LTI Discrete-Time Systems
in Transform Domain

Frequency Response
Transfer Function
Introduction to Filters

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Frequency Response of an LTI Discrete-Time System

- The well known **convolution sum** description of an LTI discrete-time system is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

- Taking the DTFT of both sides we obtain

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right) e^{-j\omega n}$$

Frequency Response of an LTI Discrete-Time System

- Or,

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega(\ell+k)} \right)$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\underbrace{\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}}_{X(e^{j\omega})} \right) e^{-j\omega k}$$

Frequency Response of an LTI Discrete-Time System

- Hence, we can write

$$Y(e^{j\omega}) = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) X(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

- The above equation relates the input and the output of an LTI system in the frequency domain

- It follows that

$$H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$$

Frequency Response of an LTI Discrete-Time System

- The function $H(e^{j\omega})$ is called the **frequency response** of the LTI discrete-time system

- $H(e^{j\omega})$ provides a frequency-domain description of the system

- $H(e^{j\omega})$ is precisely the DTFT of the impulse response $\{h[n]\}$ of the system

Frequency Response of an LTI Discrete-Time System

- $H(e^{j\omega})$, in general, is a complex function of ω with a period 2π

- It can be expressed in terms of its real and imaginary part

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + j H_{im}(e^{j\omega})$$

or, in terms of its magnitude and phase,

$$H(e^{j\omega}) = \left| H(e^{j\omega}) \right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg H(e^{j\omega})$$

Frequency Response of an LTI Discrete-Time System

- Note: Magnitude and phase functions are real functions of ω, whereas the frequency response is a complex function of ω

- If the impulse response $h[n]$ is real then it is proven that the magnitude function is an even function of ω:

$$\left| H(e^{j\omega}) \right| = \left| H(e^{-j\omega}) \right|$$

  and the phase function is an odd function of ω:

$$\theta(\omega) = -\theta(-\omega)$$

- Likewise, for a real impulse response $h[n]$, $H_{re}(e^{j\omega})$ is even and $H_{im}(e^{j\omega})$ is odd

Magnitude and Phase Response

- The function $\left| H(e^{j\omega}) \right|$ is called the **magnitude response** and the function $\theta(\omega)$ is called the **phase response** of an LTI discrete-time system

- Design specifications for an LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both

Gain Function
Attenuation/Loss Function

- In some cases, the magnitude function is specified in **decibels** as

$$G(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \quad dB$$

where $G(\omega)$ is called the **gain function**

- The negative of the gain function

$$A(\omega) = -G(\omega)$$

is called the **attenuation** or **loss function**

Eigenfunction of a System

- If the input of an LTI system is a single complex exponential function

$$x[n] = e^{j\omega_o n}, \quad -\infty < n < \infty$$

  then it follows that the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_o(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_o k} \right) e^{j\omega_o n}$$

- Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Eigenfunction of a System

- Therefore, we can write

$$y[n] = H(e^{j\omega_o}) e^{j\omega_o n}$$

- Thus for a complex exponential input signal $e^{j\omega_o n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega_o})$

- $e^{j\omega_o n}$ is called eigenfunction of the system

Frequency Response of a System Described by a Difference Equation

- Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Its frequency response is obtained by taking the DTFT of both sides of the above equation

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$$

Example 1 – Frequency Response of a Moving Average (MA) Filter

- Consider the M-point moving average filter with an impulse response given by

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its frequency response is then given by

$$H(e^{j\omega}) = \frac{1}{M} \sum_{n=0}^{M-1} e^{-j\omega n}$$

Example 1 - Moving Average Filter

- Or,

$$H(e^{j\omega}) = \frac{1}{M}\frac{1-e^{-jM\omega}}{1-e^{-j\omega}}$$

$$= \frac{1}{M}\cdot\frac{\sin(M\omega/2)}{\sin(\omega/2)}e^{-j(M-1)\omega/2}$$

Example 1 - Moving Average Filter

- Thus, the **magnitude response** of the M-point moving average filter is given by

$$\left| H(e^{j\omega}) \right| = \left| \frac{1}{M} \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$$

and the **phase response** is given by

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=1}^{\lfloor M/2 \rfloor} \mu(\omega - \frac{2\pi k}{M})$$
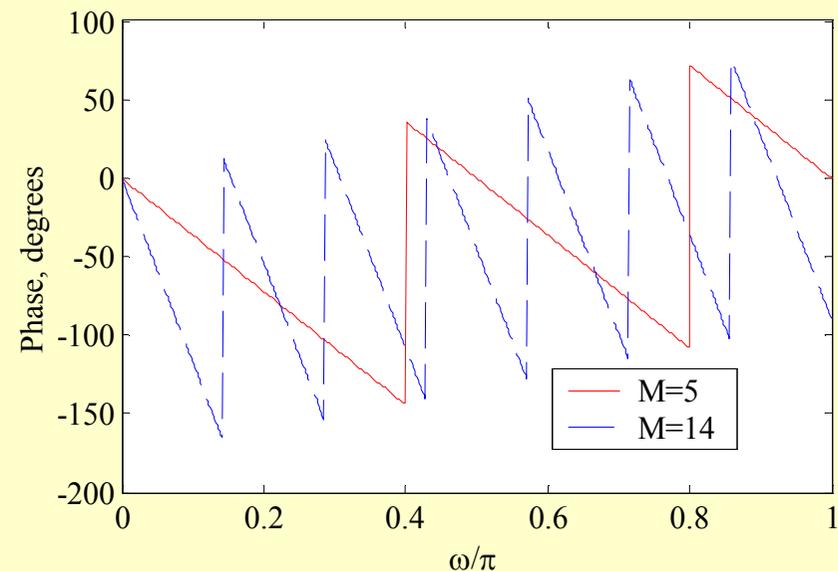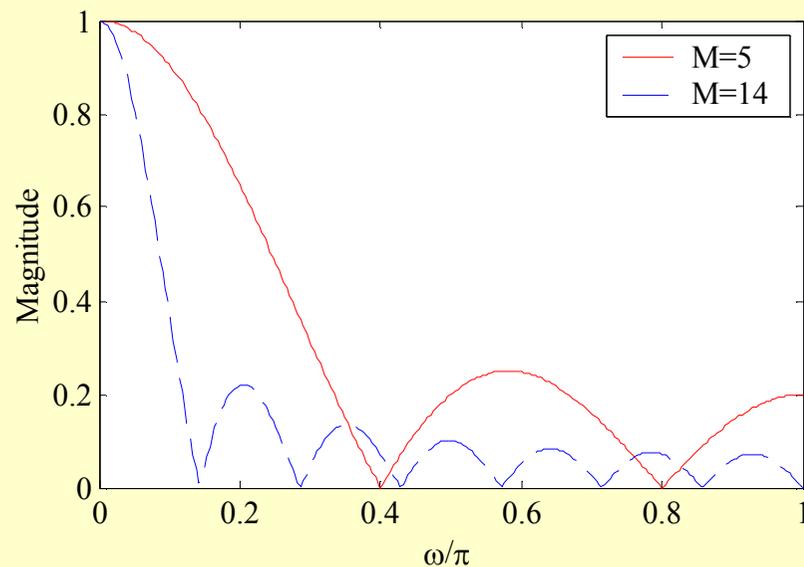
Example 1 - Moving Average Filter
Frequency Response Computation Using MATLAB

- The function **`freqz(h,w)`** can be used to determine the values of the frequency response vector **h** at a set of given frequency points **w**

- From **h**, the real and imaginary parts can be computed using the functions **real** and **imag**, and the magnitude and phase functions using the functions **abs** and **angle**

Example 1 - Moving Average Filter
Frequency Response Computation
Using MATLAB

- The magnitude and phase responses of an *M*-point moving average filter are shown below

- The jumps in the phase function occur at the zeros of the frequency response, where the *sinc* function changes sign.

Example 1 - Moving Average Filter
Frequency Response Computation Using MATLAB

- The phase response of a discrete-time system when determined by a computer may exhibit jumps by an amount 2π caused by the way the arctangent function is computed

- The phase response can be made a continuous function of ω by "unwrapping" the phase response across the jumps

Example 1 - Moving Average Filter
Frequency Response Computation
Using MATLAB

- To this end the function **unwrap** can be used, provided the computed phase is in radians

- The jumps by the amount of 2π should not be confused with the jumps caused by the zeros of the frequency response as indicated in the phase response of the moving average filter

Example 2 – Steady-State Response

- Determine the steady-state output $y[n]$ of a real coefficient LTI discrete-time system with a frequency response $H(e^{j\omega})$ for an input

$$x[n] = A\cos(\omega_o n + \phi), \quad -\infty < n < \infty$$

- Note that the frequency response determines the steady-state response of an LTI discrete-time system to a sinusoidal input

Example 2 – Steady State Response

- We can express the input $x[n]$ as

$$x[n] = g[n] + g*[n]$$

where

$$g[n] = \frac{1}{2} A e^{j\phi} e^{j\omega_o n}$$

- Now the output of the system for an input $e^{j\omega_o n}$ is simply

$$H(e^{j\omega_o}) e^{j\omega_o n}$$

Example 2 – Steady State Response

- Because of linearity, the response $v[n]$ to an input $g[n]$ is given by

$$v[n] = \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n}$$

- Likewise, the output $v^*[n]$ to the input $g^*[n]$ is

$$v^*[n] = \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n}$$

Example 2 – Steady State Response

- Combining the last two equations we get

$$y[n] = v[n] + v*[n]$$

$$= \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n} + \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n}$$

$$= \frac{1}{2} A \left| H(e^{j\omega_o}) \right| \left\{ e^{j\theta(\omega_o)} e^{j\phi} e^{j\omega_o n} + e^{-j\theta(\omega_o)} e^{-j\phi} e^{-j\omega_o n} \right\}$$

$$= A \left| H(e^{j\omega_o}) \right| \cos(\omega_o n + \theta(\omega_o) + \phi)$$

Example 2 – Steady State Response

- Thus, the output $y[n]$ has the same sinusoidal waveform as the input with two differences:

  (1) the amplitude is multiplied by $\left|H(e^{j\omega_o})\right|$, the value of the magnitude function at $\omega = \omega_o$

  (2) the output has a **phase lag** relative to the input by an amount $\theta(\omega_o)$, the value of the phase function at $\omega = \omega_o$

Example 3 – Response to a Causal Exponential Sequence

- The expression for the steady-state response developed earlier assumes that the system is initially relaxed before the application of the input

- In practice, excitation (input) to a discrete-time system is usually a right-sided sequence applied at some sample index $n = n_o$

- We develop the expression for the output for such an input

Example 3 – Response to a Causal Exponential Sequence

- Without any loss of generality, assume $x[n] = 0$ for $n < 0$

- From the input-output relation

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

we observe that for an input

$$x[n] = e^{j\omega n} \mu[n]$$

the output is given by

$$y[n] = \left(\sum_{k=0}^{n} h[k] e^{j\omega(n-k)} \right) \mu[n]$$

Example 3 – Response to a Causal Exponential Sequence

- Or, $y[n] = \left(\sum_{k=0}^{n} h[k]e^{-j\omega k} \right) e^{j\omega n}\mu[n]$

- The output for $n < 0$ is $y[n] = 0$

- The output for $n \geq 0$ is given by

$$y[n] = \left(\sum_{k=0}^{n} h[k]e^{-j\omega k} \right) e^{j\omega n}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}$$

Example 3 – Response to a Causal Exponential Sequence

- Or,

$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}$$

- The first term on the RHS is the same as that obtained when the input is applied at $n = 0$ to an initially relaxed system and is the **steady-state response**:

$$y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$$

Example 3 – Response to a Causal Exponential Sequence

- The second term on the RHS is called the **transient response**:

$$y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

- To determine the effect of the above term on the total output response, we observe

$$\left| y_{tr}[n] \right| = \left| \sum_{k=n+1}^{\infty} h[k] e^{-j\omega(k-n)} \right| \leq \sum_{k=n+1}^{\infty} \left| h[k] \right| \leq \sum_{k=0}^{\infty} \left| h[k] \right|$$

Example 3 – Response to a Causal Exponential Sequence

- For a causal, stable LTI IIR discrete-time system, $h[n]$ is absolutely summable

- As a result, the transient response $y_{tr}[n]$ is a bounded sequence

- Moreover, as $n \to \infty$,

$$\sum_{k=n+1}^{\infty} |h[k]| \to 0$$

and hence, the transient response decays to zero as n gets very large

Example 3 – Response to a Causal Exponential Sequence

- For a causal FIR LTI discrete-time system with an impulse response of length $N + 1$, $h[n] = 0$ for $n > N$

- Hence, $y_{tr}[n] = 0$ for $n > N - 1$

- Here the output reaches the steady-state value $y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$ at $n = N$

The Transfer Function

- The **transfer function** is a generalization of the frequency response function

- The convolution sum description of an LTI discrete-time system with an impulse response $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The Transfer Function

- Taking the z-transform of both sides we get

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]x[n-k] \right) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k]z^{-n} \right)$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+k)} \right)$$

The Transfer Function

- Or, $Y(z) = \displaystyle\sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell}\right)}_{X(z)} z^{-k}$

- Therefore, $Y(z) = \underbrace{\left(\displaystyle\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)}_{H(z)} X(z)$

- Thus, $\quad Y(z) = H(z)X(z)$

The Transfer Function

- Hence,

$$H(z) = Y(z) / X(z)$$

- The function $H(z)$, which is the z-transform of the impulse response $h[n]$ of the LTI system, is called the **transfer function** or the **system function**

- The inverse z-transform of the transfer function $H(z)$ yields the impulse response $h[n]$

The Transfer Function

- Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Its transfer function is obtained by taking the z-transform of both sides of the above equation

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

The Transfer Function

- Or, equivalently

$$H(z) = z^{(N-M)} \frac{\sum_{k=0}^{M} p_k z^{M-k}}{\sum_{k=0}^{N} d_k z^{N-k}}$$

- An alternate form of the transfer function is given by

$$H(z) = \frac{p_0}{d_0} \frac{\prod_{k=1}^{M} (1 - \xi_k z^{-1})}{\prod_{k=1}^{N} (1 - \lambda_k z^{-1})}$$

The Transfer Function

- Or, equivalently

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z - \xi_k)}{\prod_{k=1}^{N}(z - \lambda_k)}$$

- $\xi_1, \xi_2, \ldots, \xi_M$ are the finite **zeros**, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the finite **poles** of $H(z)$

- If $N > M$, there are additional $(N - M)$ zeros at $z = 0$

- If $N < M$, there are additional $(M - N)$ poles at $z = 0$

The Transfer Function

- If $M \geq N$ then $H(z)$ can be re-expressed as

$$H(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

  where the degree of $P_1(z)$ is less than N

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**

The Transfer Function

- **Simple Poles**: In most practical cases, the rational z-transform of interest $H(z)$ is a proper fraction with simple poles

- Let the poles of $H(z)$ be at $z = \lambda_k$, $1 \leq k \leq N$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

The Transfer Function

- The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) H(z) \big|_{z = \lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $z > |\lambda_\ell|$ and, thus has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$

The Transfer Function - Stability

- Therefore, the impulse response $h[n]$ is of infinite duration (IIR) and is given by

$$h[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Thus, the ROC is given by $|z| > \max_k |\lambda_k|$

- Furthermore, for stability, $|\lambda_k| < 1$, which means that all poles must lie inside the unit circle

The Transfer Function - Stability

- Therefore, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and the entire z-plane including the point $z = \infty$

- On the other hand, FIR digital filters with bounded impulse response are always stable

- Problem: Use the above approach to determine the inverse of a rational z-transform of a noncausal sequence

The Transfer Function - Stability

- On the other hand, an IIR filter may be unstable if not designed properly

- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation
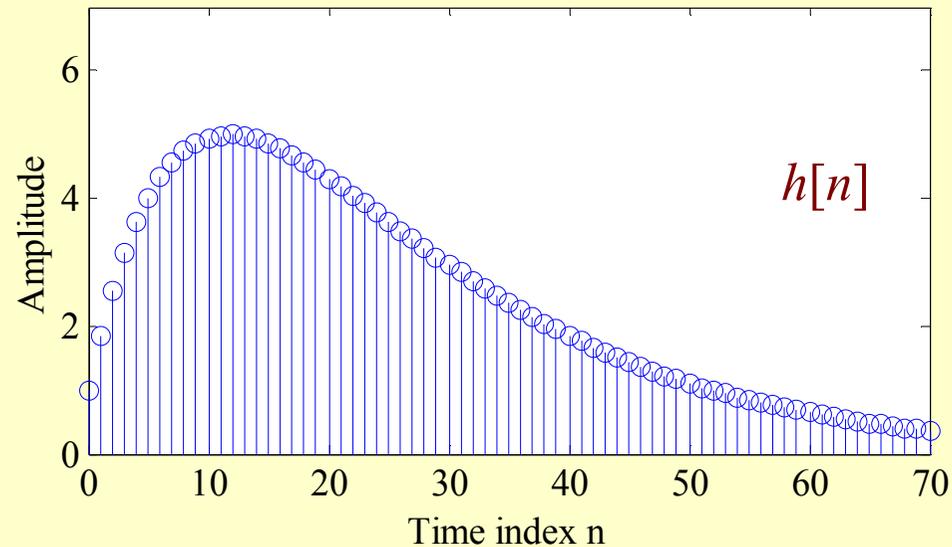
Example 4 – Effects of Quantization

- Consider the causal IIR transfer function

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

Example 4 – Effects of Quantization

- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases
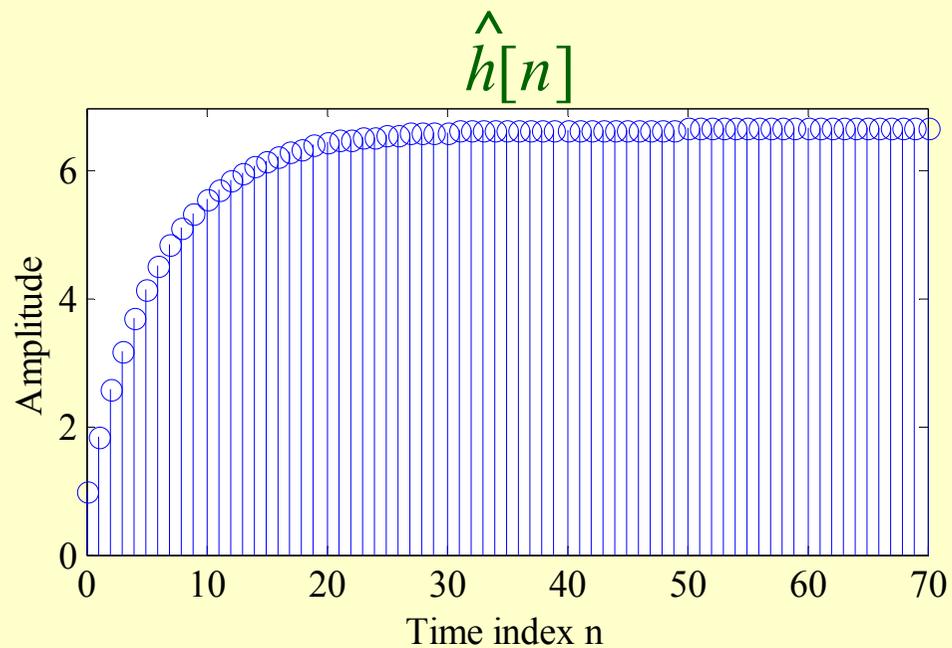
Example 4 – Effects of Quantization

- The absolute summability condition of $h[n]$ is satisfied

- Hence, $H(z)$ is a stable transfer function

- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

Example 4 – Effects of Quantization

- A plot of the impulse response of $\hat{h}[n]$ is shown below

Example 4 – Effects of Quantization

- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases

- Hence, the absolute summability condition of is violated

- Thus, $\hat{H}(z)$ is an unstable transfer function

Example 5 – Transfer Function of a Moving Average Filter

- Consider the M-point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^M - 1}{M[z^M(z-1)]}$$

Example 5 – Moving Average Filter

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \leq k \leq M - 1$

- There are $M - 1$ poles at $z = 0$ and a single pole at $z = 1$

$M = 8$

- The pole at $z = 1$ exactly cancels the zero at $z = 1$

- The ROC is the entire z-plane except $z = 0$
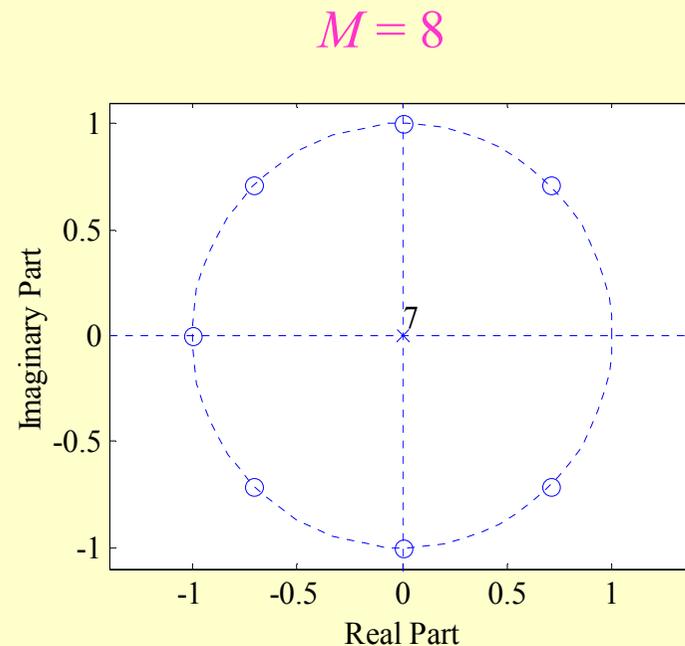
Example 6 – An IIR Filter

- A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n] = x[n-1] - 1.2\,x[n-2] + x[n-3] + 1.3\,y[n-1]$$
$$- 1.4\,y[n-2] + 0.222\,y[n-3]$$

- Its transfer function is therefore given by

$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$
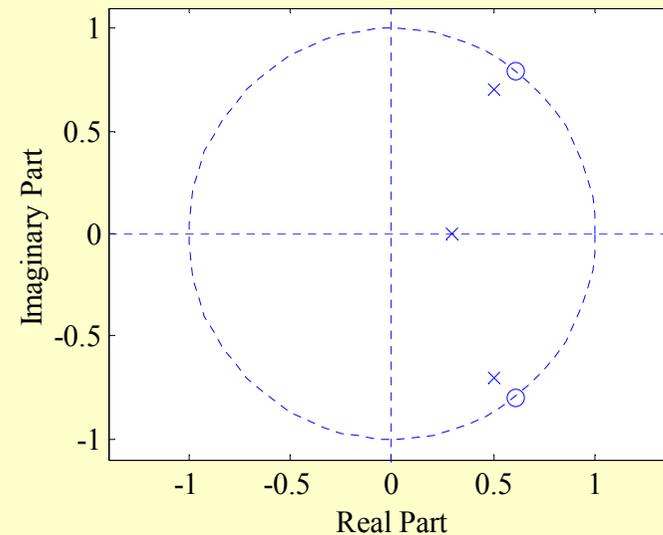
Example 6 – An IIR Filter

- Alternate forms:

$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$$

$$= \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$

- ROC: $|z| > \sqrt{0.74}$

Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z - \xi_k)}{\prod_{k=1}^{N}(z - \lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

Frequency Response from Transfer Function

- It is convenient to visualize the contributions of the **zero factor** $(z - \xi_k)$ and the **pole factor** $(z - \lambda_k)$ from the factored form of the frequency response

- The magnitude function is given by

$$\left| H(e^{j\omega}) \right| = \left| \frac{p_0}{d_0} \right| \left| e^{j\omega(N-M)} \right| \frac{\prod_{k=1}^{M} \left| e^{j\omega} - \xi_k \right|}{\prod_{k=1}^{N} \left| e^{j\omega} - \lambda_k \right|}$$

Frequency Response from Transfer Function

which reduces to

$$\left| H(e^{j\omega}) \right| = \left| \frac{p_0}{d_0} \right| \frac{\prod_{k=1}^{M} \left| e^{j\omega} - \xi_k \right|}{\prod_{k=1}^{N} \left| e^{j\omega} - \lambda_k \right|}$$

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0 / d_0) + \omega(N - M)$$

$$+ \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

Frequency Response from Transfer Function

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{p_0}{d_0}\right|^2 \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)(e^{-j\omega} - \xi_k^*)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)(e^{-j\omega} - \lambda_k^*)}$$

Geometric Interpretation of Frequency Response Computation

- The factored form of the frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

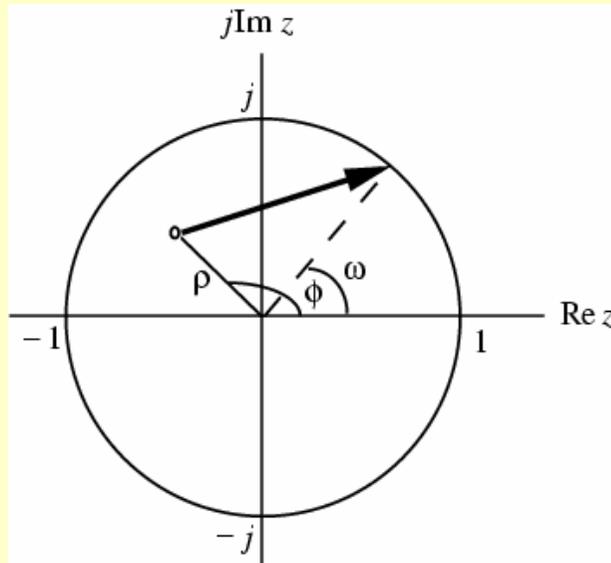# Geometric Interpretation of Frequency Response Computation

- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency

- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

where $\rho e^{j\phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

Geometric Interpretation of Frequency Response Computation

- As shown below in the z-plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$

Geometric Interpretation of Frequency Response Computation

- As ω is varied from 0 to 2π, the tip of the vector moves counterclockise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

Geometric Interpretation of Frequency Response Computation

- As indicated by

$$\left| H(e^{j\omega}) \right| = \left| \frac{p_0}{d_0} \right| \frac{\prod_{k=1}^{M} \left| e^{j\omega} - \xi_k \right|}{\prod_{k=1}^{N} \left| e^{j\omega} - \lambda_k \right|}$$

the magnitude response $|H(e^{j\omega})|$ at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

Geometric Interpretation of Frequency Response Computation

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0 / d_0) + \omega(N - M)$$

$$+ \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

we observe that the phase response at a specific value of ω is obtained by adding the phase of the term p_0 / d_0 and the linear-phase term $\omega(N - M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

Geometric Interpretation of Frequency Response Computation

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations

- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$

Geometric Interpretation of Frequency Response Computation

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range

- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range

The Concept of Filtering

- One application of an LTI discrete-time system is to **pass** certain frequency components of an input sequence without any distortion (if possible) and to **block** other frequency components

- Such systems are called digital filters and one of the main subjects of discussion in this course

The Concept of Filtering

- Let us consider the IDTFT

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) X(e^{j\omega}) e^{j\omega n} d\omega$$

- This transform expresses an arbitrary system output signal as a linear weighted sum of an infinite number of exponential sequences

The Concept of Filtering

- Thus, by appropriately choosing the values of the magnitude function $\left| H(e^{j\omega}) \right|$ of the LTI digital filter, some of the input frequency components can be selectively heavily attenuated or filtered with respect to the others

The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$\left| H(e^{j\omega}) \right| \cong \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Concept of Filtering

- We apply an input

$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n, \ 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

  to this system

- Because of linearity, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

$$+ B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

The Concept of Filtering

- As

$$\left| H(e^{j\omega_1}) \right| \cong 1, \quad \left| H(e^{j\omega_2}) \right| \cong 0$$

  the output reduces to

$$y[n] \cong A \left| H(e^{j\omega_1}) \right| \cos\left(\omega_1 n + \theta(\omega_1)\right)$$

- Thus, the system acts like a **lowpass filter**
- In the following example, we consider the design of a very simple digital filter

Example 7 - The Concept of Filtering

- An input signal consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample

- We need to design a highpass filter that will pass the high-frequency components of the input but block the low-frequency components

- For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:
$h[0] = h[2] = \alpha$, $h[1] = \beta$

Example 7 - The Concept of Filtering

- The convolution sum description of this filter is then given by

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha\, x[n] + \beta\, x[n-1] + \alpha\, x[n-2]$$

- $y[n]$ and $x[n]$ are, respectively, the output and the input sequences

- **Design Objective**: Choose suitable values of α and β so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample

Example 7 - The Concept of Filtering

- Now, the frequency response of the FIR filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$

$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$

$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega}$$

$$= (2\alpha\cos\omega + \beta)e^{-j\omega}$$

Example 7 - The Concept of Filtering

- The magnitude and phase functions are

$$\left| H(e^{j\omega}) \right| = 2\alpha \cos\omega + \beta$$

$$\theta(\omega) = -\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero

- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

Example 7 - The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$\left|H(e^{j0.1})\right| = 2\alpha\cos(0.1) + \beta = 0$$

$$\left|H(e^{j0.4})\right| = 2\alpha\cos(0.4) + \beta = 1$$

- Solving the above two equations we get

$$\alpha = -6.76195$$

$$\beta = 13.456335$$

Example 7 - The Concept of Filtering

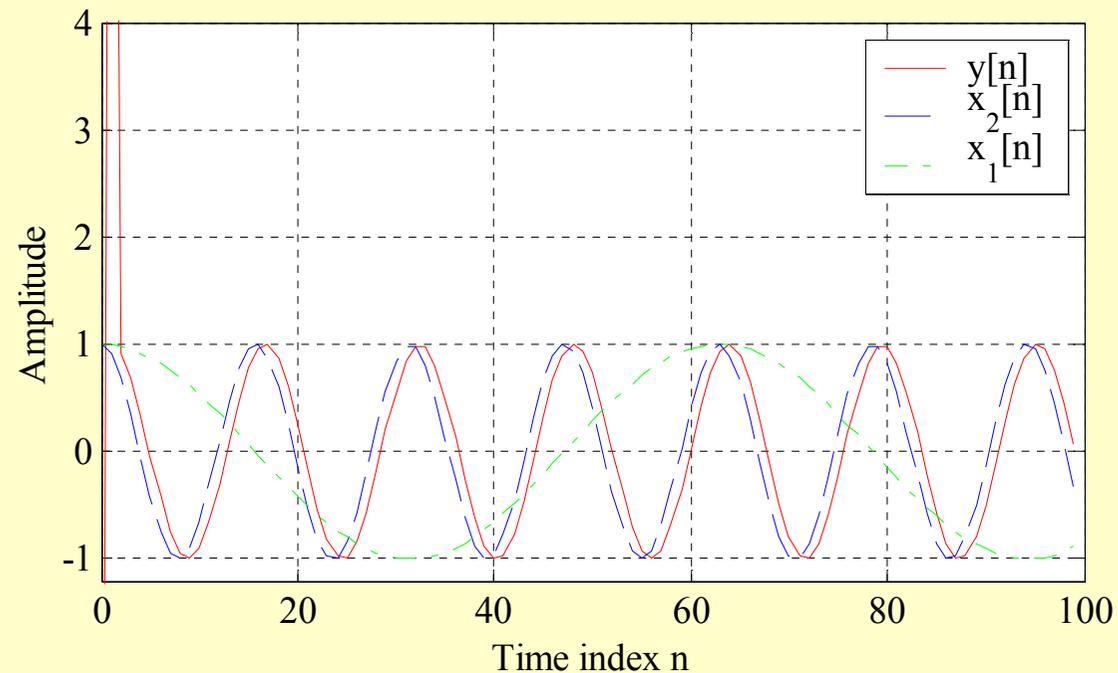- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76195(x[n] + x[n-2]) + 13.456335\, x[n-1]$$

where the input is

$$x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]$$

Example 7 - The Concept of Filtering

- Figure below shows the plots generated by running this program

Example 7 - The Concept of Filtering

- The first seven samples of the output are shown below

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	-13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	-0.0291995	0.8918614	0.3623572
5	0.8775825	-0.4161468	0.4614357	-0.0292002
6	0.8253356	-0.7373937	0.0879419	-0.4161467

Example 7 - The Concept of Filtering

- From this table, it can be seen that, neglecting the least significant digit,

$$y[n] = \cos(0.4(n-1)) \qquad \text{for } n \geq 2$$

- Computation of the present value of the output requires the knowledge of the present and two previous input samples

- Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$

Example 7 - The Concept of Filtering

- Therefore, first two output samples constitute the transient part of the output

- Since the impulse response is of length 3, the steady-state is reached at $n = N = 2$

- Note also that the output is delayed version of the high-frequency component $\cos(0.4n)$ of the input, and the delay is one sample period

Phase and Group Delays

- The output $y[n]$ of a frequency-selective LTI discrete-time system with a frequency response $H(e^{j\omega})$ exhibits some delay relative to the input $x[n]$ caused by the nonzero phase response $\theta(\omega) = \arg\{H(e^{j\omega})\}$ of the system

- For an input

$$x[n] = A\cos(\omega_o n + \phi), \quad -\infty < n < \infty$$

Phase and Group Delays

the output is

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos(\omega_o n + \theta(\omega_o) + \phi)$$

- Thus, the output lags in phase by $\theta(\omega_o)$ radians

- Rewriting the above equation we get

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos\left(\omega_o\left(n + \frac{\theta(\omega_o)}{\omega_o}\right) + \phi\right)$$

Phase and Group Delays

- This expression indicates a time delay, known as **phase delay**, at $\omega = \omega_o$ given by

$$\tau_p(\omega_o) = -\frac{\theta(\omega_o)}{\omega_o}$$

- Now consider the case when the input signal contains many sinusoidal components with different frequencies that are not harmonically related

Phase and Group Delays

- In this case, each component of the input will go through different phase delays when processed by a frequency-selective LTI discrete-time system

- Then, the output signal, in general, will not look like the input signal

- The signal delay now is defined using a different parameter

Phase and Group Delays

- To develop the necessary expression, consider a discrete-time signal $x[n]$ obtained by a double-sideband suppressed carrier (DSB-SC) modulation with a carrier frequency ω_c of a low-frequency sinusoidal signal of frequency ω_o:

$$x[n] = A\cos(\omega_o n)\cos(\omega_c n)$$

Phase and Group Delays

- The input can be rewritten as

$$x[n] = \frac{A}{2}\cos(\omega_\ell n) + \frac{A}{2}\cos(\omega_u n)$$

where $\omega_\ell = \omega_c - \omega_o$ and $\omega_u = \omega_c + \omega_o$

- Let the above input be processed by an LTI discrete-time system with a frequency response $H(e^{j\omega})$ satisfying the condition

$$H(e^{j\omega}) \cong 1 \qquad \text{for } \omega_\ell \le \omega \le \omega_u$$

Phase and Group Delays

- The output $y[n]$ is then given by

$$y[n] = \frac{A}{2}\cos\left(\omega_\ell n + \theta(\omega_\ell)\right) + \frac{A}{2}\cos\left(\omega_u n + \theta(\omega_u)\right)$$

$$= A\cos\left(\omega_c n + \frac{\theta(\omega_u) + \theta(\omega_\ell)}{2}\right)\cos\left(\omega_o n + \frac{\theta(\omega_u) - \theta(\omega_\ell)}{2}\right)$$

- Note: The output is also in the form of a modulated carrier signal with the same carrier frequency ω_c and the same modulation frequency ω_o as the input

Phase and Group Delays

- However, the two components have different phase lags relative to their corresponding components in the input

- Now consider the case when the modulated input is a **narrowband signal** with the frequencies ω_ℓ and ω_u very close to the carrier frequency ω_c, i.e. ω_o is very small

Phase and Group Delays

- In the neighborhood of ω_c we can express the unwrapped phase response $\theta_c(\omega)$ as

$$\theta_c(\omega) \cong \theta_c(\omega_c) + \left. \frac{d\,\theta_c(\omega)}{d\omega}\right|_{\omega=\omega_c} \cdot (\omega - \omega_c)$$

  by making a Taylor's series expansion and keeping only the first two terms

- Using the above formula we now evaluate the time delays of the carrier and the modulating components

Phase and Group Delays

- In the case of the carrier signal we have

$$-\frac{\theta_c(\omega_u) + \theta_c(\omega_\ell)}{2\omega_c} \cong -\frac{\theta_c(\omega_c)}{\omega_c}$$

which is seen to be the same as the phase delay if only the carrier signal is passed through the system

Phase and Group Delays

- In the case of the modulating component we have

$$-\frac{\theta_c(\omega_u) - \theta_c(\omega_\ell)}{\omega_u - \omega_\ell} \cong -\left.\frac{d\theta_c(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

- The parameter

$$\tau_g(\omega_c) = -\left.\frac{d\theta_c(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

is called the **group delay** or **envelope delay** caused by the system at $\omega = \omega_c$

Phase and Group Delays

- The group delay is a measure of the linearity of the phase function as a function of the frequency

- It is the time delay between the waveforms of underlying continuous-time signals whose sampled versions, sampled at $t = nT$, are precisely the input and the output discrete-time signals
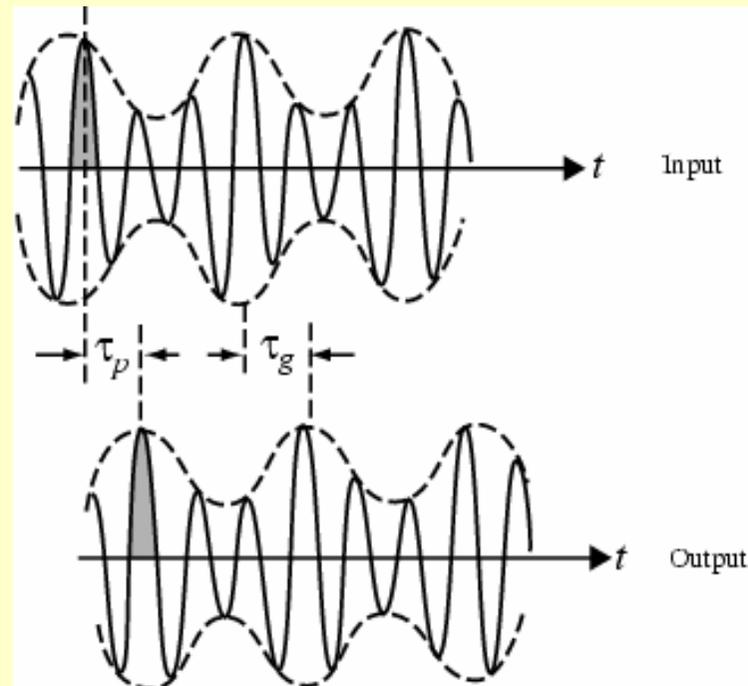
Phase and Group Delays

- If the phase function and the angular frequency ω are in radians per second, then the group delay is in seconds

- Figure below illustrates the evaluation of the phase delay and the group delay
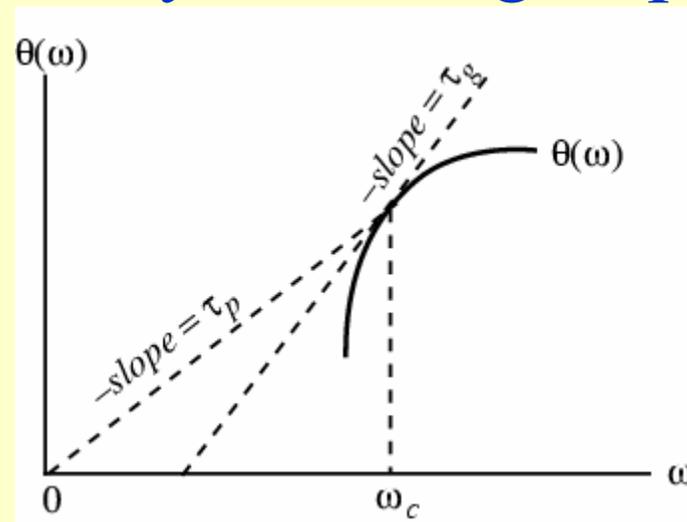
Phase and Group Delays

- Figure below shows the waveform of an amplitude-modulated input and the output generated by an LTI system

Phase and Group Delays

- Note: The carrier component at the output is delayed by the phase delay and the envelope of the output is delayed by the group delay relative to the waveform of the underlying continuous-time input signal

- The waveform of the underlying continuous-time output shows distortion when the group delay of the LTI system is not constant over the bandwidth of the modulated signal

Phase and Group Delays

- If the distortion is unacceptable, a delay equalizer is usually cascaded with the LTI system so that the overall group delay of the cascade is approximately linear over the band of interest

- To keep the magnitude response of the parent LTI system unchanged the equalizer must have a constant magnitude response at all frequencies

Example 8 - Phase and Group Delay of a Second Order FIR Filter

- The phase function of the FIR filter

$$y[n] = \alpha\, x[n] + \beta\, x[n-1] + \alpha\, x[n-2]$$

with $\beta > 2\alpha$ is $\theta(\omega) = -\omega$

- Hence its group delay is given by $\tau_g(\omega) = 1$

Example 8 - Phase and Group Delay of an *M*-order MA filter

- For the *M*-point moving-average filter

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

the phase function is

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=1}^{\lfloor M/2 \rfloor} \mu\left(\omega - \frac{2\pi k}{M}\right)$$

- Hence its group delay is

$$\tau_g(\omega) = \frac{M-1}{2}$$