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Signals and Systems

Lecture 5

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Properties of convolution

No	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t)$
1	$\boldsymbol{x(t)}$	$\boldsymbol{\delta(t-T)}$	$\boldsymbol{x(t-T)}$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t),\ \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$

More properties of convolution

7	$t^N u(t)$	$e^{\lambda t} u(t)$	$\frac{N!\,e^{\lambda t}}{\lambda^{N+1}} u(t) - \sum_{k=0}^{N} \frac{N!\,t^{N-k}}{\lambda^{k+1}(N-k)!} u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\frac{M!N!}{(M+N+1)!} t^{M+N+1} u(t)$
9	$te^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2) t e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2} u(t)$
10	$t^M e^{\lambda t} u(t)$	$t^N e^{\lambda t} u(t)$	$\frac{M!\,N!}{(N+M+1)!} t^{M+N+1} e^{\lambda t} u(t)$
11	$t^M e^{\lambda_1 t} u(t)$	$t^N e^{\lambda_2 t} u(t)$	$\sum_{k=0}^{M} \frac{(-1)^k M!(N+k)!\,t^{M-k} e^{\lambda_1 t}}{k!(M-k)!(\lambda_1 - \lambda_2)^{N+k+1}} u(t)$
	$\lambda_1 \neq \lambda_2$		$+ \sum_{k=0}^{N} \frac{(-1)^k N!(M+k)!\,t^{N-k} e^{\lambda_2 t}}{k!(N-k)!(\lambda_2 - \lambda_1)^{M+k+1}} u(t)$

More properties of convolution cont.

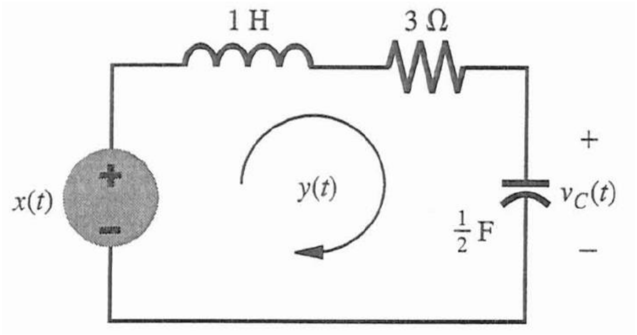
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda t}u(t)$	$\dfrac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$ $\phi=\tan^{-1}[-\beta/(\alpha+\lambda)]$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)}{\lambda_2-\lambda_1}$ $\quad \mathrm{Re}\,\lambda_2>\mathrm{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_2-\lambda_1}u(-t)$

Find the output of a system using a convolution property

- Find the loop current $y(t)$ of the RLC circuit shown below for input $x(t) = 10e^{-3t}u(t)$ when all the initial conditions are zero.
- We have seen that the system's equation is

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

- The above can be written as $(D^2 + 3D + 2)y(t) = Dx(t)$.



- We solved the equation of the above system in the previous lecture and we found that the impulse response of the system is

$$h(t) = (-e^{-t}+2e^{-2t})u(t)$$

- Therefore, $y(t) = x(t) * h(t)$. We can solve this convolution using Property 4. Proof is given in Class 3 Presentation, Problem 1(ii).

4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t),\ \lambda_1 \neq \lambda_2$

Find the output of a system using a convolution property cont.

- $y(t) = \ x(t) * h(t)$
- $x(t) = 10e^{-3t}u(t)$ and $h(t) = (-e^{-t}+2e^{-2t})u(t)$.
- $y(t) = 10e^{-3t}u(t) * (-e^{-t}+2e^{-2t})u(t) =$
  $$10e^{-3t}u(t) * (-e^{-t})u(t) + 10e^{-3t}u(t) * 2e^{-2t}u(t)$$
- For the first term we have:
  $$10e^{-3t}u(t) * (-e^{-t})u(t) = -10e^{-3t}u(t) * e^{-t}u(t)$$
  $$= -10\frac{e^{-3t} - e^{-t}}{(-3) - (-1)}u(t) = 5(e^{-3t} - e^{-t})u(t)$$
- For the second term we have:
  $$10e^{-3t}u(t) * 2e^{-2t}u(t) = 20\frac{e^{-3t} - e^{-2t}}{(-3) - (-2)}u(t) = -20(e^{-3t} - e^{-2t})u(t)$$
- $y(t) = \ x(t) * h(t) = (-15e^{-3t} + 20e^{-2t} - 5e^{-t})u(t)$

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Intuitive/graphical explanation of convolution

- Assume that the impulse response decays linearly from the value of 1 at $t = 0$ to the value of 0 at $t = 1$. See figure below left.
- The system's response at t is the convolution between $x(t)$ and $h(t)$

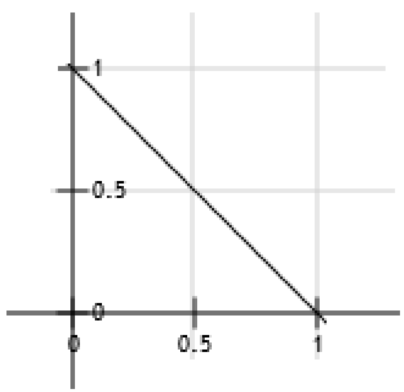
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

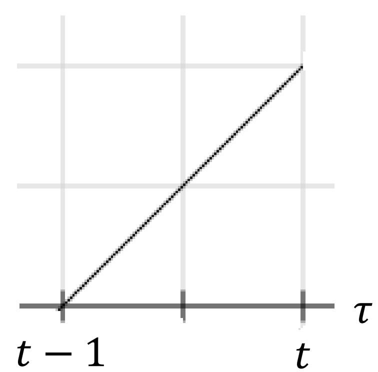
- Since $h(t) \neq 0$ if $0 \leq t \leq 1$ we see that

$h(t-\tau) \neq 0$ if $0 \leq t-\tau \leq 1 \Rightarrow -1 \leq \tau - t \leq 0 \Rightarrow t-1 \leq \tau \leq t$.

For $h(t-\tau)$ see figure below right. Therefore,

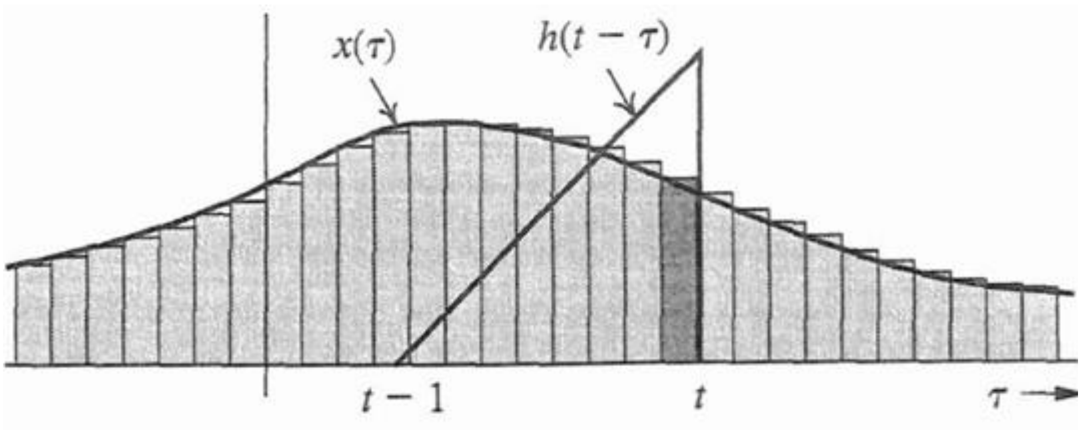
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} x(\tau)h(t-\tau)d\tau$$





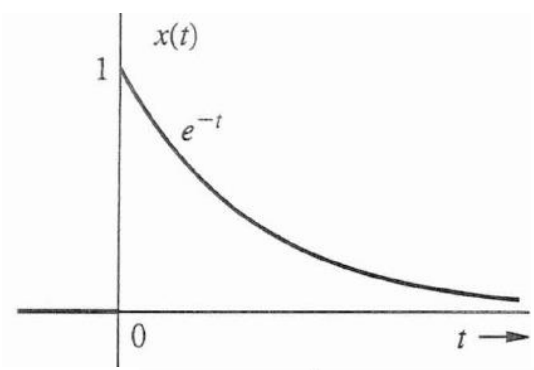
Intuitive/graphical explanation of convolution cont.

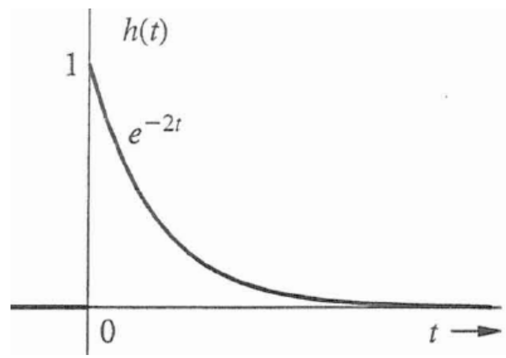
- The system's output is $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} x(\tau)h(t-\tau)d\tau$.
- We approximate, as previously, the input $x(\tau)$ as a collection of rectangular pulses (defined in the previous lecture).
- The system's response $y(t)$ at t is determined by $x(t)$ PLUS the contribution from all the previous pulses **weighted with** $x(\tau)$ within the range $[t-1, t]$ where $h(t)$ is non-zero.
- Note that $x(\tau)$ corresponds to a Dirac delayed τ units of time; therefore, its response is $x(\tau)h(t-\tau)$ due to linearity. The system's response is shown **graphically** below.
- The summation of all these weighted inputs is also shown **functionally** in the convolution integral $y(t) = x(t) * h(t)$ above.

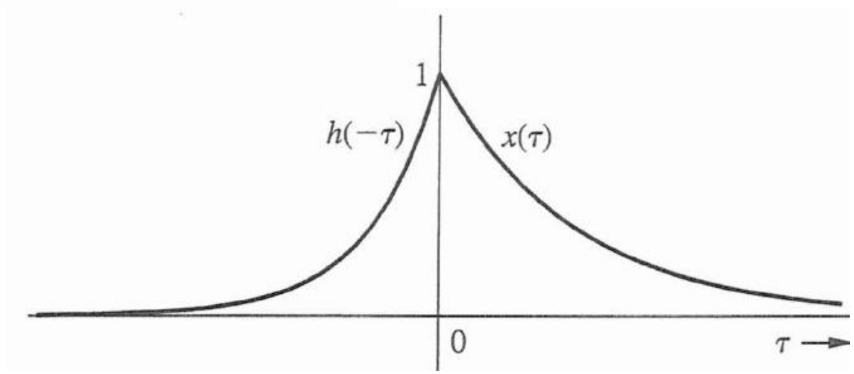


Example for graphical demonstration of convolution

- Demonstrate graphically the convolution
  $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ with
  $x(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$.
- Remember: the variable of integration is τ and not t.







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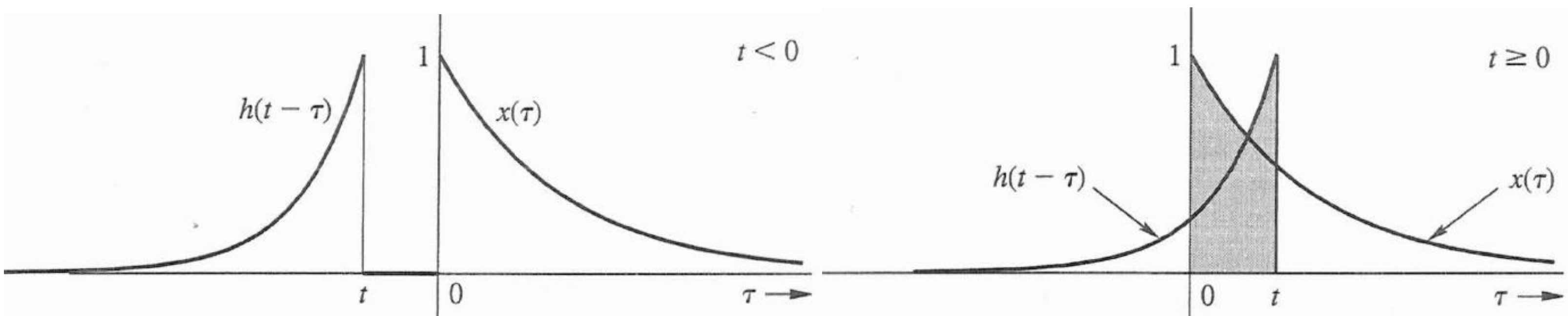
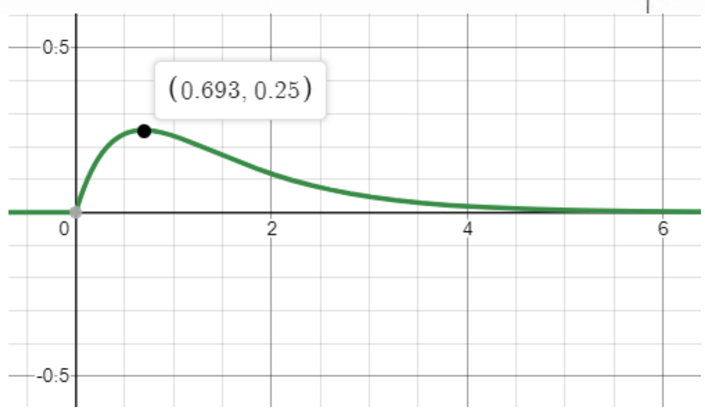
Graphical demonstration of convolution cont.

- By definition we have:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau.$$

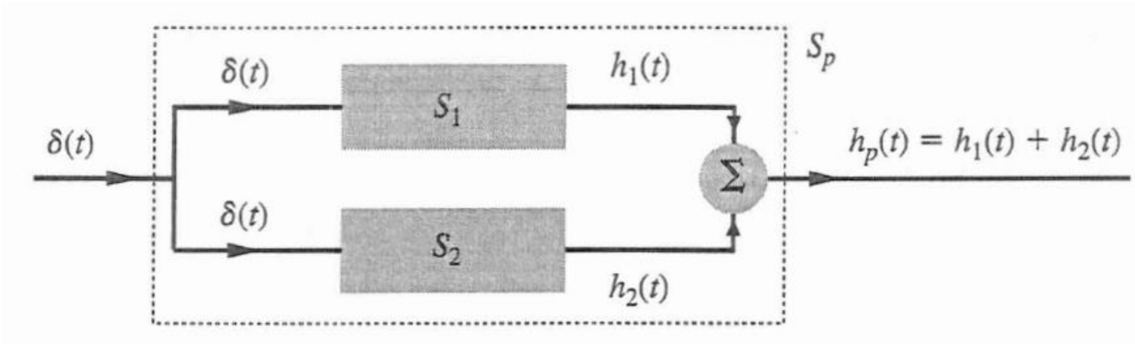
- We found previously that $y(t) = (e^{-t} - e^{-2t})u(t)$ **(Class 3, Problem 1 (ii))**
- The convolution is shown at the bottom figure with maximum of 0.25 at $0.693 = \ln(2)$.



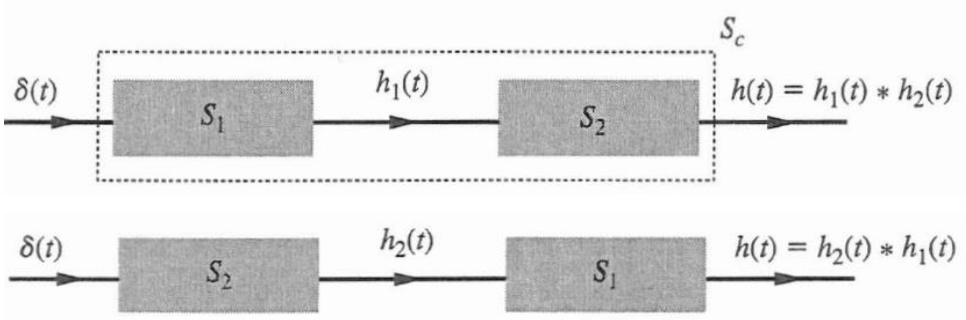


Interconnected systems

- For the parallel connection of two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ the output is $y(t) = h_1(t) * x(t) + h_2(t) * x(t)$.



- For the connection in series of two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ the output is $y(t) = h_1(t) * h_2(t) * x(t)$. The order of the connection is not important.



Integrator

- Consider an LTI system S. If the input $x(t)$ produces the output $y(t)$, the input $\int_{-\infty}^{t} x(\tau)d\tau$ produces the output $\int_{-\infty}^{t} y(\tau)d\tau$.

Proof

Consider the case where the output is the signal $\int_{-\infty}^{t} y(\tau)d\tau$. We know that since S is an LTI system we have:

$$\int_{\tau=-\infty}^{t} y(\tau)d\tau = \int_{\tau=-\infty}^{t}\left(\int_{z=-\infty}^{z=+\infty} x(\tau-z)\,h(z)dz\right)d\tau$$

$$= \int_{z=-\infty}^{z=+\infty}\int_{\tau=-\infty}^{t} x(\tau-z)\,h(z)d\tau dz = \int_{z=-\infty}^{z=+\infty} h(z)\left(\int_{\tau=-\infty}^{t} x(\tau-z)\,d\tau\right)dz$$

We define $\tau - z = \rho$ and therefore, $d\tau = d\rho$. For $\tau = t$ we see that $\rho = t - z$. Therefore, the above becomes:

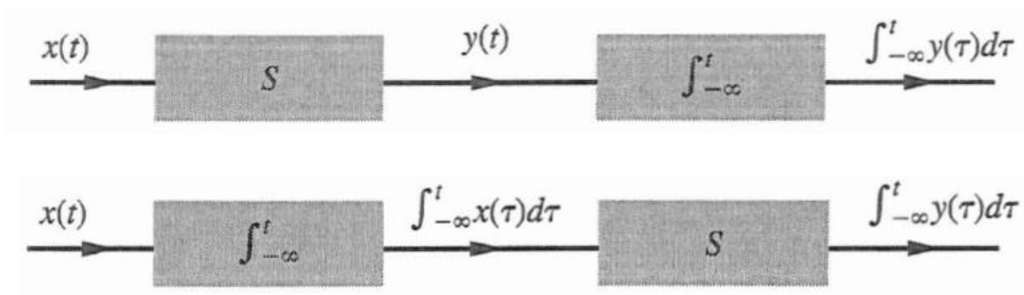
$$\int_{\tau=-\infty}^{t} y(\tau)d\tau = \int_{z=-\infty}^{z=+\infty} h(z)\left(\int_{\rho=-\infty}^{t-z} x(\rho)\,d\rho\right)dz = h(t) * \left(\int_{\rho=-\infty}^{t} x(\rho)\,d\rho\right)$$

which reveals that the output $\int_{-\infty}^{t} y(\tau)d\tau$ corresponds to an input $\int_{-\infty}^{t} x(\tau)d\tau$.

Integrator – Differentiator - Step response

- The previous analysis is depicted in the figures below.

Integration: if $x(t) \Rightarrow y(t)$ then $\int_{-\infty}^{t} x(\tau)d\tau \Rightarrow \int_{-\infty}^{t} y(\tau)d\tau$.



- Similar comments are valid for differentiation. More specifically,

Differentiation: if $x(t) \Rightarrow y(t)$ then $\frac{dx(t)}{dt} \Rightarrow \frac{dy(t)}{dt}$.

- Knowing that $\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$ and that $\delta(t) \Rightarrow h(t)$ we see that if the input of the system is the step function, i.e., $x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ then the output of the system, which is called the **step response**, must be:

$$g(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Total response

- We learnt that:

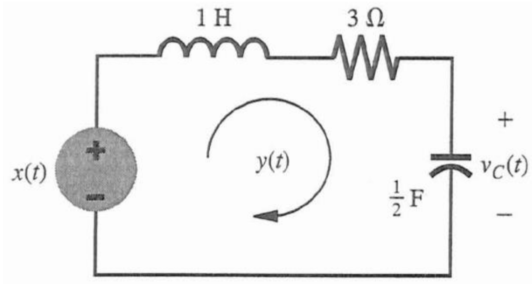
Total response = zero-input response + zero-state response

$$\sum_{k=1}^{N} c_k e^{\lambda_k t} + x(t) * h(t)$$

- We will combine everything using the same RLC circuit as an example.
- Let us assume $x(t) = 10e^{-3t}u(t),\ y(0) = 0,\ \dot{y}(0) = -5$.
- For the zero-input response look at Lecture 3, Example 1, Slides 9-10.
- The total current (which is considered to be the output of the system) is:

Total current = zero-input current + zero-state current

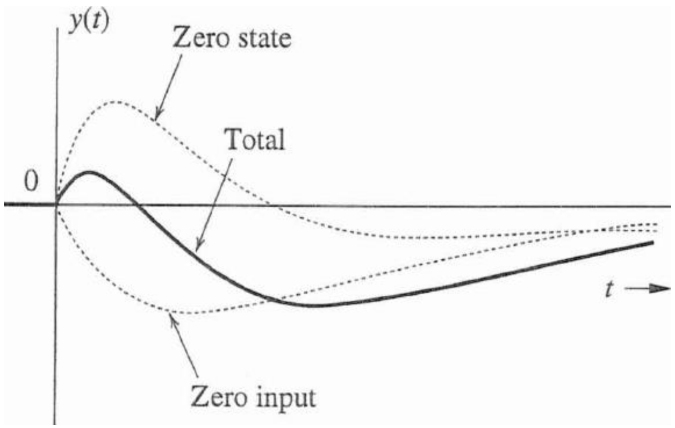
$$y(t) = (5e^{-2t} - 5e^{-t})u(t) + (-15e^{-3t} + 20e^{-2t} - 5e^{-t})u(t)$$
$$= (5e^{-2t} - 5e^{-t}) + (-15e^{-3t} + 20e^{-2t} - 5e^{-t}), \qquad t \geq 0$$

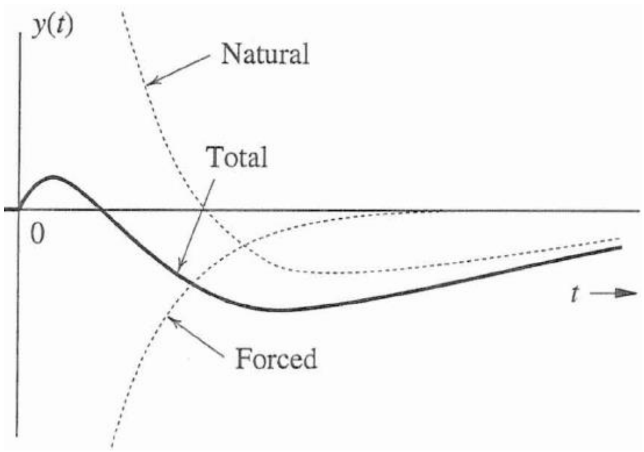


Natural versus forced responses

- Note that characteristic modes also appear in zero-state response **(obviously they have an impact on $h(t)$ as well!)**.
- We can collect the e^{-t} and e^{-2t} terms together, and call these the **natural response**.
- The remaining e^{-3t} which is not a characteristic mode is called the **forced response**.

$$y(t) = (25e^{-2t} - 10e^{-t})u(t) + (-15e^{-3t})u(t)$$





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Appendix: The output of a LTI system when the input is complex

- What happens if the input $x(t)$ of a system is complex instead of real?
  $x(t) = x_r(t) + jx_i(t)$ with $x_r(t)$, $x_i(t)$ the real and imaginary parts of the input, respectively.

- The output of the system is:
  $y(t) = \ h(t) * [x_r(t) + jx_i(t)] = h(t) * x_r(t) + j\ h(t) * x_i(t)$

- That is, we can consider the convolution on the real and imaginary components separately.