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Signals and Systems

Lecture 14

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Introduction. Time sampling theorem resume.

- We wish to perform spectral analysis using digital computers.
- Therefore, we must somehow sample the Fourier transform of the signal.
- In this Lecture we will compute a discrete version of the Fourier transform on the sampled, finite-duration signal. The transform that we will derive is known as Discrete Fourier Transform (DFT).
- The goal is to understand how DFT is related to the original Fourier transform.
- We showed that a signal bandlimited to $B Hz$ can be reconstructed from signal samples if they are obtained at a rate of $f_s > 2B$ samples per second.
- Note that the signal spectrum exists over the frequency range (in Hz) from $-B$ to B.
- The interval $2B$ is called **spectral width**.

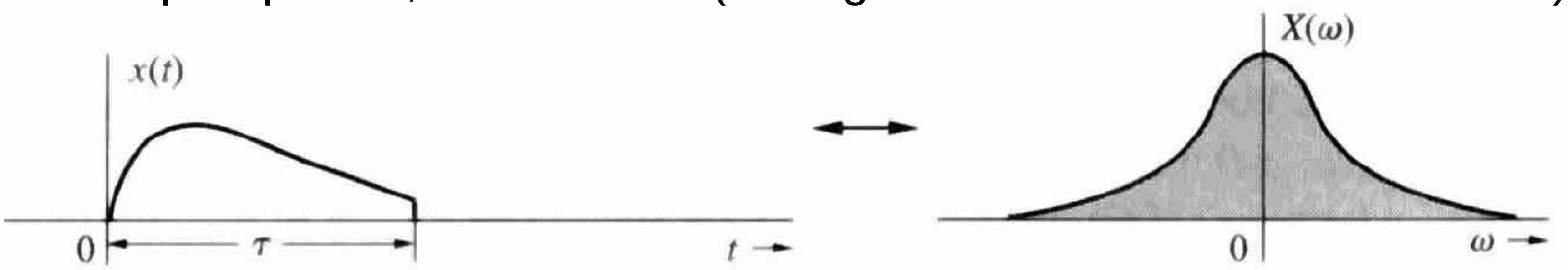
  Note the difference between spectral width ($2B$) and **bandwidth** (B).
- In time sampling theorem: $f_s > 2B$ or $f_s >$(**spectral width**).

Time sampling theorem has a dual: Spectral sampling theorem

- Consider a time-limited signal $x(t)$ with a spectrum $X(\omega)$.
- In general, a time-limited signal is 0 for $t < T_1$ and $t > T_2$. The duration of the signal is $\tau = T_2 - T_1$. Below we assume that $T_1 = 0$.
- Recall that $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\tau} x(t)e^{-j\omega t}dt$.
- The Fourier transform $X(\omega)$ is assumed real for simplicity.

Spectral sampling theorem

The spectrum $X(\omega)$ of a signal $x(t)$, time-limited to a duration of τ seconds, can be reconstructed from the samples of $X(\omega)$ taken at a rate R samples per Hz, where $R > \tau$ (the signal width or duration in seconds).



Spectral sampling theorem

- We now construct the periodic signal $x_{T_0}(t)$. This is a periodic extension of $x(t)$ with period $T_0 > \tau$.
- This periodic signal can be expressed using Fourier series.

$$x_{T_0}(t) = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$$

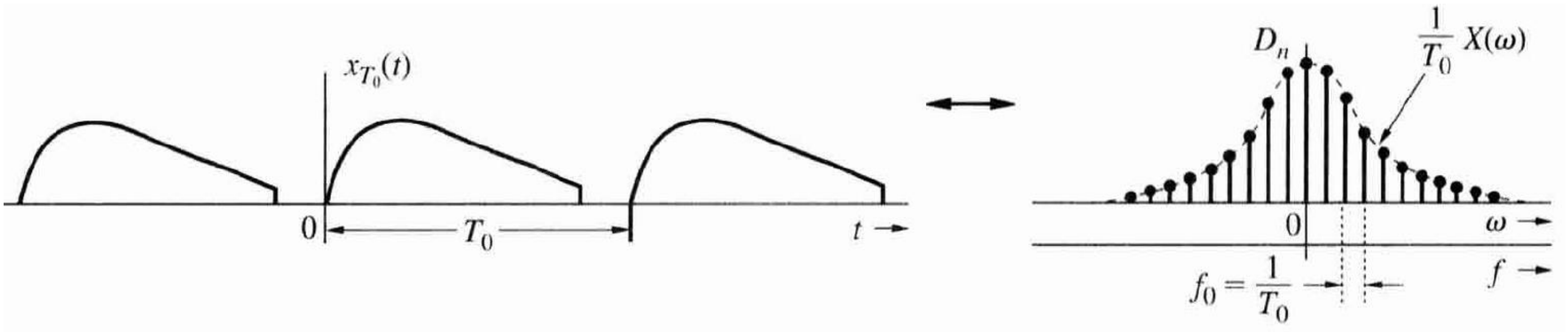
$$D_n = \frac{1}{T_0}\int_0^{T_0} x(t)\ e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_0^{\tau} x(t)\ e^{-jn\omega_0 t}dt = \frac{1}{T_0}X(n\omega_0)$$

$$\Rightarrow x(t) = \sum_{n=-\infty}^{n=\infty} \frac{1}{T_0}\ X(n\omega_0)e^{jn\omega_0 t},\ \omega_0 = \frac{2\pi}{T_0}$$

- The result indicates that the coefficients of the Fourier series for $x_{T_0}(t)$ are the values of $X(\omega)$ taken at integer multiples of ω_0 and scaled by $\frac{1}{T_0}$.
- We call "**spectrum of a periodic signal**" the weights of the exponential terms in its Fourier series representation.
- The above implies that the "spectrum" of the periodic signal $x_{T_0}(t)$ is the sampled version of spectrum $X(\omega)$.

Spectral sampling theorem cont.

- The spectrum of the periodic signal $x_{T_0}(t)$ is the sampled version of spectrum $X(\omega)$ (see figure below).



  - If successive cycles of $x_{T_0}(t)$ do not overlap, $x(t)$ can be recovered from $x_{T_0}(t)$.
  - If we know $x(t)$ we can find $X(\omega)$.
  - **The above imply that $X(\omega)$ can be reconstructed from its samples.**

- These samples are separated by the so called fundamental frequency $f_0 = \frac{1}{T_0} Hz$ or $\omega_0 = 2\pi f_0 \text{rads/s}$ of the periodic signal $x_{T_0}(t)$.
- Therefore, the condition for recovery is $T_0 > \tau \Rightarrow f_0 = \frac{1}{T_0} < \frac{1}{\tau} Hz$.

Spectral interpolation formula

- To reconstruct the spectrum $X(\omega)$ from the samples of $X(\omega)$, the samples should be taken at frequency intervals $f_0 < \frac{1}{\tau} Hz$. If the sampling rate is R **frequency samples/Hz** we have:

$$R = \frac{1}{f_0} > \tau \text{ frequency samples}/Hz$$

- In the previous lecture we proved that the continuous version of a signal can be recovered from its sampled version through the so called **signal interpolation formula**:

$$x(t) = \sum_n x(nT_s)h(t - nT_s) = \sum_n x(nT_s)\text{sinc}\left(\frac{\pi t}{T_s} - n\pi\right)$$

- We use the dual of the approach employed to derive the signal interpolation formula shown above, to obtain the **spectral interpolation formula** as follows. We assume that $x(t)$ is time-limited to τ and centred at T_c. We can prove that:

$$X(\omega) = \sum_{n=-\infty} X(n\omega_0)\text{sinc}\left(\frac{\omega T_0}{2} - n\pi\right) e^{-j(\omega - n\omega_0)T_c},\ \omega_0 = \frac{2\pi}{T_0},\ T_0 > \tau$$

Spectral interpolation formula: Proof.

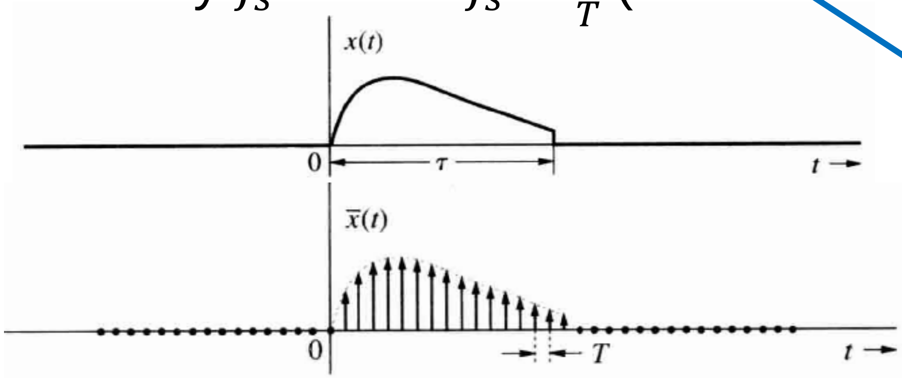
- We know that $x_{T_0}(t) = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$
- Therefore, $\mathcal{F}\{x_{T_0}(t)\} = 2\pi \sum_{n=-\infty}^{n=\infty} D_n\, \delta(\omega - n\omega_0)$
  [It is easier to prove that $\mathcal{F}^{-1}\{2\pi \sum_{n=-\infty}^{n=\infty} D_n\, \delta(\omega - n\omega_0)\} = x_{T_0}(t)$]
- We can write $x(t) = x_{T_0}(t) \cdot \mathrm{rect}\left(\frac{t-\boldsymbol{T_c}}{T_0}\right)$ (1)
  [We were given that $x(t)$ is centred at $\boldsymbol{T_c}$.]
- We know that $\mathcal{F}\left\{\mathrm{rect}\left(\frac{t}{T_0}\right)\right\} = T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right)$.
- Therefore, $\mathcal{F}\left\{\mathrm{rect}\left(\frac{t-T_c}{T_0}\right)\right\} = T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right) e^{-j\omega T_c}$.
- From (1) we see that $X(\omega) = \frac{1}{2\pi} \mathcal{F}\{x_{T_0}(t)\} * \mathcal{F}\left\{\mathrm{rect}\left(\frac{t-T_c}{T_0}\right)\right\}$
- $X(\omega) = \frac{1}{2\pi} 2\pi [\sum_{n=-\infty}^{n=\infty} D_n\, \delta(\omega - n\omega_0)] * T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right) e^{-j\omega T_c}$

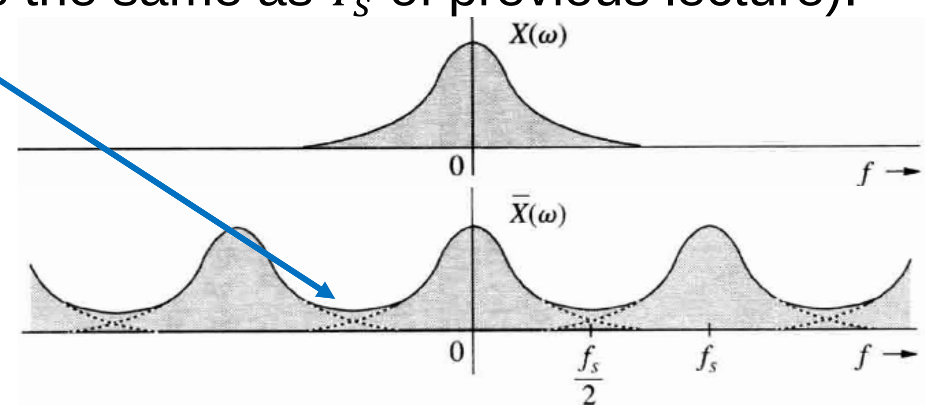
$$X(\omega) = \sum_{n=-\infty} D_n T_0 \mathrm{sinc}\left[\frac{(\omega - n\omega_0)T_0}{2}\right] e^{-j(\omega - n\omega_0)T_c},\ \omega_0 = \frac{2\pi}{T_0},\ T_0 > \tau$$

$$X(\omega) = \sum_{n=-\infty} X(n\omega_0) \mathrm{sinc}\left(\frac{\omega T_0}{2} - n\pi\right) e^{-j(\omega - n\omega_0)T_c}$$

Discrete Fourier Transform DFT

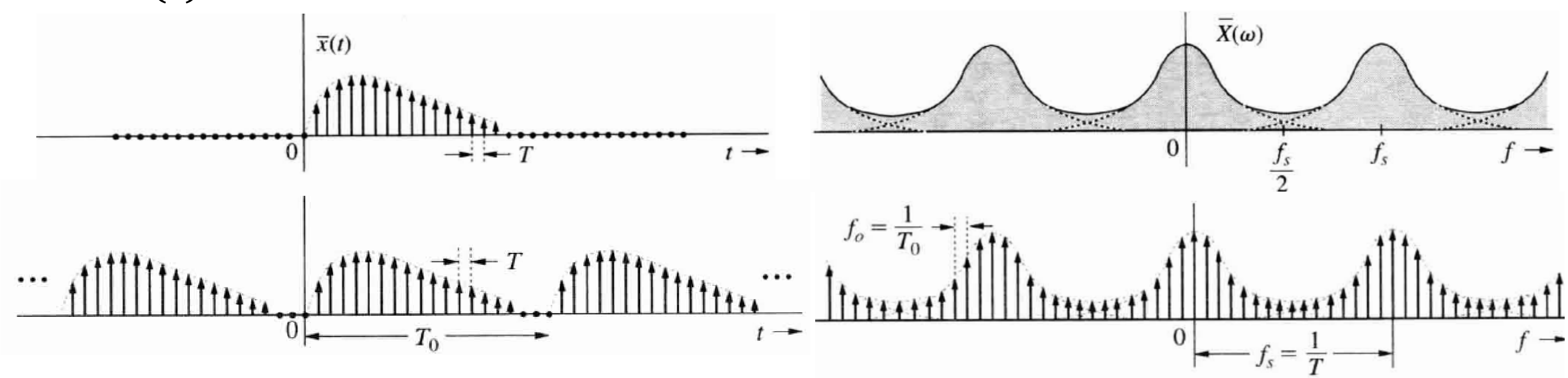
- The numerical computation of the Fourier transform requires samples of $x(t)$ since computers can work only with discrete values.
- Furthermore, the Fourier transform can only be computed at some discrete values of ω.
- **The goal of what follows is to relate the samples of $X(\omega)$ with the samples of $x(t)$.**
- Consider a time-limited signal $x(t)$. Its spectrum $X(\omega)$ will not be bandlimited **(try to think why)**. **In other words aliasing after sampling cannot be avoided.**
- The spectrum $\bar{X}(\omega)$ of the sampled signal $\bar{x}(t)$ consist of $X(\omega)$ repeating every $f_s Hz$ with $f_s = \frac{1}{T}$ (note that T is the same as T_s of previous lecture).





Discrete Fourier Transform DFT cont.

- Suppose now that the sampled signal $\bar{x}(t)$ is repeated periodically every T_0 seconds.
- According to the spectral sampling theorem, this operation results in sampling the spectrum at a rate of T_0 samples/Hz. This means that the samples are spaced at $f_0 = \frac{1}{T_0} Hz$.
- **Therefore, when a signal is sampled and periodically repeated, its spectrum is also sampled and periodically repeated.**
- The goal of what follows is to relate the samples of $X(\omega)$ with the samples of $x(t)$.



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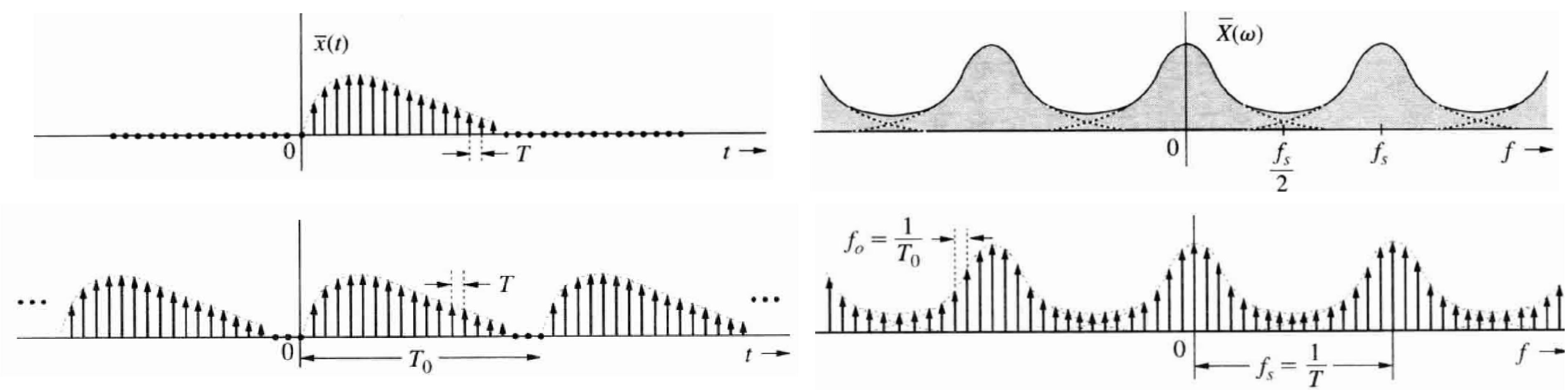
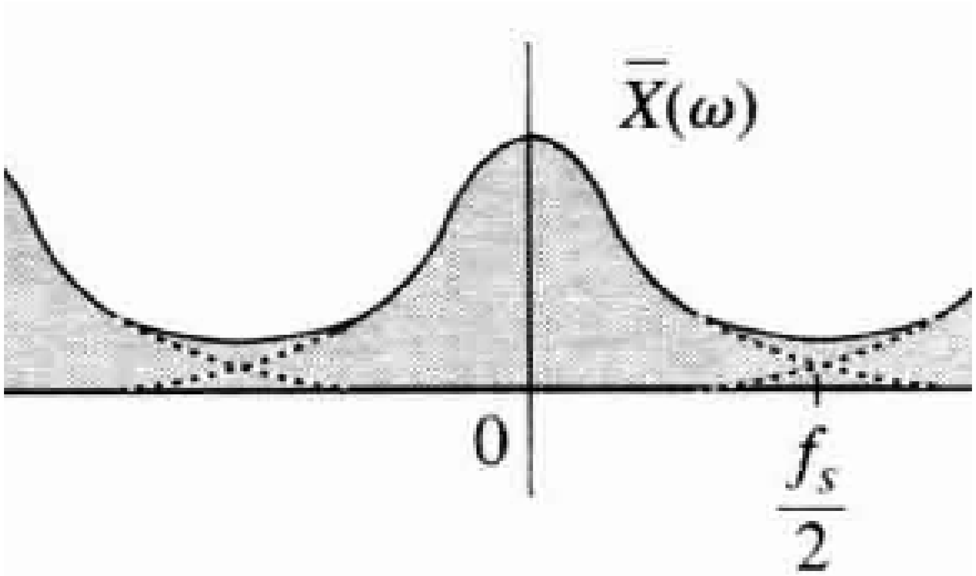
Discrete Fourier Transform DFT cont.

- The number of samples of the discrete signal in one period T_0 is $N_0 = \frac{T_0}{T}$ (figure below left).
- The number of samples of the discrete spectrum in one period is $N_0' = \frac{f_s}{f_0}$.
- We see that $N_0' = \frac{f_s}{f_0} = \frac{\frac{1}{T}}{\frac{1}{T_0}} = \frac{T_0}{T} = N_0$.
- **This is an interesting observation: the number of samples in a period of time is identical to the number of samples in a period of frequency.**



Aliasing and leakage effects

- Since $X(\omega)$ is not bandlimited, we will get some aliasing effect:



- Furthermore, if $x(t)$ is not time limited, we need to truncate $x(t)$ with a window function. This leads to leakage effect as discussed in previous lecture (sampling).

Formal definition of DFT

- If $x(nT)$ and $X(r\omega_0)$ are the n^{th} and r^{th} samples of $x(t)$ and $X(\omega)$ respectively, we define:

$$x_n = Tx(nT) = \frac{T_0}{N_0}x(nT)$$

$$X_r = X(r\omega_0),\ \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- It can be shown that x_n and X_r are related by the following equations:

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jnr\Omega_0} \qquad (1)$$

$$x_n = \frac{1}{N_0}\sum_{r=0}^{N_0-1} X_r e^{jrn\Omega_0},\ \Omega_0 = \omega_0 T = \frac{2\pi}{N_0} \qquad (2)$$

- The equations (1) and (2) above are the direct and inverse Discrete Fourier Transforms respectively, known as DFT and IDFT.
- In the above equations, the summation is performed from 0 to $N_0 - 1$. It can be shown that the summation can be performed over any successive N_0 values of n or r.

Proof of DFT relationships

- For the sampled signal we have:

$$\overline{x(t)} = \sum_{n=0}^{N_0-1} x(nT)\delta(t-nT).$$

- Since $\delta(t-nT) \Leftrightarrow e^{-jn\omega T}$

$$\overline{X(\omega)} = \sum_{n=0}^{N_0-1} x(nT)e^{-jn\omega T}$$

- For $|\omega| \leq \frac{\omega_s}{2}$, $\overline{X(\omega)}$ the Fourier transform of $\overline{x(t)}$ is $\frac{X(\omega)}{T}$, i.e.,

$$X(\omega) = T\overline{X(\omega)} = T\sum_{n=0}^{N_0-1} x(nT)e^{-jn\omega T},\ |\omega| \leq \frac{\omega_s}{2}$$

$$X_r = X(r\omega_0) = T\sum_{n=0}^{N_0-1} x(nT)e^{-jnr\omega_0 T}$$

- If we let $\omega_0 T = \Omega_0$ then $\Omega_0 = \omega_0 T = 2\pi f_0 T = \frac{2\pi}{N_0}$ and also $Tx(nT) = x_n$.
- Therefore, $X_r = \sum_{n=0}^{N_0-1} x_n e^{-jnr\Omega_0}$

Proof of DFT relationships cont.

- To prove the inverse relationship write:

$$\sum_{r=0}^{N_0-1} X_r e^{jrm\Omega_0} = \sum_{r=0}^{N_0-1}\left[\sum_{n=0}^{N_0-1} x_n e^{-jnr\Omega_0}\right] e^{jrm\Omega_0} \Rightarrow$$
$$\sum_{r=0}^{N_0-1} X_r e^{jrm\Omega_0} = \sum_{n=0}^{N_0-1} x_n\left[\sum_{r=0}^{N_0-1} e^{-jr(n-m)\Omega_0}\right]$$

- $\sum_{r=0}^{N_0-1} e^{-jr(n-m)\Omega_0} = \sum_{r=0}^{N_0-1} e^{-jr(n-m)\frac{2\pi}{N_0}} = \begin{cases} N_0 & n-m = kN_0, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

- Since $0 \le m, n \le N_0 - 1$ the only multiple of N_0 that the term $(n-m)$ can be is 0. Therefore:

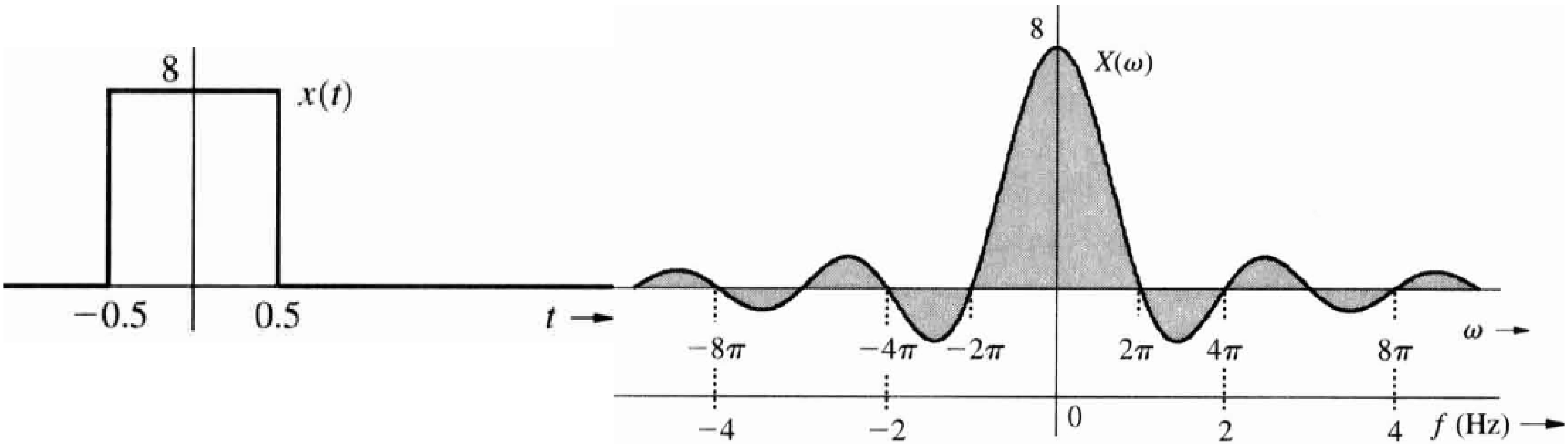
$$\sum_{r=0}^{N_0-1} e^{-jr(n-m)\frac{2\pi}{N_0}} = \begin{cases} N_0 & n-m = 0 \Rightarrow n = m \\ 0 & \text{otherwise} \end{cases}$$

- Therefore,

$$x_m = \frac{1}{N_0}\sum_{r=0}^{N_0-1} X_r e^{jrm\Omega_0},\ \Omega_0 = \frac{2\pi}{N_0}$$

Example

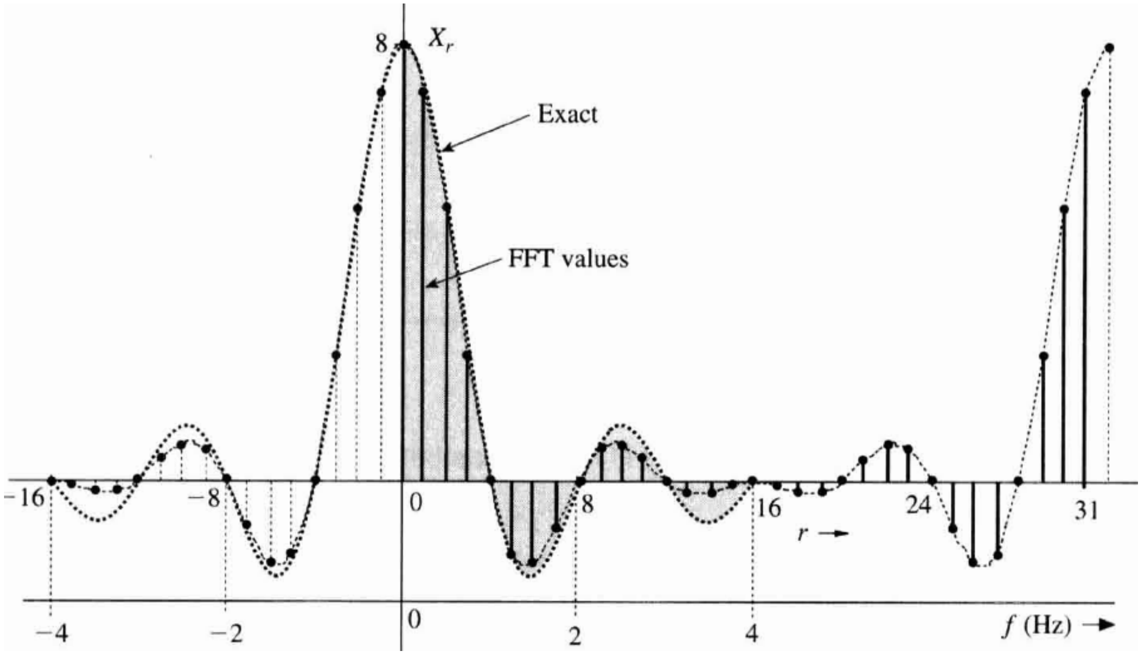
- Use DFT to compute the Fourier transform of $8\text{rect}(t)$ (Lathi page 808.)



- The essential bandwidth B (calculated by finding where the amplitude response drops to 1% of its peak value) is well above $16Hz$. However, we select $B = 4Hz$:
  - To observe the effects of aliasing.
  - In order not to end up with a huge number of samples in time.

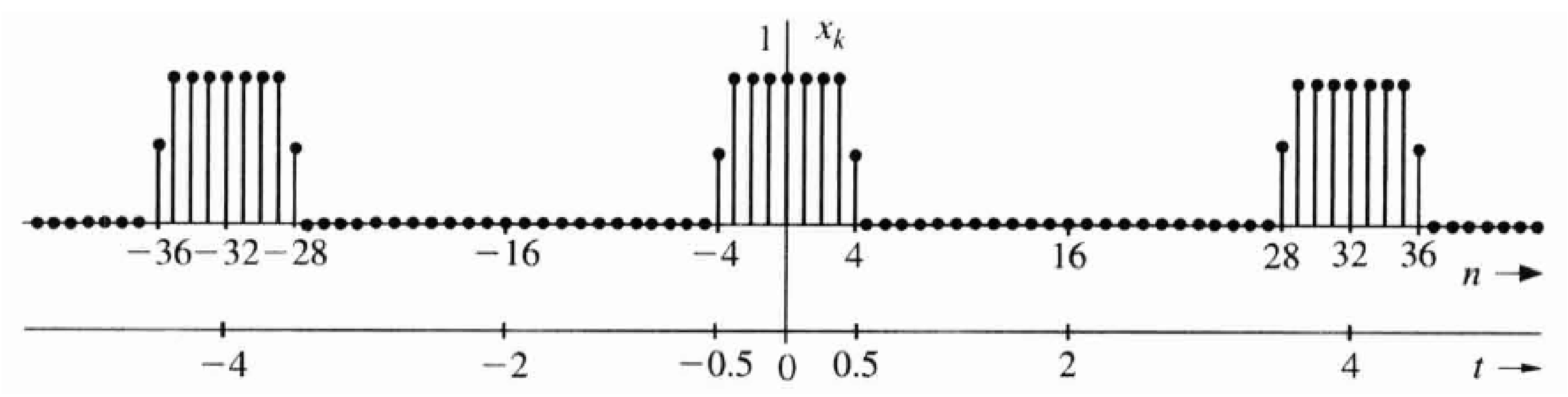
Example cont.

- $B = 4Hz,\ f_s = 2B = 8Hz,\ T = \frac{1}{f_s} = \frac{1}{8}s.$
- For the frequency resolution we choose $f_0 = \frac{1}{4}Hz$. This choice gives us 4 samples in each lobe of $X(\omega)$ and $T_0 = \frac{1}{f_0} = 4s$.
- Recall that T_0 is the period of the periodically extended original signal $x(t)$.

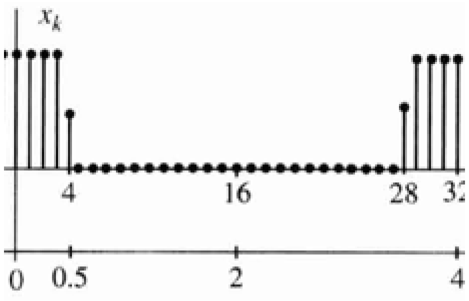


Example cont.

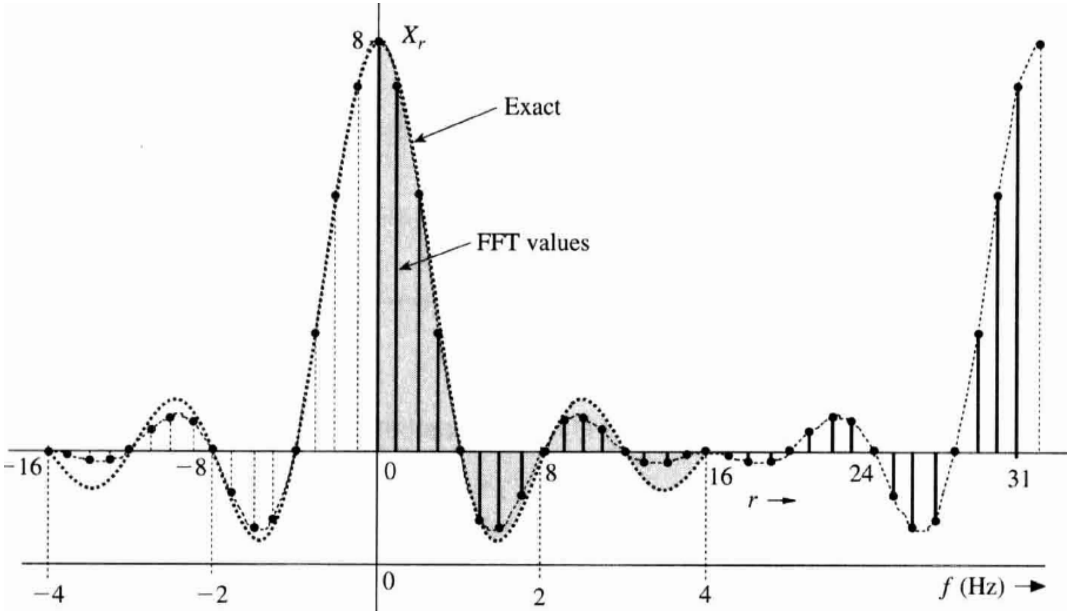
- $N_0 = \frac{T_0}{T} = \frac{4}{1/8} = 32$. Therefore, we must repeat $x(t)$ every $4s$ and take samples every $\frac{1}{8}s$. This yields 32 samples in a period.
- $x_n = Tx(nT) = \frac{1}{8}x(\frac{n}{8})$ with $x(t) = 8\text{rect}(t)$.
- The DFT of the signal x_n is obtained by taking any full period of x_n (i.e., N_0 samples) and not necessarily N_0 over the interval $(0, T_0)$ as we assumed in the theoretical analysis of DFT.



Example cont.



- $x_n = \begin{cases} 1 & 0 \le n \le 3 \quad \text{and} \quad 29 \le n \le 31 \\ 0 & 5 \le n \le 27 \\ 0.5 & n = 4{,}28 \end{cases}$
- $\Omega_0 = \frac{2\pi}{32} = \frac{\pi}{16}$
- $X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n} = \sum_{n=0}^{31} x_n e^{-jr(\pi/16)n}$. See figure below.



Example cont.

- Observe that X_r is periodic.
- The dotted curve depicts the Fourier transform of $x(t) = 8\text{rect}(t)$.
- The aliasing error is quite visible when we use a single graph to compare the superimposed plots. The error increases rapidly with r.