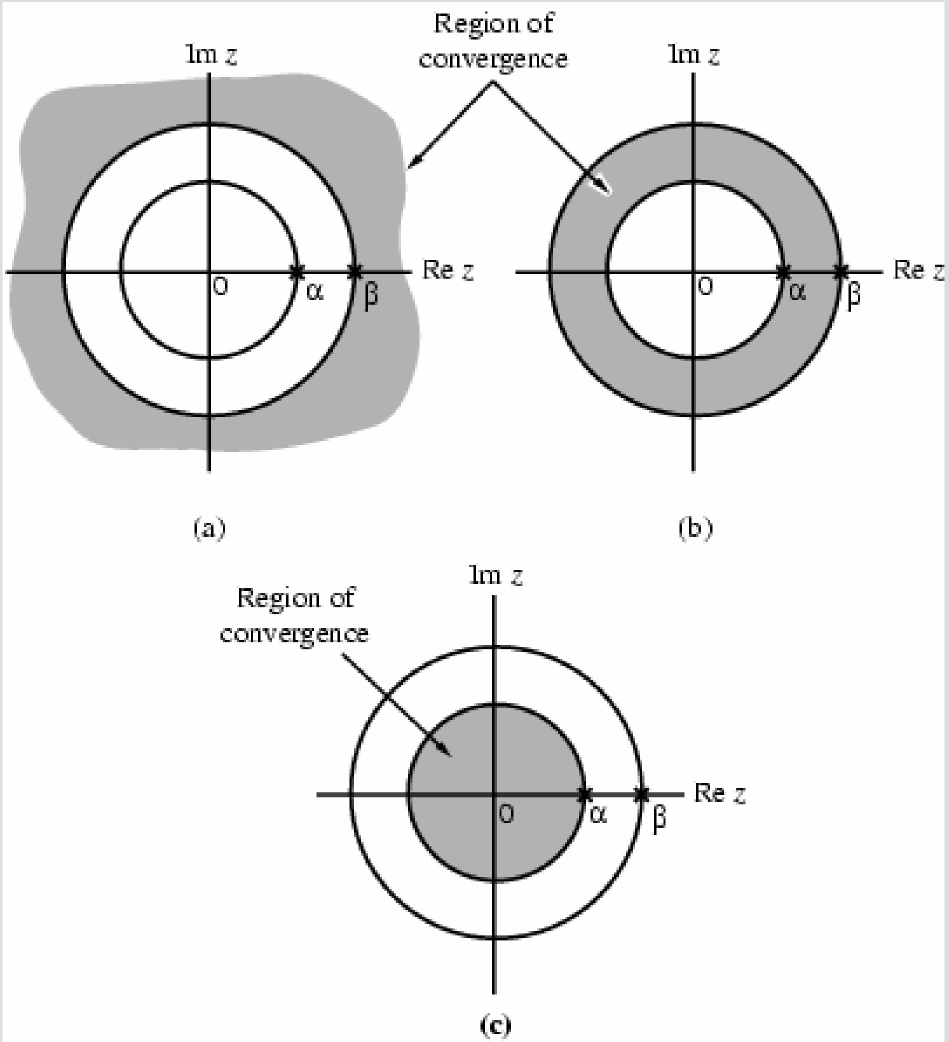
ROC of a Rational z-Transform

- Likewise, the *z*-transform of a left-sided sequence
  $$x[n] = \left(r_1\alpha^n + r_2\beta^n\right)\mu[-n - N_o], \quad |\alpha| < |\beta|$$
  has an ROC defined by $0 \le |z| < |\alpha|$
- Finally, for a two-sided sequence, some of the poles contribute to terms in the parent sequence for $n < 0$ and the other poles contribute to terms $n \ge 0$

ROC of a Rational z-Transform

- The ROC is thus bounded on the outside by the pole with the smallest magnitude that contributes for $n < 0$ and on the inside by the pole with the largest magnitude that contributes for $n \geq 0$
- There are three possible ROCs of a rational z-transform with poles at $z = \alpha$ and $z = \beta$ $(|\alpha| < |\beta|)$

ROC of a Rational z-Transform



ROC of a Rational *z*-Transform

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has $R+1$ ROCs
- Thus, there are $R+1$ distinct sequences with the same z-transform
- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform

ROC of a Rational *z*-Transform

- The ROC of a rational *z*-transform can be easily determined using MATLAB

```
[z,p,k] = tf2zp(num,den)
```

determines the zeros, poles, and the gain constant of a rational *z*-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

ROC of a Rational *z*-Transform

- `[num,den] = zp2tf(z,p,k)` implements the reverse process
- The factored form of the *z*-transform can be obtained using `sos = zp2sos(z,p,k)`
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix `sos`

ROC of a Rational z-Transform

$$sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

ROC of a Rational *z*-Transform

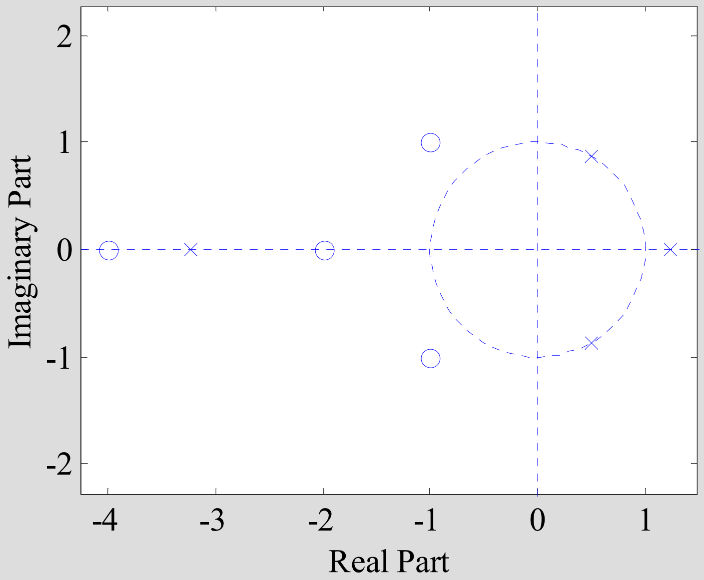
- The pole-zero plot is determined using the function `zplane`
- The *z*-transform can be either described in terms of its `zeros` and `poles`: `zplane(zeros,poles)`
- or, it can be described in terms of its numerator and denominator coefficients: `zplane(num,den)`

ROC of a Rational z-Transform

- Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below



$\times$ – pole

o – zero

Inverse z-Transform

- **General Expression**: Recall that, for $z = r\,e^{j\omega}$, the z-transform $G(z)$ given by

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g[n]r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}d\omega$$

Inverse z-Transform

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z)\, z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

Inverse z-Transform

- But the integral remains unchanged when is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum \left[\begin{array}{l} \text{residues of } G(z) z^{n-1} \\ \text{at the poles inside } C \end{array} \right]$$

- The above equation needs to be evaluated at all values of n and is not pursued here

Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**
- Example - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Inverse Transform by Partial-Fraction Expansion

- **Simple Poles**: In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of $G(z)$ be at $z = \lambda_k, 1 \le k \le N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

Inverse Transform by Partial-Fraction Expansion

- The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) G(z) \Big|_{z=\lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $z > |\lambda_\ell|$ and, thus has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Note: The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence

Inverse Transform by Partial-Fraction Expansion

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

Inverse Transform by Partial-Fraction Expansion

- Now

$$\rho_1 = (1 - 0.2\,z^{-1})H(z)\big|_{z=0.2} = \left.\frac{1 + 2\,z^{-1}}{1 + 0.6\,z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1 + 0.6\,z^{-1})H(z)\big|_{z=-0.6} = \left.\frac{1 + 2\,z^{-1}}{1 - 0.2\,z^{-1}}\right|_{z=-0.6} = -1.75$$

Inverse Transform by Partial-Fraction Expansion

- Hence

$$H(z) = \frac{2.75}{1 - 0.2\,z^{-1}} - \frac{1.75}{1 + 0.6\,z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- **Multiple Poles**: If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, $1 \leq \ell \leq N - L$

Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z)\right]_{z=\nu},$$

$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Partial-Fraction Expansion Using MATLAB

- `[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors `num` and `den`
- Vector `r` contains the residues
- Vector `p` contains the poles
- Vector `k` contains the constants η_ℓ

Partial-Fraction Expansion Using MATLAB

- `[num,den]=residuez(r,p,k)` converts a *z*-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division

Inverse z-Transform via Long Division

- Example - Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result

$$\{h[n]\} = \{\underset{\uparrow}{1} \quad 1.6 \quad -0.52 \quad 0.4 \quad -0.2224 \quad \cdots\}, \quad n \geq 0$$

Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational z-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

Table: z-Transform Properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g-}R_{h-} < |z| < R_{g+}R_{h+}$.

z-Transform Properties

- Example - Consider the two-sided sequence

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1]$$

- Let $x[n] = \alpha^n \mu[n]$ and $y[n] = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms
- Now $X(z) = \dfrac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$

and $Y(z) = \dfrac{1}{1 - \beta z^{-1}}, \quad |z| < |\beta|$

z-Transform Properties

- Using the linearity property we arrive at

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist

z-Transform Properties

- Example - Determine the z-transform and its ROC of the causal sequence

$$x[n] = r^n (\cos \omega_o n) \mu[n]$$

- We can express $x[n] = v[n] + v^*[n]$ where

$$v[n] = \frac{1}{2} r^n e^{j\omega_o n} \mu[n] = \frac{1}{2} \alpha^n \mu[n]$$

- The z-transform of $v[n]$ is given by

$$V(z) = \frac{1}{2} \cdot \frac{1}{1 - \alpha z^{-1}} = \frac{1}{2} \cdot \frac{1}{1 - r e^{j\omega_o} z^{-1}}, \quad |z| > |\alpha| = r$$

z-Transform Properties

- Using the conjugation property we obtain the z-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2} \cdot \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \cdot \frac{1}{1 - r e^{-j\omega_o} z^{-1}},$$

$$|z| > |\alpha|$$

- Finally, using the linearity property we get

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1 - r e^{j\omega_o} z^{-1}} + \frac{1}{1 - r e^{-j\omega_o} z^{-1}} \right)$$

z-Transform Properties

- or,

$$X(z) = \frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}, \quad |z| > r$$

- <u>Example</u> - Determine the *z*-transform $Y(z)$ and the ROC of the sequence

$$y[n] = (n+1)\alpha^n\mu[n]$$

- We can write $y[n] = n\,x[n] + x[n]$ where

$$x[n] = \alpha^n\mu[n]$$

z-Transform Properties

- Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

- Using the differentiation property, we arrive at the z-transform of $n\,x[n]$ as

$$-z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \qquad |z| > |\alpha|$$

z-Transform Properties

- Using the linearity property we finally obtain

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$= \frac{1}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$