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Signals and Systems

Lectures 6-7

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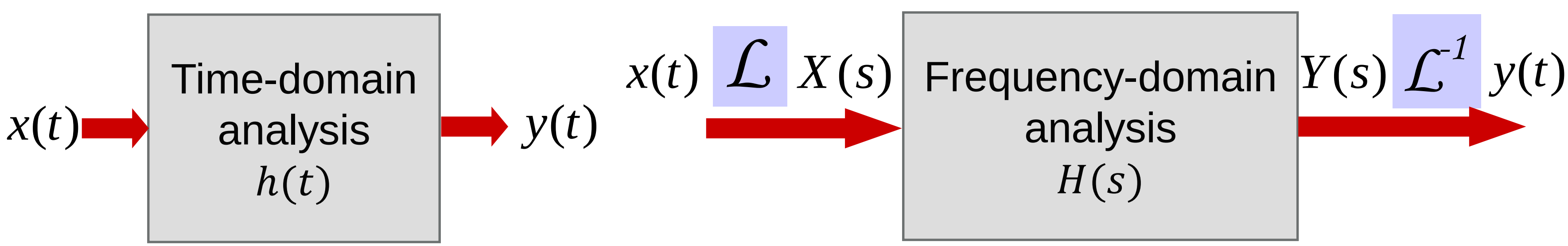
Laplace Transform

- Laplace Transform is the dual (or complement) of the time-domain analysis for analysing signals and systems.
- In time-domain analysis of an LTI system, we break the input $x(t)$ into a sum of impulse-like components and add the system's response to all these components.
- In frequency-domain analysis, we break the input $x(t)$ into exponential components of the form e^{st} where s is the complex frequency:

$$s = \alpha + j\omega$$

- The Laplace Transform is the tool to map signal and system behaviours from the time-domain into the frequency domain.
- The Laplace Transform is a generalisation of the Fourier Transform.



Laplace Transform

- For a signal $x(t)$ the two-sided Laplace transform is defined by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- The signal $x(t)$ is said to be the inverse Laplace transform of $X(s)$. It can be shown that

$$x(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{c-jT}^{c+jT} X(s)e^{st}ds$$

  where the integration is done along the vertical line $Re\{s\} = c$ in the complex plane such that c is greater than the real part of all singularities of $X(s)$ and $X(s)$ is bounded on the line, for example if contour path is in the region of convergence.
- If all singularities are in the left half-plane, or $X(s)$ is an entire function, then c can be set to zero and the above inverse integral formula becomes identical to the inverse Fourier transform.
- In practice, computing the complex integral can be done by using the so-called Cauchy residue theorem.

Laplace Transform

- For the purpose of 2nd year curriculum, we assume that we mainly use the Laplace transform in the case of causal signals (one-sided Laplace transform). In the case of a causal signal the Laplace Transform will be defined as:

$$X(s) = \mathcal{L}\{x(t)\} = \int_{0}^{\infty} x(t)e^{-st}dt$$

- Furthermore, to accommodate special type of functions, as for example the Dirac function, we sometimes use the definition:

$$X(s) = \mathcal{L}\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- Note that the Laplace transform is a linear transform, i.e.,

$$\mathcal{L}\{k_1x_1(t) + k_2x_2(t)\} = k_1\mathcal{L}\{x_1(t)\} + k_2\mathcal{L}\{x_2(t)\}$$

Examples: Laplace transforms of Dirac and unit step function

- Find the Laplace transform of the Dirac function $\delta(t)$.

$$X(s) = \mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-s0} = 1$$

- Find the Laplace transform of the unit step function $u(t)$.

$$X(s) = \mathcal{L}\{u(t)\} = \int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = \frac{1}{-s}e^{-st}|_0^{\infty} =$$

$$= \frac{1}{-s}[(e^{-s\cdot\infty}) - (e^{-s\cdot 0})] = \frac{1}{-s}(0-1) = \frac{1}{s}$$

  - Note that in order to have $e^{-s\cdot\infty} = 0$ the real part of s must be positive, i.e., $\text{Re}\{s\} > 0$.
  - The above condition implies that the Laplace transform of a function might exist for certain values of s and not all values of s. The set of these values consists the so-called Region-of-Convergence (ROC) of the Laplace transform.

Example: Laplace transform of causal exponential function

- Find the Laplace transform of the causal function $x(t) = e^{at}u(t)$.

$X(s) = \mathcal{L}\{x(t)\}$

$$= \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{at}e^{-st}dt = \int_{0}^{\infty} e^{-(s-a)t}dt = \frac{1}{-(s-a)}e^{-(s-a)t}\Big|_0^{\infty}$$

$$= \frac{1}{-(s-a)}\left[\left(e^{-(s-a)\cdot\infty}\right) - \left(e^{-(s-a)\cdot 0}\right)\right] = \frac{1}{-(s-a)}(0-1) = \frac{1}{s-a}.$$

  - Note that in order to have $e^{-(s-a)\cdot\infty} = 0$ the real part of $s-a$ must be positive, i.e., $\mathrm{Re}\{s-a\} > 0 \Rightarrow \mathrm{Re}\{s\} > \mathrm{Re}\{a\}$.

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Example: Laplace transform of a non-causal exponential function

- Find now the Laplace transform of the anti-causal function $x(t) = -e^{at}u(-t)$.

$$X(s) = \mathcal{L}\{x(t)\}$$

$$= \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{at}e^{-st}dt = -\int_{-\infty}^{0} e^{-(s-a)t}dt$$

$$= \frac{-1}{-(s-a)}e^{-(s-a)t}\Big|_{-\infty}^{0} = \frac{1}{(s-a)}[\left(e^{-(s-a)\cdot 0}\right) - \left(e^{-(s-a)\cdot(-\infty)}\right)]$$

$$= \frac{1}{(s-a)}[\left(e^{-(s-a)\cdot 0}\right) - \left(e^{(s-a)\cdot(\infty)}\right)] = \frac{1}{(s-a)}(1-0) = \frac{1}{s-a}.$$

- Note that in order to have $e^{(s-a)\cdot\infty} = 0$ the real part of $s-a$ must be negative, i.e., $\mathrm{Re}\{s-a\} < 0 \Rightarrow \mathrm{Re}\{s\} < \mathrm{Re}\{a\}$.

Two functions with same Laplace transform but different ROCs

- In the two previous slides we proved that the following two functions:
  - $x(t) = e^{at}u(t)$
  - $x(t) = -e^{at}u(-t)$

  have the same Laplace transform but entirely different ROCs.

- This verifies that the transform function alone is not sufficient to describe the function in the Laplace domain.

- If you are given that the Laplace transform of a function is $\frac{1}{s-a}$, can you tell which is the function in time? (Answer: NO)

- If you are given that the Laplace transform of a causal (or non-causal) function is $\frac{1}{s-a}$ can you tell which is the function in time? (Answer: YES)

- What is the union of the ROCs of the above two functions? (Answer: s −plane)

Summary of previous slides

- Please look carefully and assimilate the material presented in previous slides.

- It is clearly shown that two different functions have exactly the same Laplace transform as far as the analytical expression of the transform is concerned.

- What makes the two transforms different is the Region-of-Convergence (ROC) of the two transforms.

- The two ROCs are complementary and their union comprises the entire s −plane.

Example: Laplace transform of causal cosine function

- Find the Laplace transform of the function

$x(t) = \cos(\omega_0 t)u(t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})u(t).$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{j\omega_0 t}e^{-st}dt = \int_{0}^{\infty} e^{-(s-j\omega_0)t}dt = \frac{1}{-(s-j\omega_0)}e^{-(s-j\omega_0)t}\Big|_0^{\infty}$$

$$= \frac{1}{-(s-j\omega_0)}\left[\left(e^{-(s-j\omega_0)\cdot\infty}\right) - \left(e^{-(s-j\omega_0)\cdot 0}\right)\right] = \frac{1}{-(s-j\omega_0)}(0-1) = \frac{1}{(s-j\omega_0)}, \ \mathrm{Re}\{s\} > 0$$

$$\int_{-\infty}^{\infty} e^{-j\omega_0 t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-j\omega_0 t}e^{-st}dt = \int_{0}^{\infty} e^{-(s+j\omega_0)t}dt = \frac{1}{-(s+j\omega_0)}e^{-(s+j\omega_0)t}\Big|_0^{\infty}$$

$$= \frac{1}{-(s+j\omega_0)}\left[\left(e^{-(s+j\omega_0)\cdot\infty}\right) - \left(e^{-(s+j\omega_0)\cdot 0}\right)\right] = \frac{1}{-(s+j\omega_0)}(0-1) = \frac{1}{(s+j\omega_0)}, \ \mathrm{Re}\{s\} > 0$$

Example: Laplace transform of causal cosine function cont.

- Based on the previous analysis we have:

$$x(t) = \cos(\omega_0 t)u(t) = \frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})u(t).$$

$$\mathcal{L}\{x(t)\} = \frac{1}{2}\left[\frac{1}{(s-j\omega_0)} + \frac{1}{(s+j\omega_0)}\right] = \frac{s}{s^2+{\omega_0}^2},\ \ \mathrm{Re}\{s\} > 0$$

Summary of some important Laplace transform pairs

No	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$

Example: Inverse Laplace transform of ratios of polynomials

- Find the inverse Laplace transform of $\frac{7s-6}{s^2-s-6}$
- By factorising the denominator we get $s^2 - s - 6 = (s-3)(s+2)$.
- Therefore we can write: $\frac{7s-6}{s^2-s-6} = \frac{7s-6}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$.

  $A + B = 7, 2A - 3B = -6$

  $A = 3, B = 4$

  Finally, $\frac{7s-6}{s^2-s-6} = \frac{3}{s-3} + \frac{4}{s+2}$.
- We saw that the Laplace transforms of both $e^{at}u(t)$ and $-e^{at}u(-t)$ are $\frac{1}{s-a}$ with different ROCs.
- Therefore, the inverse Laplace transform of $\frac{3}{s-3}$ is
  - $3e^{3t}u(t)$ with $\mathrm{Re}\{s\} > 3$ or
  - $-3e^{3t}u(-t)$ with $\mathrm{Re}\{s\} < 3$
- Similarly, the inverse Laplace transform of $\frac{4}{s+2}$ is
  - $4e^{-2t}u(t)$ with $\mathrm{Re}\{s\} > -2$ or
  - $-4e^{-2t}u(-t)$ with $\mathrm{Re}\{s\} < -2$

Example: Inverse Laplace transform of ratios of polynomials cont.

We found the Laplace Transform $\frac{7s-6}{s^2-s-6} = \frac{3}{s-3} + \frac{4}{s+2}$.

- The inverse Laplace transform of $\frac{3}{s-3}$ is
  - $3e^{3t}u(t)$ with $\mathrm{Re}\{s\} > 3$ or
  - $-3e^{3t}u(-t)$ with $\mathrm{Re}\{s\} < 3$
- The inverse Laplace transform of $\frac{4}{s+2}$ is
  - $4e^{-2t}u(t)$ with $\mathrm{Re}\{s\} > -2$ or
  - $-4e^{-2t}u(-t)$ with $\mathrm{Re}\{s\} < -2$
- We have 4 possible combinations for the function in time.
  - Function 1: $3e^{3t}u(t) + 4e^{-2t}u(t)$ ROC: $\mathrm{Re}\{s\} > 3 \cap \mathrm{Re}\{s\} > -2 = \mathrm{Re}\{s\} > 3$
    Function 1 is causal but increases continuously with positive time.
  - Function 2: $3e^{3t}u(t) - 4e^{-2t}u(-t)$ ROC: $\mathrm{Re}\{s\} > 3 \cap \mathrm{Re}\{s\} < -2 = \emptyset$
    Function 2 is not an option.
  - Function 3: $-3e^{3t}u(-t) + 4e^{-2t}u(t)$ ROC: $\mathrm{Re}\{s\} < 3 \cap \mathrm{Re}\{s\} > -2$
    Function 3 is non-causal but does not possess convergence problems.
  - Function 4: $-3e^{3t}u(-t) - 4e^{-2t}u(-t)$ ROC: $\mathrm{Re}\{s\} < 3 \cap \mathrm{Re}\{s\} < -2 = \mathrm{Re}\{s\} < -2$
    Function 4 is anti-causal and increases continuously with negative time.

Example: Inverse Laplace transform of ratios of polynomials cont.

- Find the inverse Laplace transform of $\frac{2s^2+5}{(s+1)(s+2)}$. The function in time is causal.
- We observe that the power of the numerator is the same as the power of the denominator. In that case the partial fraction expression is different. We basically have to add the coefficient of the highest power of the numerator to it.
- Therefore, we can write: $\frac{2s^2+5}{(s+1)(s+2)} = 2 + \frac{A}{s+1} + \frac{B}{s+2}$

$6 + A + B = 0$
$4 + 2A + B = 5$
$A = 7, B = -13$

Finally, $\frac{s^2+5}{(s+1)(s+2)} = 2 + \frac{7}{s+1} - \frac{13}{s+2}$

- The inverse Laplace transform of $\frac{7}{s+1}$ is $7e^{-t}u(t)$ with $\mathrm{Re}\{s\} > -1$.
- The inverse Laplace transform of $\frac{-13}{s+2}$ is $-13e^{-2t}u(t)$ with $\mathrm{Re}\{s\} > -2$.
- The inverse Laplace transform of 2 is $2\delta(t)$.
- The total inverse Laplace transform is $2\delta(t) + (7e^{-t} - 13e^{-2t})u(t), \mathrm{Re}\{s\} > -1$

Time-shifting property of the Laplace transform

- Remember that in this course we mainly deal with causal signals.

- Consider a causal signal $x(t)$ with Laplace transform $X(s)$. Suppose that we delay $x(t)$ by t_0 units of time to obtain $x(t-t_0)$ with $t_0 \geq 0$. The new signal will have Laplace transform

$$X(s) = \mathcal{L}\{x(t-t_0)\} = \int_0^{\infty} x(t-t_0)e^{-st}dt$$

Let $t - t_0 = v$. In that case $dt = dv$ and

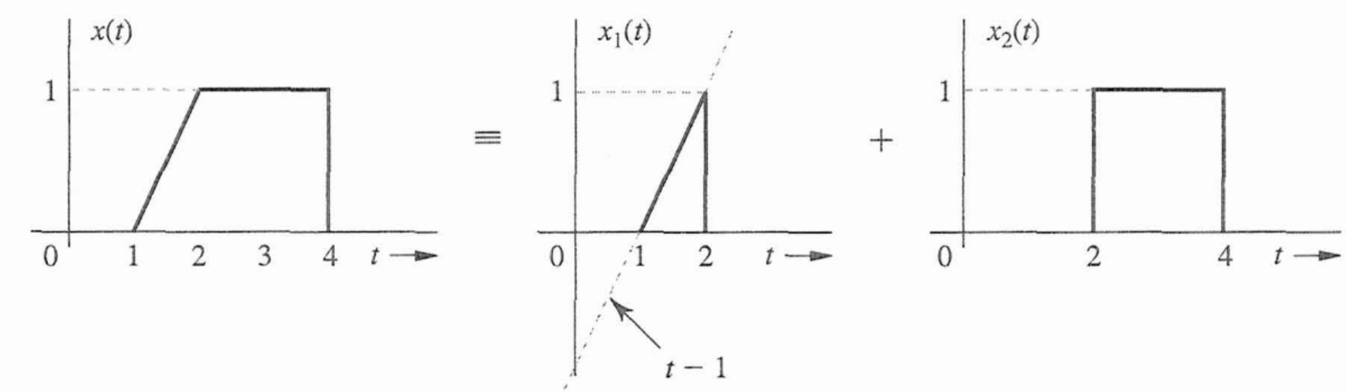
$$\mathcal{L}\{x(t-t_0)\} = \int_{-t_0}^{\infty} x(v)e^{-s(v+t_0)}dv = e^{-st_0}\int_{-t_0}^{\infty} x(v)e^{-sv}dv$$

$$= e^{-st_0}\int_0^{\infty} x(v)e^{-sv}dv = e^{-st_0}X(s)$$

- **Observe that delaying a signal introduces only an exponential component in the new Laplace transform which becomes just a phase component if $s = j\omega$.**

Example on using the Laplace transform properties

- Consider the signal $x(t)$ shown below. This can be broken into signals $x_1(t)$ and $x_2(t)$.



- $x_1(t) = (t-1)[u(t-1) - u(t-2)]$ and $x_2(t) = u(t-2) - u(t-4)$.
- Therefore,

  $x(t) = x_1(t) + x_2(t) = (t-1)[u(t-1) - u(t-2)] + u(t-2) - u(t-4)$ or
  $x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$

- $\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$, $\mathcal{L}\{(t-1)u(t-1)\} = e^{-s}\frac{1}{s^2}$, $\mathcal{L}\{(t-2)u(t-2)\} = e^{-2s}\frac{1}{s^2}$

  $\mathcal{L}\{u(t)\} = \frac{1}{s} \Rightarrow \mathcal{L}\{u(t-4)\} = e^{-4s}\frac{1}{s}$

- $X(s) = e^{-s}\frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$

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Frequency shifting property

- Consider the signal $x(t)$ with Laplace transform $X(s)$. What is the signal that corresponds to the Laplace transform $X(s-s_0)$? By definition, this should be

$$\frac{1}{2\pi j}\int_{c-jT}^{c+jT} X(s-s_0)e^{st}ds = \frac{1}{2\pi j}\int_{c-jT}^{c+jT} X(s-s_0)e^{s_0 t}e^{(s-s_0)t}ds$$

$$= \frac{1}{2\pi j}\int_{c-jT}^{c+jT} X(s-s_0)e^{s_0 t}e^{(s-s_0)t}d(s-s_0)$$

$$= e^{s_0 t}\frac{1}{2\pi j}\int_{c-jT}^{c+jT} X(s-s_0)e^{(s-s_0)t}d(s-s_0)$$

- If we set $s-s_0=v$ then $(s=c\pm jT \Rightarrow v=c\pm jT-s_0)$

$$\mathcal{L}^{-1}\{X(s-s_0)\} = e^{s_0 t}\frac{1}{2\pi j}\int_{c-s_0-jT}^{c-s_0+jT} X(v)e^{vt}dv = e^{s_0 t}x(t) \text{ or}$$

$$\mathcal{L}\{e^{s_0 t}x(t)\} = X(s-s_0)$$

Application of frequency shifting property

- Given that $\mathcal{L}\{\cos(bt)u(t)\} = \frac{s}{s^2+b^2}$ show $\mathcal{L}\{e^{-at}\cos(bt)u(t)\} = \frac{s+a}{(s+a)^2+b^2}$.
- Recall the frequency shifting property $\mathcal{L}\{e^{s_0 t}x(t)\} = X(s - s_0)$.
- Using the shifting property with $s_0 = -a$ we have that if

$\mathcal{L}\{\cos(bt)u(t)\} = \frac{s}{s^2+b^2}$ then

$\mathcal{L}\{e^{-at}\cos(bt)u(t)\} = \frac{s+a}{(s+a)^2+b^2}$.

Time-differentiation property

- Given that $\mathcal{L}\{x(t)\} = X(s)$ show $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

- Partial integration states that: $\int_a^b u dv = uv|_a^b - \int_a^b v du$.

- Using partial integration with $u = e^{-st} \Rightarrow du = -se^{-st}dt$ and $v = x(t) \Rightarrow dv = \frac{dx(t)}{dt}dt$ we have:

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} d(x(t)) = e^{-st}x(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)(-s)e^{-st}dt =$$

$$[e^{-s\infty}x(\infty) - e^{-s0^-}x(0^-)] + s\int_{0^-}^{\infty} x(t)e^{-st}dt$$

- The first term in the brackets goes to zero (as long as $x(t)$ doesn't grow faster than an exponential which is a condition for the existence of the transform). Therefore, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = -e^{-s0^-}x(0^-) + s\int_{0^-}^{\infty} x(t)e^{-st}dt = sX(s) - x(0^-)$.

Time-differentiation property cont.

- Repeated application of the differentiation property yields the Laplace transform of the higher order derivatives of a signal.

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s[sX(s) - x(0^-)] - \dot{x}(0^-) = s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\vdots$$

$$\mathcal{L}\left\{\frac{d^nx(t)}{dt^n}\right\} = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$

with $x^{(r)}(0^-) = \left.\frac{d^rx(t)}{dt^r}\right|_{t=0^-}$

Differentiation in frequency

- Let us now take the derivative of the Laplace transform:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$$

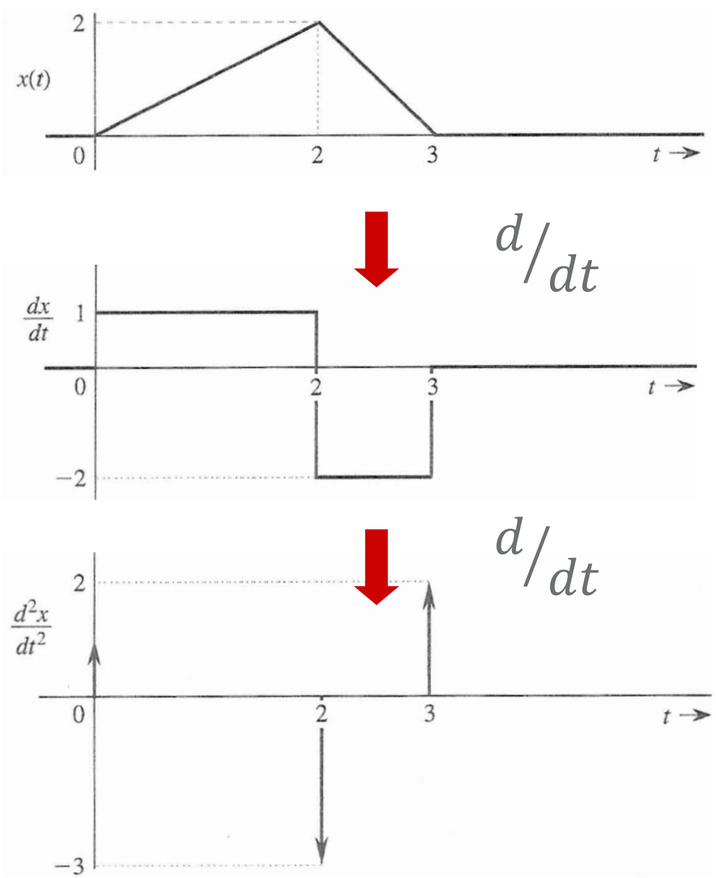
$$\frac{dX(s)}{ds} = \int_0^{\infty}(-t)x(t)e^{-st}dt = -\int_0^{\infty} tx(t)e^{-st}dt$$

- From the above we see that:

$$\mathcal{L}\{tx(t)\} = -\frac{dX(s)}{ds}$$

Application of time-differentiation

- Find the Laplace transform of the signal $x(t)$ using time differentiation and time shifting properties.



- $\frac{d^2x(t)}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$
- $\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = \mathcal{L}\{\delta(t) - 3\delta(t-2) + 2\delta(t-3)\}$ $= 1 - 3e^{-2s} + 2e^{-3s}$
- $\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s) - sx(0^-) - \dot{x}(0^-) =$ $s^2X(s) - 0 - 0 = s^2X(s)$
- Therefore,
  $s^2X(s) = 1 - 3e^{-2s} + 2e^{-3s} \Rightarrow$
  $X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$

Time- and frequency- integration property and scaling property

- If $\mathcal{L}\{x(t)\} = X(s)$ the following properties hold.
- **Time-integration** property.

$$\mathcal{L}\left\{\int_{0^-}^{\infty} x(\tau)d\tau\right\} = \frac{X(s)}{s}$$

- The dual property of time-integration is the **frequency-integration** property.

$$\mathcal{L}\left\{\frac{x(t)}{t}\right\} = \int_{s}^{\infty} X(z)dz$$

- Scaling property.

$$\mathcal{L}\{x(at)\} = \frac{1}{a}X(\frac{s}{a})$$

The above shows that time compression of a signal by a factor a causes expansion of its Laplace transform in s by the same factor.

Time-convolution and frequency-convolution properties

- If $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$ the following properties hold.

- Time-convolution property.

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s)$$

  As seen, convolution in time domain is equivalent to multiplication in Laplace domain.

- Frequency-convolution property.

$$\mathcal{L}\{x_1(t)x_2(t)\} = \frac{1}{2\pi j} X_1(s) * X_2\,(s)$$

  As seen, convolution in Laplace domain is equivalent to multiplication in time domain weighted by the complex scalar $2\pi j$.

Application of the convolution properties

- **Problem**

Determine the convolution $c(t) = e^{at}u(t) * e^{bt}u(t)$.

Solution

We use the Laplace transforms:

$$\mathcal{L}\{e^{at}u(t)\} = \frac{1}{(s-a)} \text{ and } \mathcal{L}\{e^{bt}u(t)\} = \frac{1}{(s-b)}$$

Therefore, $\mathcal{L}\{e^{at}u(t) * \ e^{bt}u(t)\} = \frac{1}{(s-a)(s-b)}$.

We see that $\frac{1}{(s-a)(s-b)} = \frac{1}{(a-b)}[\frac{1}{s-a} - \frac{1}{s-b}]$.

Therefore, $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(a-b)}[\frac{1}{s-a} - \frac{1}{s-b}]\right\} = \frac{1}{(a-b)}$
$\mathcal{L}^{-1}\left\{\frac{1}{s-a} - \frac{1}{s-b}\right\} \Rightarrow$

$$c(t) = e^{at}u(t) * \ e^{bt}u(t) = \frac{1}{(a-b)}(e^{at} - e^{bt})u(t)$$

Relationship with time-domain analysis

- If $h(t)$ is the impulse response of an LTI system, then we have seen in previous lectures that the system's response to an input $x(t)$ is $x(t) * h(t)$.

- Assuming causality and that $\mathcal{L}\{h(t)\} = H(s)$ and $\mathcal{L}\{x(t)\} = X(s)$ then the Laplace transform of the output of the system is:
$$Y(s) = X(s)H(s)$$

- The response $y(t)$ is the zero-state response of the LTI system to the input $x(t)$. The **transfer function** of the system $H(s)$ is defined as:
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}\{\text{zero-state response}\}}{\mathcal{L}\{\text{input}\}}$$

The everlasting exponential $e^{s_0 t}$

- Suppose that the input to an LTI system is an everlasting exponential $e^{s_0 t}$ which starts at $t = -\infty$.

- If $h(t)$ is the unit impulse response of an LTI system then:
  $y(t) = h(t) * e^{s_0 t} = \int_{-\infty}^{\infty} h(\tau) e^{s_0 (t-\tau)} d\tau = e^{s_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau = e^{s_0 t} H(s_0)$

- Therefore, $y(t) = e^{s_0 t} H(s_0)$ where $H(s_0)$ is the Laplace transform of $h(t)$ evaluated at s_0.

- What we proved above is the very important result that the output of an LTI system to a single everlasting exponential is the same as the input multiplied by the constant $H(s_0)$.

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Summary of Laplace transform properties

	$x(t)$	$X(s)$
Addition	$x_1(t) + x_2(t)$	$X_1(s) + X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s) - s^2x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$

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Summary of Laplace transform properties cont.

	$x(t)$	$X(s)$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0 t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\dfrac{dX(s)}{ds}$
Frequency integration	$\dfrac{x(t)}{t}$	$\int_s^{\infty} X(z)\,dz$
Scaling	$x(at),\ a \geq 0$	$\dfrac{1}{a}X\left(\dfrac{s}{a}\right)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\dfrac{1}{2\pi j}X_1(s) * X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \quad (n > m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s) \quad$ [poles of $sX(s)$ in LHP]

Initial and final value theorems

Problem: How to find the initial and final values of a function $x(t)$ ($t \to 0^+$ and $t \to \infty$) if we know its Laplace Transform $X(s)$ and do not want to compute the inverse?

- **Initial Value Theorem**

  $\lim_{t\to 0^+} x(t) = x(0^+) = \lim_{s\to\infty} sX(s)$, $x(t)$ must be bounded on $(0,\infty)$.

  Conditions

  - Laplace transforms of $x(t)$ and $\frac{dx(t)}{dt}$ must exist.
  - The power of the numerator of $X(s)$ is less than the power of its denominator.

- **Final Value Theorem**

  $\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)$, $x(t)$ must be bounded on $(0,\infty)$.

  Conditions

  - Laplace transforms of $x(t)$ and $\frac{dx(t)}{dt}$ must exist.
  - The poles of $sX(s)$ are all on the left plane or origin.

Initial and final value theorems: Example

- **Example:** Find the initial and final values of $y(t)$ if $Y(s)$ is given by:

$$Y(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$$

- **Initial Value**

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{10(2s+3)}{s(s^2+2s+5)} = \lim_{s\to\infty}\frac{10(2s+3)}{(s^2+2s+5)} = 0$$

- **Final Value**

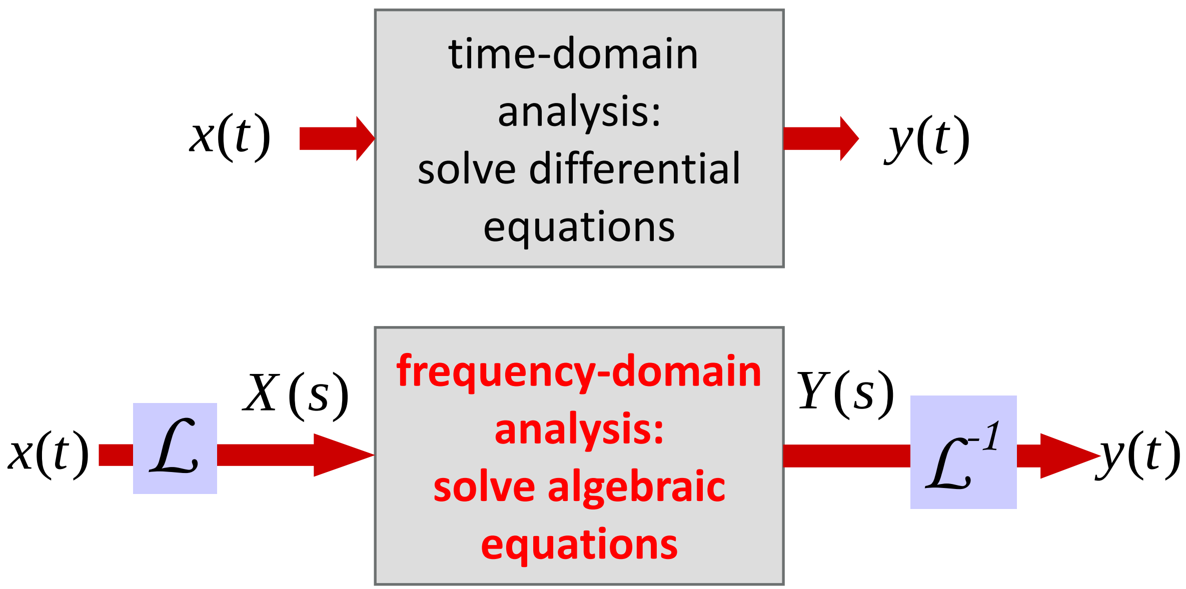
$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{10(2s+3)}{s(s^2+2s+5)} = \lim_{s\to 0}\frac{10(2s+3)}{(s^2+2s+5)} = 6$$

Laplace transform for solving differential equations

- Recall the time-differentiation property of the Laplace transform:

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}\,(0^-)$$

- We exploit the above in order to **solve differential equations as algebraic equations!**



Laplace transform for solving differential equations: Example

- **Problem:** Solve the following second-order linear differential equation:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+x(t)$$

given that $y(0^-)=2, \dot{y}(0^-)=1$ and input $x(t)=e^{-4t}u(t)$.

Time Domain	Laplace (Frequency) Domain
$\frac{dy(t)}{dt}$	$sY(s)-y(0^-)=sY(s)-2$
$\frac{d^2y(t)}{dt^2}$	$s^2Y(s)-sy(0^-)-\dot{y}(0^-)=s^2Y(s)-2s-1$
$x(t)=e^{-4t}u(t)$	$X(s)=\frac{1}{s+4}$
$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)=\frac{s}{s+4}-0=\frac{s}{s+4}$

Example cont.

Time Domain	Laplace (Frequency) Domain
$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$	$(s^2Y(s) - \mathbf{2s} - \mathbf{1}) + 5(sY(s) - \mathbf{2}) + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$ $\Rightarrow (s^2+5s+6)Y(s) - (\mathbf{2s+11}) = \frac{s+1}{s+4}$ $(s^2+5s+6)Y(s) = \frac{s+1}{s+4} + (\mathbf{2s+11}) = \frac{s+1+2s^2+8s+11s+44}{s+4} = \frac{2s^2+20s+45}{s+4} \Rightarrow Y(s) = \frac{2s^2+20s+45}{(s+2)(s+3)(s+4)}$
$y(t) = (\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t})u(t)$	$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$

Zero-input and zero-state responses

- Let us think where the terms in the previous example come from.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

- The output can be written as:

$$Y(s) = \underbrace{\frac{\mathbf{2s+11}}{s^2+5s+6}}_{\text{zero-input component}} + \underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state component}}$$

- Notice that the zero-input component of the response occurs due to the initial conditions and the zero-state component occurs due to the input itself.

$$Y(s) = \left(\frac{7}{s+2} - \frac{5}{s+3}\right) + \left(\frac{-\frac{1}{2}}{s+2} + \frac{2}{s+3} - \frac{\frac{3}{2}}{s+4}\right)$$

$$y(t) = \underbrace{(7e^{-2t} - 5e^{-3t})u(t)}_{\text{zero-input response}} + \underbrace{(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t})u(t)}_{\text{zero-state response}}$$

Laplace transform and transfer function

- Let $x(t)$ be the input of a LTI system. If $h(t)$ is the impulse response of the system then we proved previously that the output of the system $y(t)$ is given by:

$$y(t) = x(t) * h(t)$$

- It can be proven that $\mathcal{L}\{x(t) * h(t)\} = X(s)H(s)$. Therefore,

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\} = X(s)H(s) \Rightarrow$$

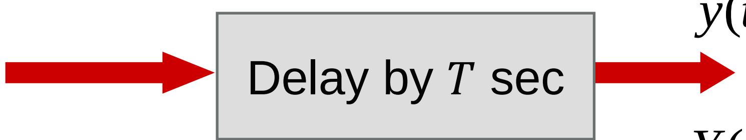
$$H(s) = \frac{Y(s)}{X(s)} \text{ with } H(s) = \mathcal{L}\{h(t)\}$$

- The Laplace transform of the impulse response of a system is called the transfer function of the system.

- Knowing the transfer function of a system we can fully determine the behaviour of the system.

Transfer function examples

$x(t) \Updownarrow X(s)$



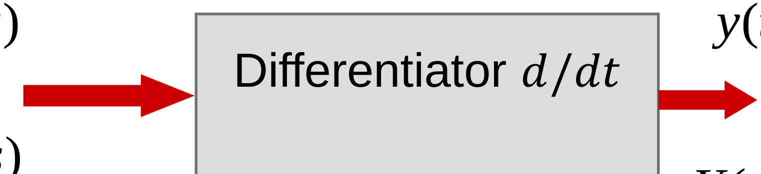
$y(t) = x(t-T) \Updownarrow Y(s) = X(s)e^{-sT}$

$$\Updownarrow$$

$$H(s) = Y(s) \big/ X(s) = e^{-sT}$$

shifting property

$x(t) \Updownarrow X(s)$



$y(t) = dx/dt \Updownarrow Y(s) = sX(s)$

$$\Updownarrow$$

$$H(s) = s$$

differentiation property

$x(t) \Updownarrow X(s)$

$y(t) = \int_{0^-}^{t} x(\tau)d\tau \Updownarrow Y(s) = \frac{1}{s}X(s)$

$$\Updownarrow$$

$$H(s) = \frac{1}{s}$$

integration property

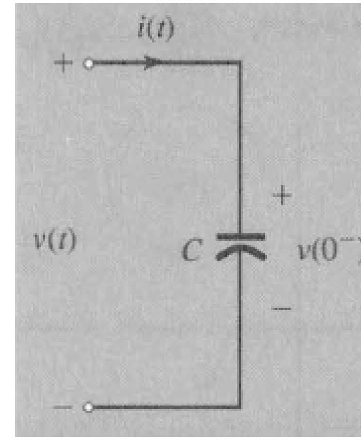
Initial conditions in systems (1)

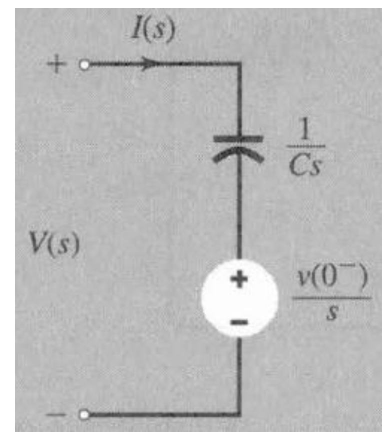
- In circuits, initial conditions may not be zero. For example, capacitors may be charged; inductors may have an initial current.
- How should these be represented in the Laplace (frequency) domain?
- Consider a capacitor C with an initial voltage $v(0^-)$. The equation $i(t) = C\frac{dv(t)}{dt}$ holds.
- Now take Laplace transform on both sides $I(s) = C(sV(s) - v(0^-))$.
- Rearrange the above to give $V(s) = \frac{1}{Cs}I(s) + \frac{v(0^-)}{s}$.

voltage across charged capacitor

voltage across capacitor with no charge

effect of the initial charge ≡ voltage source





Initial conditions in systems (2)

- Similarly, consider an inductor L with an initial current $i(0^-)$. The equation $v(t) = L\frac{di(t)}{dt}$ holds.
- Now take Laplace transform on both sides

$$V(s) = L(sI(s) - i(0^-)) = LsI(s) - Li(0^-)$$

voltage across inductor

voltage across inductor with no initial current

effect of the initial current ≡ voltage source