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Signals and Systems

Tutorial Sheet 2

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Problem 1

A Linear Time Invariant (LTI) system is specified by the system equation:

$$(D^2 + 4D + 4)y(t) = Df(t)$$

(i) Find the characteristic polynomial, characteristic equation, characteristic roots and characteristic modes of this system.

Answer

The characteristic polynomial is $(\lambda^2 + 4\lambda + 4)$.
The characteristic equation is $(\lambda^2 + 4\lambda + 4) = 0 = (\lambda + 2)^2$.
This equation has repeated roots of $\lambda = -2$ (two roots).
Therefore, the characteristic modes are: e^{-2t} and te^{-2t}.

(ii) Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0) = 3, \dot{y}_0(0) = -4$.

Answer

$y_0(t) = (c_1 + c_2 t)e^{-2t}$, $\dot{y}_0(t) = (-2c_1 - 2c_2 t + c_2)e^{-2t}$.
$y_0(0) = 3 \Rightarrow c_1 = 3$ and $\dot{y}_0(0) = -6 + c_2 = -4 \Rightarrow c_2 = 2$.
Therefore, $y_0(t) = (3 + 2t)e^{-2t}$.

Problem 2

A Linear Time Invariant (LTI) system is specified by the system equation:

$$D(D+1)y(t) = (D+2)f(t)$$

(i) Find the characteristic polynomial, characteristic equation, characteristic roots and characteristic modes of this system.

Answer

The characteristic polynomial is $(\lambda^2 + \lambda)$.
The characteristic equation is $(\lambda^2 + \lambda) = 0 = \lambda(\lambda + 1)$.
This equation has roots of $\lambda = 0$ and $\lambda = -1$.
Therefore, the characteristic modes are: $e^{0t} = 1$ and e^{-t}.

(ii) Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0) = 1, \dot{y}_0(0) = 1$.

Answer

$y_0(t) = c_1 + c_2 e^{-t}$, $\dot{y}_0(t) = -c_2 e^{-t}$.
$y_0(0) = c_1 + c_2 = 1$ and $\dot{y}_0(0) = -c_2 = 1 \Rightarrow c_1 = 2$, $c_2 = -1$.
Therefore, $y_0(t) = 2 - e^{-t}$.

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Problem 3

A Linear Time Invariant (LTI) system is specified by the system equation:

$$(D^2 + 9)y(t) = (3D + 2)f(t)$$

(i) Find the characteristic polynomial, characteristic equation, characteristic roots and characteristic modes of this system.

Answer

The characteristic polynomial is $(\lambda^2 + 9)$.
The characteristic equation is $(\lambda^2 + 9) = 0 = (\lambda - j3)(\lambda + j3)$.
This equation has roots of $\lambda = j3$ and $\lambda = -j3$.
Therefore, the characteristic modes are: e^{-j3t} and e^{j3t}.

(ii) Find $y_0(t)$, the zero-input component of the response $y(t)$ for $t \geq 0$, if the initial conditions are $y_0(0) = 0, \dot{y}_0(0) = 6$.

Answer

$y_0(t) = c\cos(3t + \theta)$, $\dot{y}_0(t) = -3c\sin(3t + \theta)$
$y_0(0) = c\cos(\theta) = 0$, $\dot{y}_0(0) = -3c\sin(\theta) = 6 \Rightarrow c\sin(\theta) = -2$
$c = 2$ and $\theta = -\frac{\pi}{2}$. Therefore, $y_0(t) = 2\cos\left(3t - \frac{\pi}{2}\right) = 2\sin(3t)$.

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Problem 4

Evaluate the following integrals. This question is easy.

(i) $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$

(ii) $\int_{-\infty}^{\infty} \delta(\tau)f(t-\tau)d\tau = f(t-0) = f(t)$

(iii) $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega 0} = 1$

(iv) $\int_{-\infty}^{\infty} \delta(t-2)\sin(\pi t)\, dt = \sin(2\pi) = 0$

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Problem 5

Find the unit impulse response of the LTI system specified by the equation:
$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 5x(t)$ (written also as $Q(D)y(t) = P(D)x(t)$).

Answer

Characteristic polynomial: $(\lambda^2 + 4\lambda + 3)$.
Characteristic equation: $(\lambda^2 + 4\lambda + 3) = 0 = (\lambda + 1)(\lambda + 3)$. Roots $\lambda = -1$ and $\lambda = -3$.
Characteristic modes: e^{-t} and e^{-3t}.
The impulse response $h(t)$ is given by $h(t) = [P(D)y_n(t)]u(t)$.
$y_n(t)$ is what we previously called $y_0(t)$ with $y_n(0) = 0$, $\dot{y}_n(0) = 1$.
$y_n(t) = c_1e^{-t} + c_2e^{-3t}$, $\dot{y}_n(t) = -c_1e^{-t} - 3c_2e^{-3t}$.
$y_n(0) = c_1 + c_2 = 0$, $\dot{y}_n(0) = -c_1 - 3c_2 = 1$, $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$.

$$y_n(t) = c_1e^{-t} + c_2e^{-3t} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$h(t) = \left[\frac{dy_n(t)}{dt} + 5y_n(t)\right]u(t) = (-\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t} - \frac{5}{2}e^{-3t})u(t) =$
$(2e^{-t} - e^{-3t})u(t)$.

Problem 6

Find the unit impulse response of the LTI system specified by the equation.

$$(D+1)(D^2+5D+6)y(t) = (5D+9)f(t) \text{ or}$$

$$Q(D)y(t) = P(D)x(t)).$$

Answer

The characteristic polynomial is $(\lambda+1)\ (\lambda^2+5\lambda+6)$.

The characteristic equation is $(\lambda+1)\ (\lambda^2+5\lambda+6) = 0 = (\lambda+1)\ (\lambda+2)\ (\lambda+3)$.

This equation has roots of $\lambda = -1$, $\lambda = -2$ and $\lambda = -3$.

Therefore, the characteristic modes are: e^{-t}, e^{-2t} and e^{-3t}.

The impulse response $h(t)$ is given by

$$h(t) = [P(D)y_n(t)]u(t)$$

where $u(t)$ is the unit step function. $y_n(t)$ is what we previously called $y_0(t)$ with $y_n(0) = 0$, $\dot{y}_n(0) = 0$ and $\ddot{y}_n(0) = 1$.

$y_n(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t}$

$\dot{y}_n(t) = -c_1e^{-t} - 2c_2e^{-2t} - 3c_3e^{-3t}$

$\ddot{y}_n(t) = c_1e^{-t} + 4c_2e^{-2t} + 9c_3e^{-3t}$

Problem 6 cont.

Input-output differential equation cont.

$$(D+1)(D^2+5D+6)y(t)=(5D+9)f(t)$$

Answer cont.

$y_n(t)=c_1e^{-t}+c_2e^{-2t}+c_3e^{-3t} \Rightarrow y_n(0)=c_1+c_2+c_3=0$ (1)

$\dot{y}_n(t)=-c_1e^{-t}-2c_2e^{-2t}-3c_3e^{-3t} \Rightarrow \dot{y}_n(0)=-c_1-2c_2-3c_3=0$ (2)

$\ddot{y}_n(t)=c_1e^{-t}+4c_2e^{-2t}+9c_3e^{-3t} \Rightarrow \ddot{y}_n(0)=c_1+4c_2+9c_3=1$ (3)

Adding (1) and (2) we obtain $-c_2-2c_3=0 \Rightarrow -2c_2-4c_3=0$ (4)

Adding (3) and (2) we obtain $2c_2+6c_3=1$ (5)

Adding (4) and (5) we obtain $c_3=\frac{1}{2}$ and replacing in (4) we get $-2c_2-2=0$ $\Rightarrow c_2=-1$

We replace c_2 and c_3 in (1) and we obtain $c_1-1+\frac{1}{2}=0 \Rightarrow c_1=\frac{1}{2}$

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Problem 6 cont.

Input-output differential equation cont.

$$(D+1)(D^2+5D+6)y(t) = (5D+9)f(t)$$

Answer cont.

We found that

$$y_n(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t} = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$$

$$\frac{dy_n(t)}{dt} = -\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t}$$

The impulse response $h(t)$ is given by

$h(t) = [P(D)y_n(t)]u(t)$ with $P(D) = 5D + 9$. Therefore,

$5\frac{dy_n(t)}{dt} = -\frac{5}{2}e^{-t} + 10e^{-2t} - \frac{15}{2}e^{-3t}$ and $9y_n(t) = \frac{9}{2}e^{-t} - 9e^{-2t} + \frac{9}{2}e^{-3t}$

$5\frac{dy_n(t)}{dt} + 9y_n(t) = 2e^{-t} + e^{-2t} - 3e^{-3t}$

Therefore,

$$h(t) = (2e^{-t} + e^{-2t} - 3e^{-3t})u(t)$$