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Signals and Systems

Lecture 4
Zero-state response

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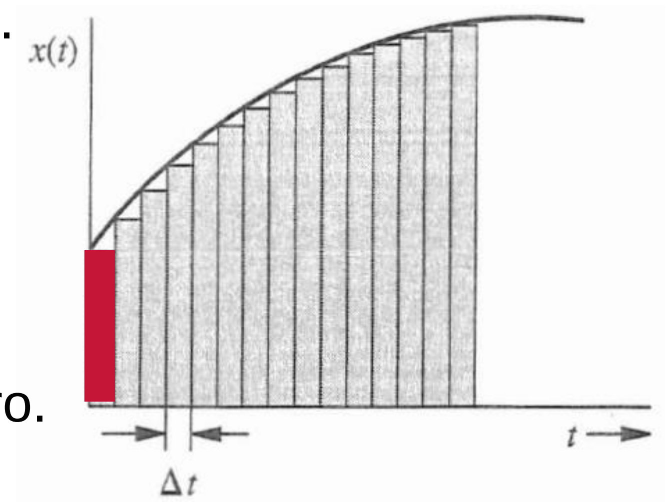
The importance of impulse response

- Zero-state response assumes that the system is in “rest” state, i.e. all internal system variables are zero.
- Deriving and understanding the zero-state response relies on knowing the so called unit impulse response $h(t)$.
- **Definition:** The unit impulse response $h(t)$ is the system’s response when the input is the Dirac function, i.e., $x(t) = \delta(t)$, with all the initial conditions being zero at $t = 0^-$.
- Any input $x(t)$ can be broken into a sequence of narrow rectangular pulses. Each pulse produces a system response.
- If a system is linear and time invariant, the system’s response to $x(t)$ is the sum of its responses to all narrow pulse components.
- $h(t)$ is the system’s response to the rectangular pulse at $t = 0$ as the pulse width approaches zero.



How to determine the unit impulse response $h(t)$?

- Given that a system is specified by the following differential equation, determine its unit impulse response $h(t)$.

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$$
$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t),\ M \le N$$

- Remember the general equation of a system:

$$Q(D) y(t) = P(D) x(t)$$

- It can be shown **(proof is out of the scope of this course)** that the impulse response $h(t)$ is given by

$$h(t) = [P(D) y_n(t)] u(t)$$

where $u(t)$ is the unit step function. But what is $y_n(t)$?

How to determine the unit impulse response $h(t)$?

- $y_n(t)$ is the solution to the homogeneous differential equation (what we called $y_0(t)$ in the previous lecture)

$$Q(D)y_n(t) = 0$$

with the following initial conditions:

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots = y_n^{(N-2)}(0) = 0,\ y_n^{(N-1)}(0) = 1$$

- We use $y_n(t)$ instead of $y_0(t)$ to associate $y_n(t)$ with the specific set of initial conditions mentioned above.

- Remember that $y_n(t)$ is a linear combination of the characteristic modes of the system.

$$y_n(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}$$

- The constants c_i are determined from the initial conditions.

- Note that $y_n^{(k)}(0)$ is the k^{th} derivative of $y_n(t)$ at $t = 0$.

Example

- Determine the impulse response for the system:
$$(D^2 + 3D + 2)y(t) = Dx(t)$$
- This is a second-order system (i.e., $N = 2,\ M = 1$) and the characteristic polynomial is:
$$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$$
- The characteristic roots are $\lambda_1 = -1$ and $\lambda_2 = -2$.
- Therefore, $y_n(t) = c_1 e^{-t} + c_2 e^{-2t}$.
- Differentiating the above equation yields $\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$.
- The initial conditions are $\dot{y}_n(0) = 1$ and $y_n(0) = 0$.

Example cont.

- Setting $t = 0$ and substituting the initial conditions yields:

$$0 = c_1 + c_2$$
$$1 = -c_1 - 2c_2$$

- The solution of the above set of equations is:

$$c_1 = 1$$
$$c_2 = -1$$

- Therefore, we obtain:

$$y_n(t) = e^{-t} - e^{-2t}$$

- Remember that $h(t)$ is given by:

$$h(t) = [P(D)y_n(t)]u(t)$$

  with $P(D) = D$ in this case.

- Therefore:

$$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t}+2e^{-2t})u(t)$$

  [Note that: $P(D)y_n(t) = D(t)y_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$]

Zero-state response

- Consider a **Linear Time-Invariant** system with impulse response $h(t)$.
- The output at time t due to a shifted impulse with amplitude a located at time instant τ is the impulse amplitude a multiplied by a shifted impulse response located at τ as well.
- In other words:

$$\delta(t) \rightarrow h(t)$$
$$a\delta(t) \rightarrow ah(t)$$
$$a\delta(t-\tau) \rightarrow ah(t-\tau)$$

- If we generalize the above observation we can say that the output of a linear system to an input $x(t) = \sum_{i=1}^{n} a_i \delta(t-\tau_i)$ is

$$y(t) = \sum_{i=1}^{n} a_i h(t-\tau_i).$$

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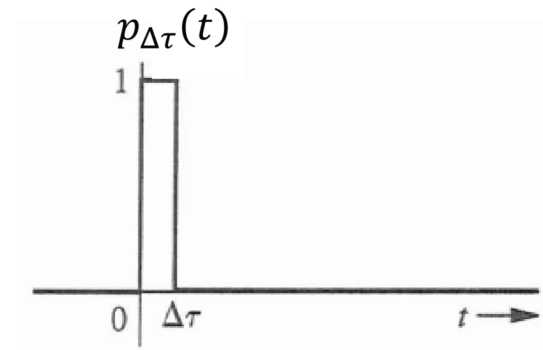
Zero-state response cont.

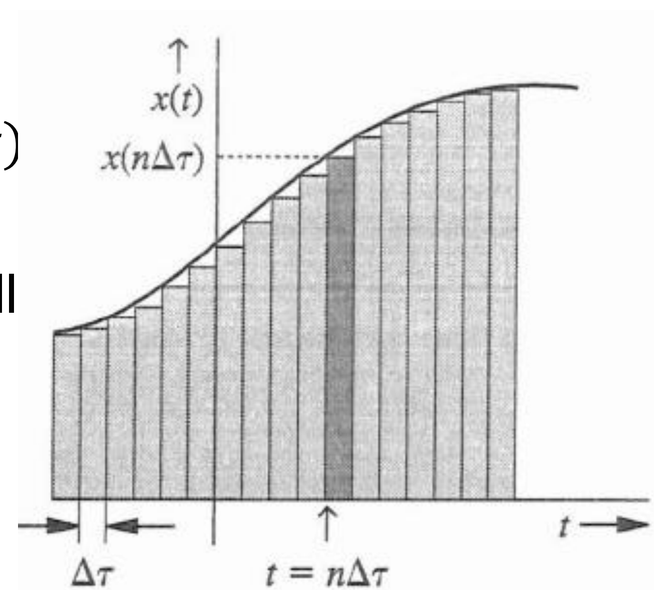
- We now consider how to determine the system's response $y(t)$ to any input $x(t)$ when the system is in the zero state (initial conditions are zero).
- Define a pulse $p_{\Delta\tau}(t)$ of height equal to 1 and width $\Delta\tau$ starting at $t = 0$ (see top figure on the right).
- Any input $x(t)$ can be approximated by a sum of narrow and shifted rectangular pulses.
- The pulse starting at $t = n\Delta\tau$ has a height $x(n\Delta\tau)$

  It can be expressed as $x(n\Delta\tau)p_{\Delta\tau}(t - n\Delta\tau)$.
- Therefore, $x(t)$ is approximated by the sum of all such pulses as follows:

$x(t) = \lim_{\Delta\tau\to 0} \sum_n x(n\Delta\tau)p_{\Delta\tau}(t - n\Delta\tau)$ or

$$x(t) = \lim_{\Delta\tau\to 0} \sum_n \Delta\tau \left[\frac{x(n\Delta\tau)}{\Delta\tau}\right] p_{\Delta\tau}(t - n\Delta\tau)$$





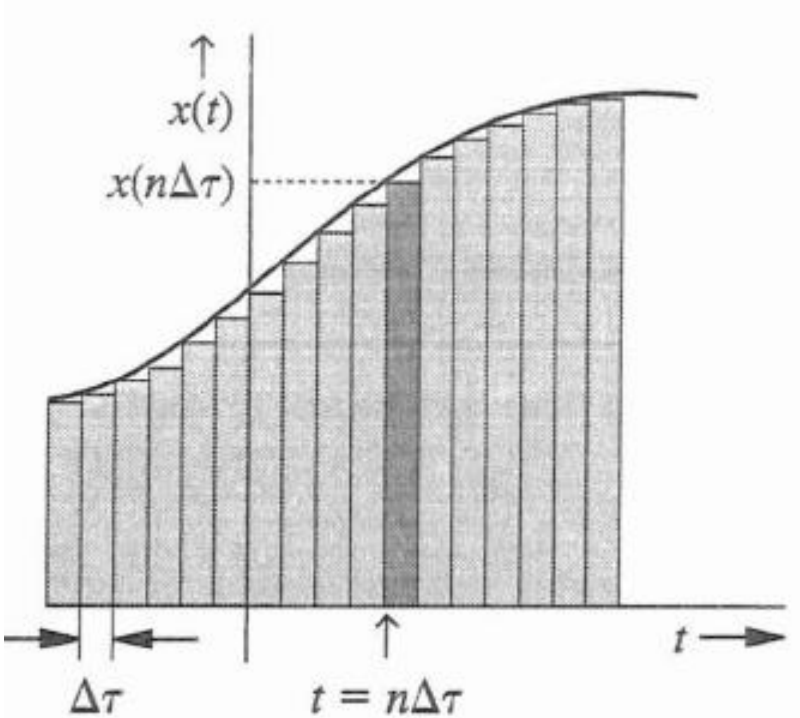
Zero-state response cont.

- The term $\frac{x(n\Delta\tau)}{\Delta\tau} p_{\Delta\tau}(t - n\Delta\tau)$ represents a pulse $p(t - n\Delta\tau)$ with height $\frac{x(n\Delta\tau)}{\Delta\tau}$.
- As $\Delta\tau \to 0$, the height of the pulse $\to \infty$ and the width of the pulse $\to 0$ but the area remains $x(n\Delta\tau)$ and $\frac{x(n\Delta\tau)}{\Delta\tau} p_{\Delta\tau}(t - n\Delta\tau) \to x(n\Delta\tau)\delta(t - n\Delta\tau)$.

Therefore,

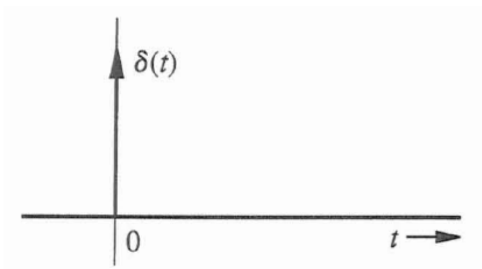
$$x(t) = \lim_{\Delta\tau \to 0} \sum_n \left[\frac{x(n\Delta\tau)}{\Delta\tau}\right] \Delta\tau p_{\Delta\tau}(t - n\Delta\tau)$$

$$x(t) = \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\Delta\tau\delta(t - n\Delta\tau)$$

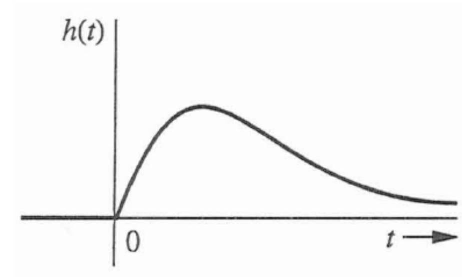


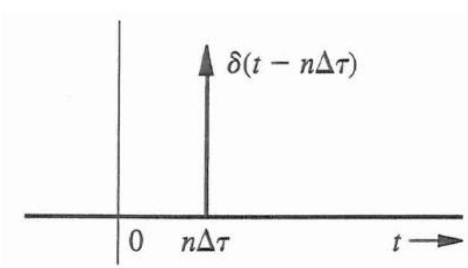
Zero-state response cont.

- Given the relationship $x(t) = \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\Delta\tau\delta(t - n\Delta\tau)$ and the fact that the system is linear, time-invariant, we have:

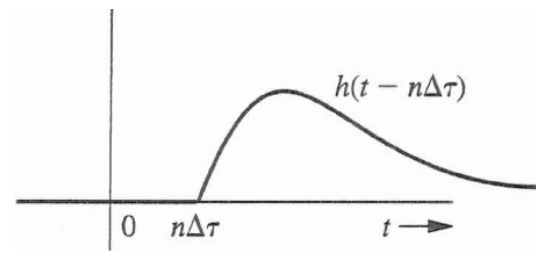


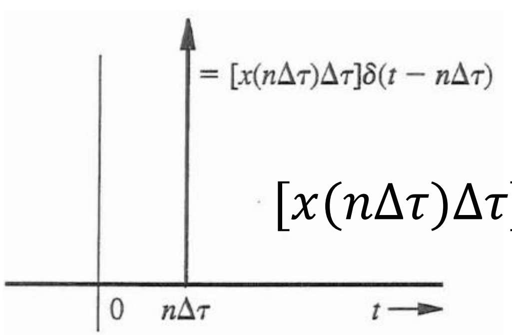
$$\delta(t) \Rightarrow h(t)$$



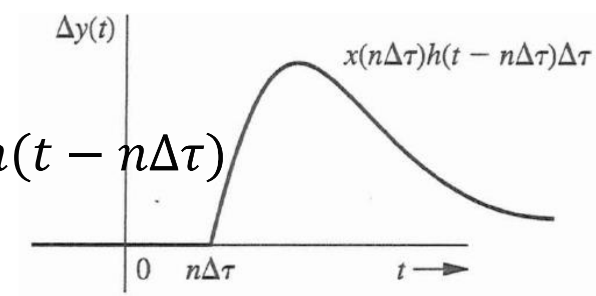


$$\delta(t - n\Delta\tau) \Rightarrow h(t - n\Delta\tau)$$





$$[x(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau) \Rightarrow [x(n\Delta\tau)\Delta\tau]h(t - n\Delta\tau)$$

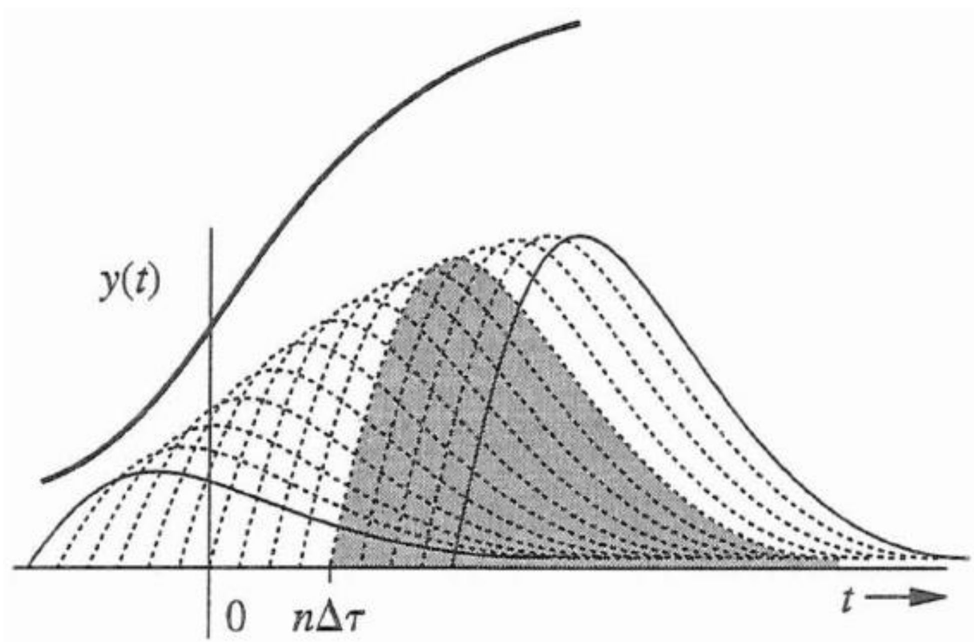


Zero-state response cont.

- Based on the previous analysis, the input-output relationship of an LTI system as a function of the impulse response is shown below.

$$\lim_{\Delta\tau\to 0}\sum_n x(n\Delta\tau)\Delta\tau\delta(t-n\Delta\tau) \Rightarrow \lim_{\Delta\tau\to 0}\sum_n x(n\Delta\tau)\Delta\tau h(t-n\Delta\tau)$$

$$\underbrace{\lim_{\Delta\tau\to 0}\sum_n x(n\Delta\tau)\delta(t-n\Delta\tau)\Delta\tau}_{x(t)} \Rightarrow \underbrace{\lim_{\Delta\tau\to 0}\sum_n x(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau}_{y(t)}$$

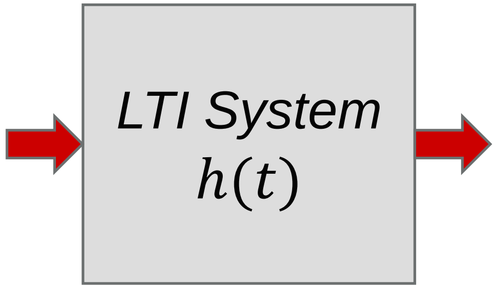


Zero-state response cont.

- Therefore,

$$y(t) = \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Knowing $h(t)$, we can determine the response $y(t)$ to any input $x(t)$.
- Observe the all-pervasive nature of the system's characteristic modes, which determines the impulse response of the system.

- $x(t)$  $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

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The convolution integral

- The previously derived integral equation occurs frequently in physical sciences, engineering and mathematics.
- It is given the name the **convolution integral**.
- The convolution integral (known simply as convolution) of two functions $x_1(t)$ and $x_2(t)$ is denoted symbolically as $x_1(t) * x_2(t)$.
- This is defined as

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

Convolution properties

- **Commutative property**: The order of operands does not matter.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau$$

Let $z = t - \tau$. In that case $\tau = t - z$ and $d\tau = -dz$ and $\tau \to \pm\infty \Rightarrow z \to \mp\infty$.

Therefore,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{\infty}^{-\infty} x_1(t-z)x_2(z)(-dz)$$

$$= -\int_{\infty}^{-\infty} x_1(t-z)x_2(z)dz = \int_{-\infty}^{\infty} x_1(t-z)x_2(z)dz = x_2(t) * x_1(t)$$

- **Associative property**

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

- **Distributive property**

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Convolution properties cont.

- **Shift property**

  Consider $x_1(t) * x_2\ (t) = c(t)$

  Then $x_1(t) * x_2\ (t-T) = x_1(t-T) * x_2\ (t) = c(t-T)$

  Furthermore,

$$x_1(t-T_1) * x_2\ (t-T_2) = c(t-T_1-T_2)$$

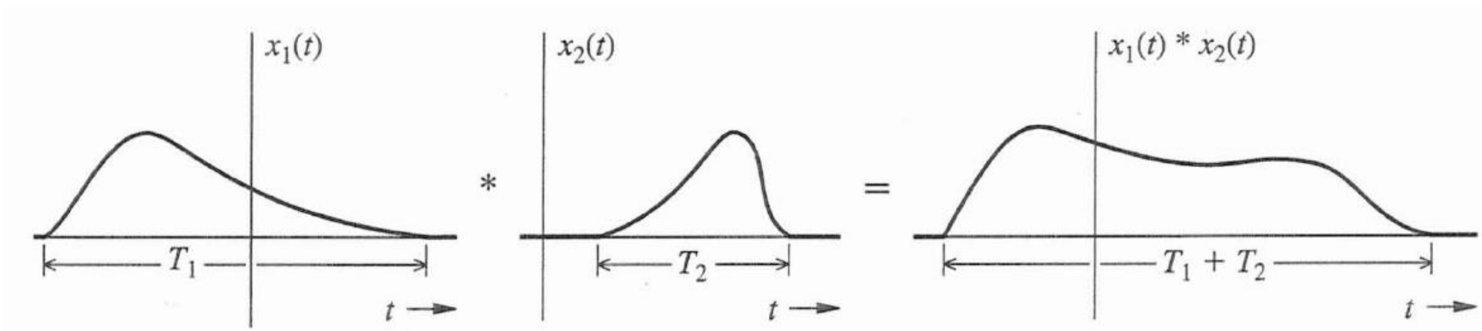
- **Convolution with an impulse**: The convolution of a function with the unit impulse function is the function itself. Therefore, the unit impulse function acts as an identity (neutral) element for convolution.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Convolution properties cont.

- **Width (duration) property**: Consider two functions $x_1(t)$ and $x_2(t)$ with durations T_1 and T_2 respectively.

  Then, the duration of the convolution function $x_1(t) * x_2\,(t)$ is $T_1 + T_2$.



- **Causality property**: If both system's impulse response $h(t)$ and input $x(t)$ are causal then:

$$y(t) = x(t) * h(t) = \begin{cases} \int_0^t x(\tau)h(t-\tau)d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}$$

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Example

- For an LTI system with unit impulse response $h(t) = e^{-2t}u(t)$ determine the response $y(t)$ for the input $x(t) = e^{-t}u(t)$.
- By definition we have:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

- Since $u(\tau) \neq 0$ for $\tau \geq 0$ and $u(t-\tau) \neq 0$ for $(t-\tau) \geq 0 \Rightarrow \tau \leq t$, we see that $y(t) \neq 0$ if $0 \leq \tau \leq t$ which also makes sense only if $t \geq 0$.
- Therefore,

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau = \int_{0}^{t} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau$$

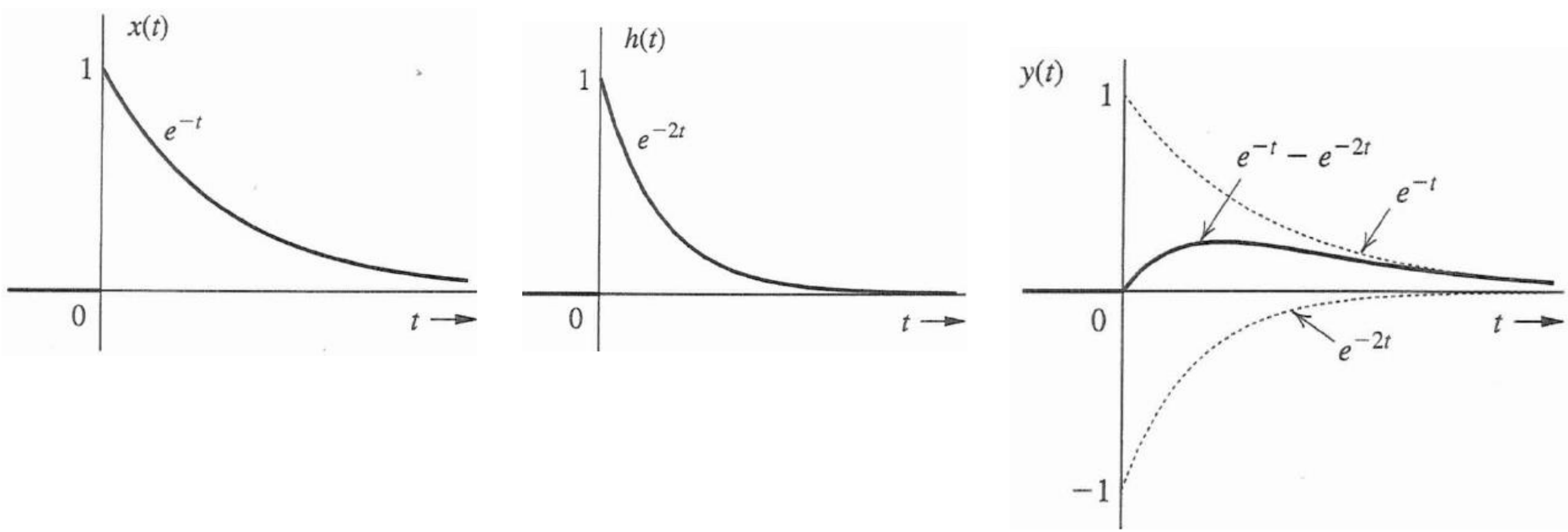
$$= \int_0^t e^{-\tau}e^{-2(t-\tau)}d\tau = e^{-2t}\int_0^t e^{-\tau}e^{2\tau}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-2t}\,(e^{t}-1)$$

$$= e^{-t} - e^{-2t},\ t \geq 0$$

- Therefore, $y(t) = (e^{-t} - e^{-2t})u(t)$.

Example cont.

- $h(t) = e^{-2t}u(t)$
- $x(t) = e^{-t}u(t)$
- $y(t) = (e^{-t} - e^{-2t})u(t)$



Relation to other courses

- Convolution has been introduced last year in the Signals and Communications course. We will emphasize into convolution and its physical implication in the next lecture.
- Zero-state response (as determined through the convolution operation) is very important, and is intimately related to the zero-input response and the characteristic modes of the system.
- All these are relevant to the 2nd year Control course.
- You will also come across convolution again in your 2nd year Communications course and third year DSP course.