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Signals and Systems

Lecture 9

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Connecting Laplace and Fourier transform

- In this lecture we will start connecting the Laplace and the Fourier transform.

- Recall that when we studied the Laplace transform we mentioned the concept of the **R**egion **O**f **C**onvergence (ROC) of the transform.

- This concept did not exist in the Fourier transform.

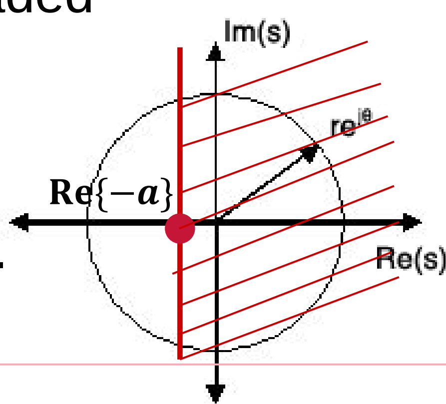
Example: Laplace transform of a causal exponential function

- Recall the Laplace transform of the causal exponential function $x(t) = e^{-at}u(t)$.
Note that from this point forward I will interchange $e^{-at}u(t)$ with $e^{at}u(t)$.

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \frac{1}{-(s+a)} e^{-(s+a)t}\Big|_0^{\infty} = \frac{-1}{(s+a)}[(e^{-(s+a)\cdot\infty}) - (e^{-(s+a)\cdot 0})] = \frac{-1}{(s+a)}(0-1) = \frac{1}{s+a}.$$

  - Note that in order to have $e^{-(s+a)\cdot\infty} = 0$, the real part of $s+a$ must be positive, i.e., $\mathrm{Re}\{s+a\} > 0 \Rightarrow \mathrm{Re}\{s\} > -\mathrm{Re}\{a\} = \mathrm{Re}\{-a\}$.
  [ROC is shown in the figure below right with the shaded area.]
  - Note that the value $s = -a$ makes the denominator of the Laplace transform 0.
  - Note that for a decaying exponential we want $a > 0$.



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Example: Fourier transform of a causal exponential function

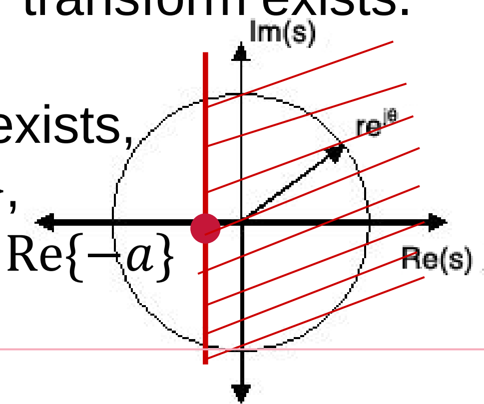
- Let us find now the Fourier transform of the causal exponential function: $x(t) = e^{-at}u(t)$.

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+a)t}dt$$

$$= \frac{1}{-(j\omega+a)} e^{-(j\omega+a)t}\Big|_{0}^{\infty} = \frac{-1}{(j\omega+a)}\left[\left(e^{-(j\omega+a)\cdot\infty}\right) - \left(e^{-(j\omega+a)\cdot 0}\right)\right]$$

$$= \frac{-1}{(j\omega+a)}(0-1) = \frac{1}{j\omega+a}.$$

  - Note that in order to have $e^{-(j\omega+a)\cdot\infty} = 0$, the real part of a must be positive. This is the only case for which the Fourier transform exists.
  - In that case $\mathrm{Re}\{-a\}$ must be negative.
  - Notice that if the Fourier transform of the function exists, the ROC of the Laplace transform, $\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$, includes the imaginary axis $s = j\omega$.
  - **Notice that $X(\omega) = X(j\omega) = X(s)|_{s=j\omega}$.**



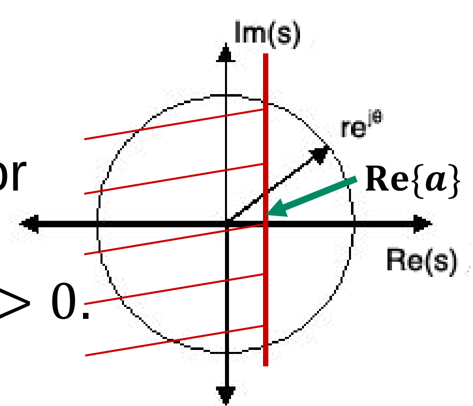
Example: Laplace transform of an anti-causal exponential function

- Recall now the Laplace transform of the function $x(t) = -e^{at}u(-t)$.
  Note that from this point forward I will interchange $e^{-at}u(-t)$ with $e^{at}u(-t)$.

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{(a-s)t}dt$$

$$= \frac{-1}{(a-s)}e^{(a-s)t}\Big|_{-\infty}^{0} = \frac{-1}{(a-s)}[(e^{(a-s)\cdot 0}) - (e^{(a-s)\cdot -\infty})] = \frac{-1}{(a-s)}(1-0) = \frac{1}{s-a}.$$

  - Note that in order to have $e^{(a-s)\cdot -\infty} = e^{(s-a)\cdot \infty} = 0$,
    the real part of $s - a$ must be negative,
    i.e., $\mathrm{Re}\{s-a\} < 0 \Rightarrow \mathrm{Re}\{s\} < \mathrm{Re}\{a\}$.
  - Note that the value $s = a$ makes the denominator of the Laplace transform 0.
  - Note that for a decaying exponential we want $a > 0$.



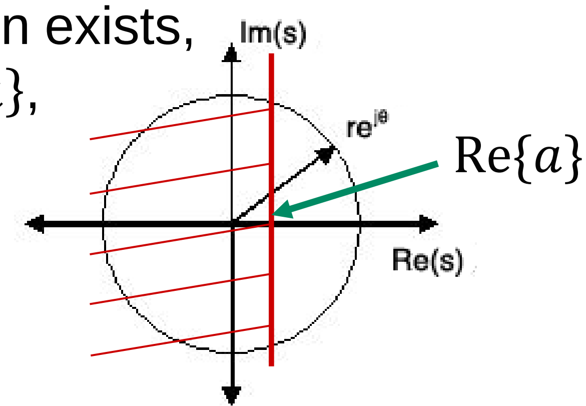
Example: Fourier transform of an anti-causal exponential function

- Let us find now the Fourier transform of the anti-causal exponential function $x(t) = -e^{at}u(-t)$.

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-j\omega t}dt = -\int_{-\infty}^{0} e^{(-j\omega+a)t}dt$$

$$= \frac{-1}{(-j\omega+a)} e^{(-j\omega+a)t}\Big|_{-\infty}^{0} = \frac{-1}{(-j\omega+a)}\left[\left(e^{(-j\omega+a)\cdot 0}\right) - \left(e^{(-j\omega+a)\cdot -\infty}\right)\right] = \frac{-1}{(-j\omega+a)}(1-0) = \frac{1}{j\omega - a}.$$

  - Note that in order to have $e^{(-j\omega+a)\cdot -\infty} = 0$, the real part of a must be positive. This is the only case for which the Fourier transform exists.
  - Notice that if the Fourier transform of the function exists, the ROC of the Laplace transform, $\mathrm{Re}\{s\} < \mathrm{Re}\{a\}$, includes the imaginary axis $s = j\omega$.
  - **Notice that $X(\omega) = X(j\omega) = X(s)|_{s=j\omega}$.**



Connection between Laplace transform and Fourier transform

- Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Setting $s = j\omega$ in the equation of the Laplace transform yields:

$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ where $X(j\omega) = X(s)|_{s=j\omega}$

- Is it true that $X(\omega) = X(j\omega) = X(s)|_{s=j\omega}$?

Yes, only if $x(t)$ is absolutely integrable which means that:

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

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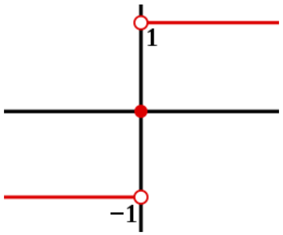
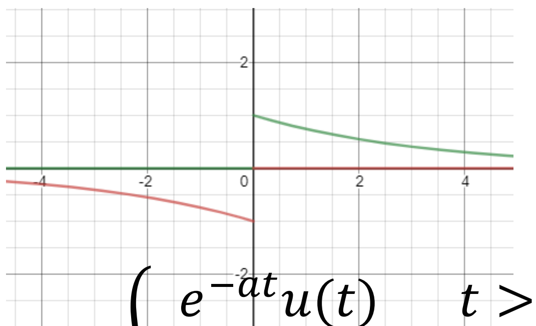
Sign function: Fourier transform

- Consider the **sign** function: $\text{sign}(t) = u(t) - u(-t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



  (Alternatively $\text{sign}(t) = \frac{d}{dt}|t|,\ t \neq 0$).
- The function $\text{sign}(t)$ DOES NOT have a Fourier transform because it is not absolutely integrable.
- We consider $\text{sign}(t)$ to be the limit of a function $\text{sign}_a(t)$, $\boldsymbol{a > 0}$ defined as:

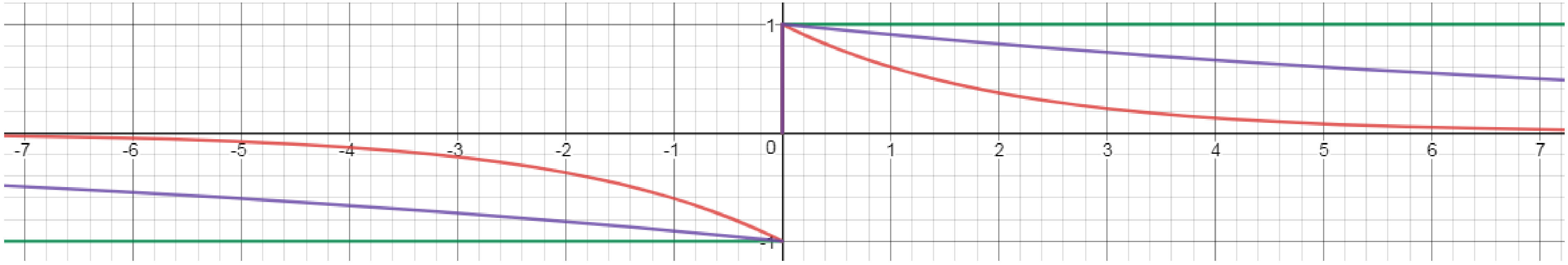
$$\text{sign}_a(t) = \begin{cases} e^{-at}u(t) & t > 0 \\ 0 & t = 0 \\ -e^{at}u(-t) & t < 0 \end{cases}$$



- We write: $\text{sign}(t) = \lim_{a \to 0} \text{sign}_a(t) = \lim_{a \to 0} \begin{cases} e^{-at}u(t) & t > 0 \\ 0 & t = 0 \\ -e^{at}u(-t) & t < 0 \end{cases}$

Sign function: Fourier transform cont.

- $\text{sign}(t) = \lim_{a \to 0} \text{sign}_a(t) = \lim_{a \to 0} \begin{cases} e^{-at}u(t) & t > 0 \\ 0 & t = 0 \\ -e^{at}u(-t) & t < 0 \end{cases}$
- The above is also written as $\text{sign}(t) = \lim_{a \to 0} \text{sign}_a(t)$ $= \lim_{a \to 0} (e^{-at}u(t) - e^{at}u(-t))$
- The **red** figure depicts $\mathbf{sign_{0.5}(t)}$.
- The **purple** figure depicts $\mathbf{sign_{0.1}(t)}$.
- The **green** figure depicts $\mathbf{sign(t)}$.



Sign function: Fourier transform cont.

- We are looking for $X(\omega) = \mathcal{F}[\text{sign}(t)] = \lim_{a\to 0} X_a(\omega) = \lim_{a\to 0} \mathcal{F}[\text{sign}_a(t)]$
- $\text{sign}_a(t) = e^{-at}u(t) - e^{at}u(-t)$
- $X_a(\omega) = \mathcal{F}[\text{sign}_a(t)] = \mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)\ ] = \frac{1}{a+j\omega} - \frac{1}{a-j\omega}$
$= \frac{-2j\omega}{a^2+\omega^2}$
- $X(\omega) = \mathcal{F}[\text{sign}(t)] = \lim_{a\to 0} X_a(\omega) = \frac{-2j\omega}{\omega^2} = \frac{2}{j\omega}$
- This method is quite popular: we attempt to find the **"near Fourier transform"** of a function which is not absolutely integrable by approximating the function as the limit of another, absolutely integrable function.

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Unit step function: Fourier transform

- Consider the unit step function $u(t)$.
- The unit step function DOES NOT have a Fourier transform because it is not absolutely integrable.
- We can however, write it as:

$$u(t) = \frac{1}{2}(1 + \mathrm{sign}(t))$$

$$X(\omega) = \mathcal{F}[u(t)] = \mathcal{F}\left\{\frac{1}{2}\left(1 + \mathrm{sign}(t)\right)\right\} = \mathcal{F}\left\{\frac{1}{2}\right\} + \mathcal{F}\left\{\frac{1}{2}\mathrm{sign}(t)\right\} =$$

$$= \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathrm{sign}(t)\} = \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

- The unit step function has the Laplace transform:

$$X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

- **You see immediately that $X(j\omega) = X(s)|_{s=j\omega} = \frac{1}{j\omega} \neq \frac{1}{j\omega} + \pi\delta(\omega) = X(\omega)$!**

Connection between Laplace transform and Fourier transform

- When $x(t)$ is absolutely integrable the ROC of its Laplace transform includes the imaginary axis. Therefore, both transforms exist and $X(\omega) = X(j\omega) = X(s)|_{s=j\omega}$.
- For the case of the function $x(t) = e^{at}u(t)$ this implies that $\mathrm{Re}\{a\} < 0$. In that case we have a decaying exponential which, by intuition, we suspect that it is an integrable function, whereas if $\mathrm{Re}\{a\} > 0$ the functions grows very quickly with time and obviously is not integrable.
- When $x(t)$ is not absolutely integrable the ROC does not include the imaginary axis and the Fourier transform of $x(t)$ **<u>may</u>** not exist.
- If $x(t)$ is not absolutely integrable but has a Fourier transform then this may differ from its Laplace transform. As already mentioned, a representative example is given by the unit step function.
- Note: the reason for this peculiar behaviour is related to the nature of convergence of the Laplace and Fourier integrals and is beyond the scope of this course. The core message for us is that the equivalence exists only when $x(t)$ is absolutely integrable.

System stability

- A system is BIBO stable (Bounded Input Bounded Output) if every bounded input produces a bounded output.
- For the output of an LTI system we have:

$$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Therefore, $|y(t)| \leq \int_{\tau=-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau$.
- Moreover, if $x(t)$ is bounded then $|x(t-\tau)| \leq K \leq \infty$ and

$$|y(t)| \leq K \int_{\tau=-\infty}^{\infty} |h(\tau)|d\tau$$

- Hence, BIBO stability exists when

$$\int_{\tau=-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

System stability cont.

- BIBO stability exists when

$$\int_{\tau=-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

- Recall that very often $h(t)$ is a linear combination of causal exponential functions of the form $x(t) = e^{at}u(t)$.
- For stability we require that $\mathrm{Re}\{a\} < 0$.
- The above function contributes with the term $\frac{1}{s-a}$ to the transfer function in the Laplace domain. The constant a which zeroes the denominator of $\frac{1}{s-a}$ or, in other words, makes the term $\frac{1}{s-a}$ infinite, is called a **pole** of the transfer function.
- **Therefore, in order to achieve stability, the poles of the transfer function of a causal system must lie on the left half of the s −plane.**

Summary of previous analysis

- An LTI system is BIBO stable if $h(t)$ is absolutely integrable (note that one can show that this condition is not only sufficient but also necessary).
- We can draw a second interesting conclusion from this derivation:

  "**An LTI system is stable if and only if the ROC of the transfer function's Laplace transform includes the imaginary axis.**"
- For causal systems with rational Laplace transforms, stability can be characterized in terms of the locations of the poles.
- Consider for example the system:

$$H(s) = \frac{c}{(s-a_1)(s-a_2)} = \frac{A}{(s-a_1)} + \frac{B}{(s-a_2)}$$

- Based on second bullet point and previous analysis:

  "**An LTI system is stable if and only if its poles lie on the left half of the s −plane, i.e., all the poles have negative real parts.**"

Laplace transform versus Fourier transform

- Do we really need to know both transforms?
- Laplace transform:
  - Usually unilateral, therefore useful for transient behaviours, for systems with initial conditions.
  - Defined also when the Fourier transform does not exist (e.g., growing causal exponentials).
  - More useful for system transient behaviour and for stability (i.e., Laplace transform exists for both stable and unstable systems).
- Fourier transform:
  - Bilateral, therefore more useful for "global trends".
  - Usually used for signal analysis and for filter design.
  - Amenable to a more intuitive interpretation: high ω, fast oscillations in the signals (or systems).