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Signals and Systems

Revision Lecture 2

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Sampling

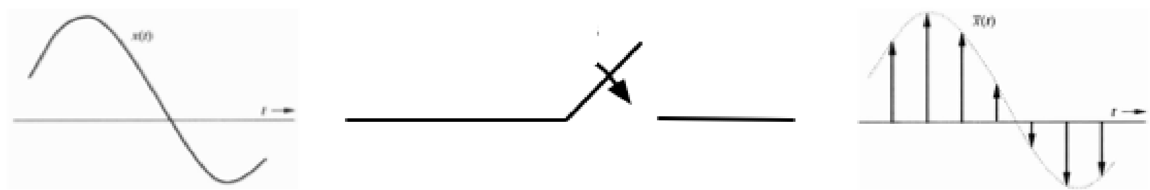
In this revision lecture we will focus on:

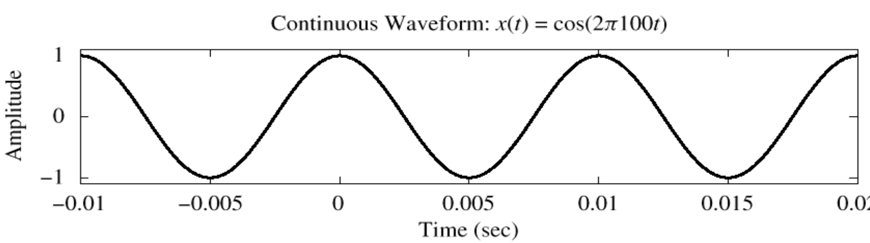
- Sampling.
- The $z-$ transform.

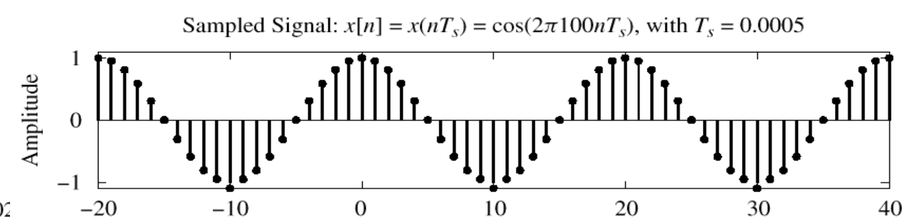
Sampling

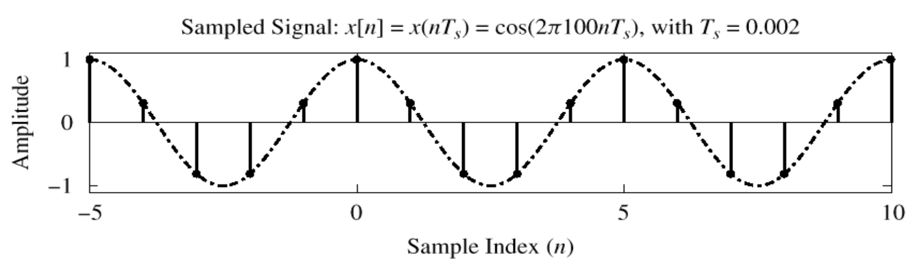
- The sampling process converts continuous signals $x(t)$ into numbers $x[n]$.
- A sample is kept every T_s units of time. This process is called **uniform sampling** and $x[n] = x(nT_s)$.

$x(t)$ $\qquad\qquad$ $T_s = 1/f_s$ $\qquad$ $x[n]$





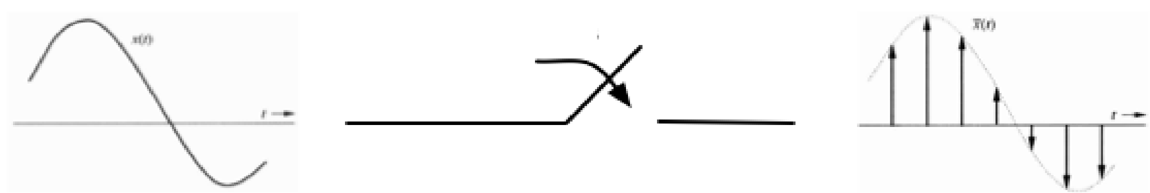




Sampling theorem

- **<u>Sampling theorem</u>** is the bridge between continuous-time and discrete-time signals.
- It states how often we must sample in order not to loose any information.

$x(t)$ $\qquad\qquad$ $T_s = 1/f_s$ $\qquad\qquad$ $x[n]$



Sampling theorem

A continuous-time lowpass signal $x(t)$ with frequencies no higher than $f_{max}\,Hz$ can be perfectly reconstructed from samples taken every T_s units of time, $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

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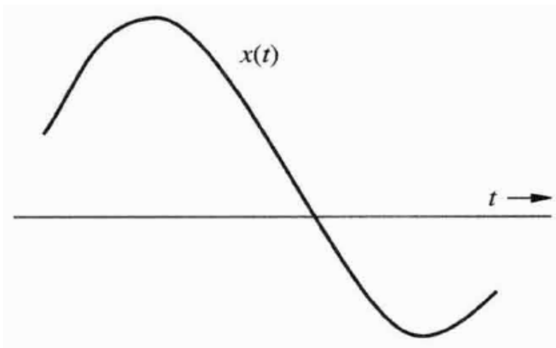
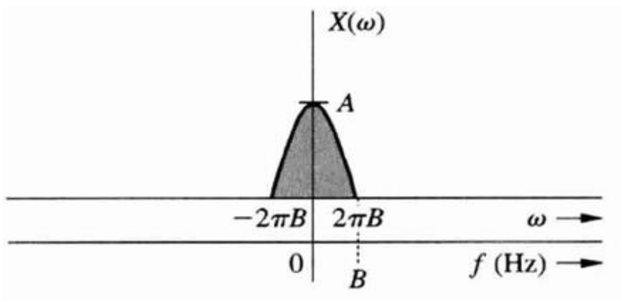
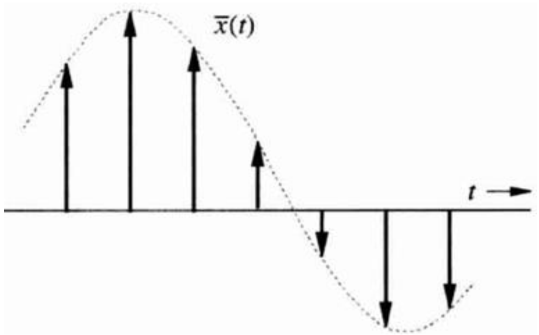
Recall: spectrum of sampled signal

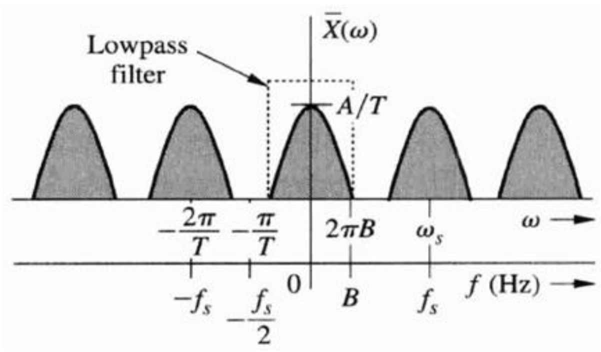
- Consider a signal, bandlimited to BHz, with Fourier transform $X(\omega)$ (depicted real for convenience).





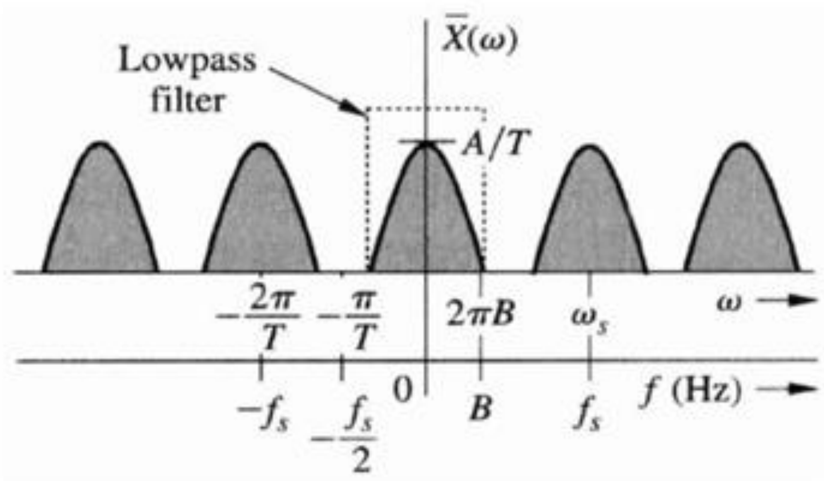
- The sampled signal has the following spectrum.





Reconstruction of the original signal

- The gap between two adjacent spectral repetitions is $(f_s - 2B) Hz$.
- **In theory**, in order to reconstruct the original signal $x(t)$, we can use an ideal lowpass filter on the sampled spectrum which has a bandwidth of any value between B and $(f_s - BHz)$.

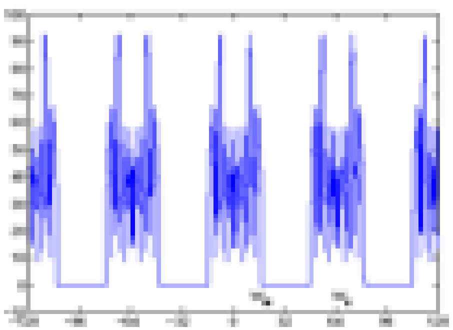


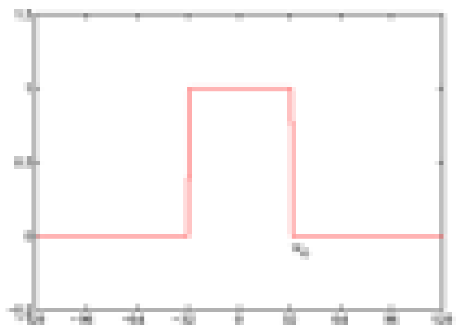
- Reconstruction process is possible only if the shaded parts do not overlap. This means that f_s must be greater that twice B.
- We can also visually verify the sampling theorem in the above figure.

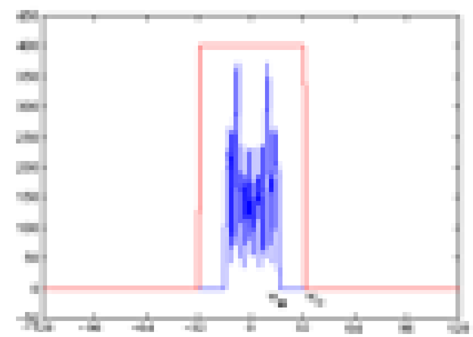
Reconstruction generic example

- **Frequency domain**

The signal $x(nT_s) =< x(t), \delta(t - nT_s) >$ has a spectrum $X_s(\omega)$ which is multiplied with a rectangular pulse in frequency domain.







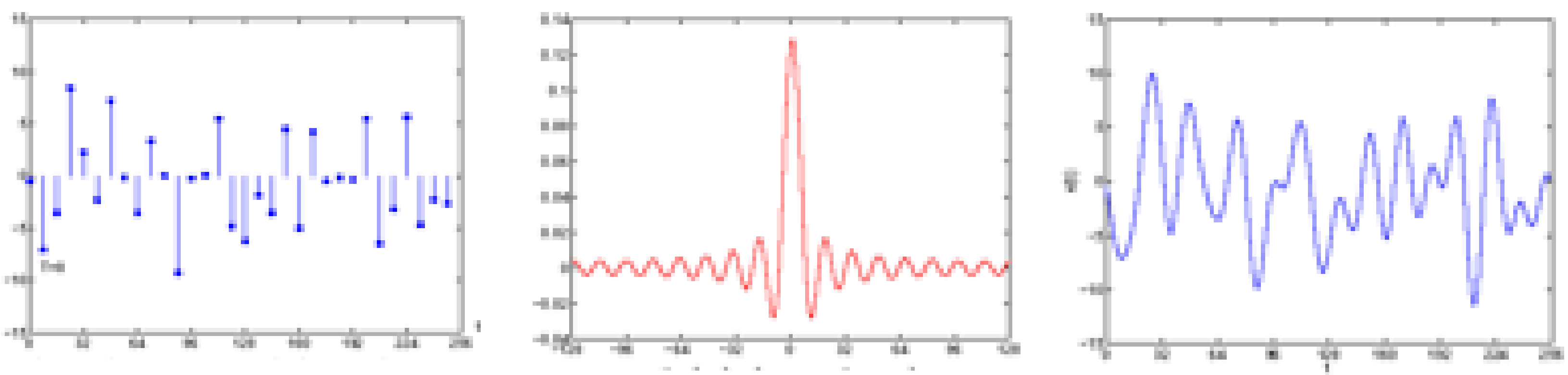
Reconstruction generic example cont.

- **Time domain**

  The signal $x(nT_s) = < x(t), \delta(t - nT_s) >$ is convolved with a sinc function, which is the time domain version of a rectangular pulse in frequency domain centred at the origin.

  $$< x(t), \delta(t - nT_s) > * \ \mathrm{sinc}(at) = \sum_n x[n] \, \mathrm{sinc}[a(t - nT_s)]$$

  [Recall that convolution with a shifted Delta function, shifts the original function at the location of the Delta function].



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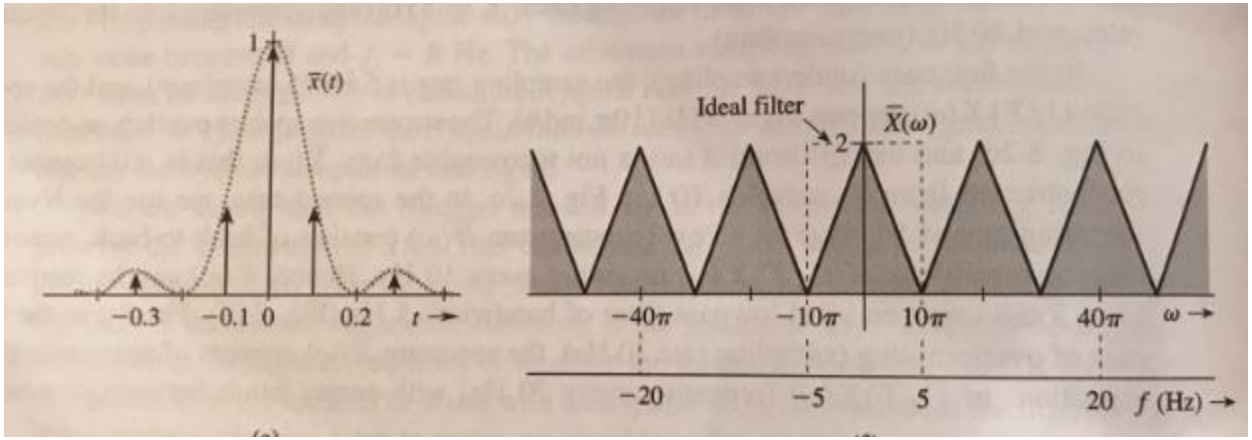
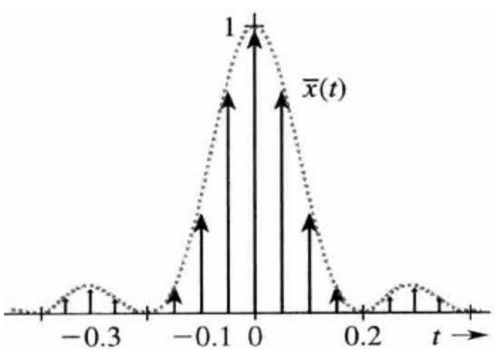
Recall Nyquist sampling: Just about the correct sampling rate

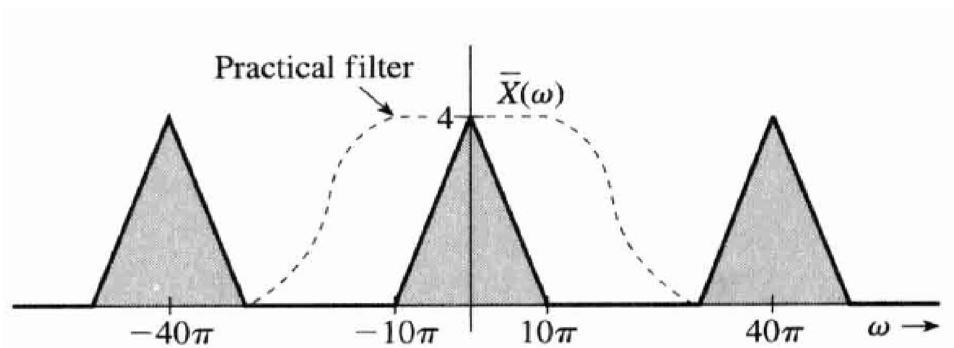
- In that case we use the Nyquist sampling rate of $10Hz = 2BHz$.
- The spectrum $\bar{X}(\omega)$ consists of back-to-back, non-overlapping repetitions of $\frac{1}{T_s}X(\omega)$ repeating every $10Hz$.
- In order to recover $X(\omega)$ from $\bar{X}(\omega)$ we must use an ideal lowpass filter of bandwidth $5Hz$. This is shown in the right figure below with the dotted line.



Recall oversampling: What happens if we sample too quickly?

- Sampling at higher than the Nyquist rate (in this case $20Hz$) makes reconstruction easier.
- The spectrum $\bar{X}(\omega)$ consists of non-overlapping repetitions of $\frac{1}{T_s}X(\omega)$, repeating every $20Hz$ with empty bands between successive cycles.
- In order to recover $X(\omega)$ from $\bar{X}(\omega)$ we can use a practical lowpass filter and not necessarily an ideal one. This is shown in the right figure below with the dotted line.





- The filter we use for reconstruction must have gain T_s and bandwidth of any value between B and $(f_s - B)Hz$.

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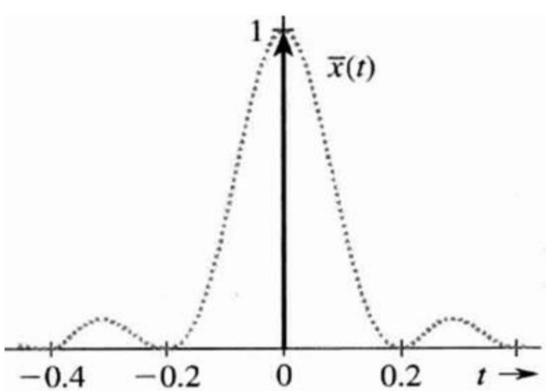
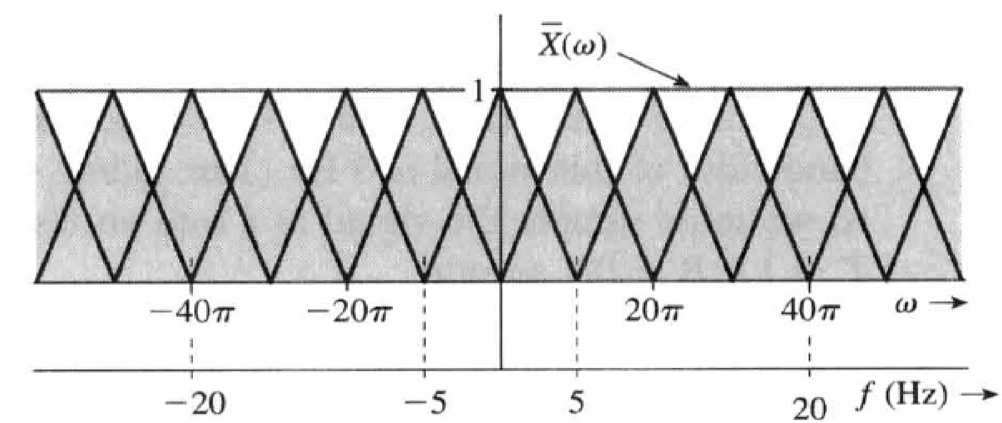
Recall undersampling: What happens if we sample too slowly?

- Sampling at lower than the Nyquist rate (in this case $5Hz$) makes reconstruction impossible.
- The spectrum $\bar{X}(\omega)$ consists of overlapping repetitions of $\frac{1}{T_S}X(\omega)$ repeating every $5Hz$.
- $X(\omega)$ is not recoverable from $\bar{X}(\omega)$.
- Sampling below the Nyquist rate corrupts the signal. This type of distortion is called **<u>aliasing</u>**.





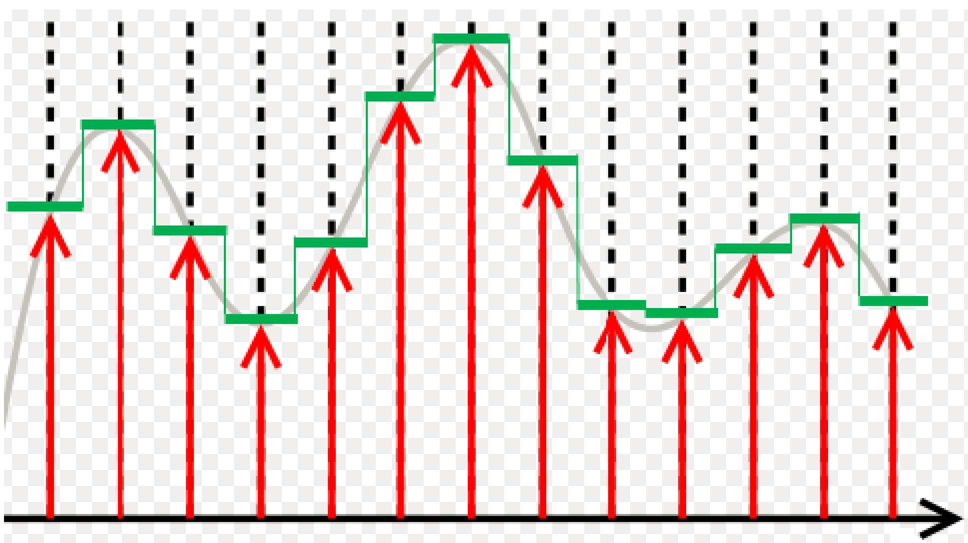
Problem: Reconstruction of the continuous time using the sample and hold operation

- As already mentioned, in order to reconstruct the original signal $x(t)$ in theory, we can use an ideal lowpass filter on the sampled spectrum.
- In practice, there are various circuits/devices which facilitate reconstruction of the original signal from its sampled version.
- Consider a continuous-time, band-limited signal $x(t)$, limited to bandwidth $|\omega| \leq 2\pi \times B$ rad/sec. We sample $x(t)$ uniformly with sampling frequency f_s to obtain the discrete-time signal $x[n]$.
- A possible technique to reconstruct the continuous-time signal from its samples is the so called **sample and hold** circuit. This is an analogue device which samples the value of a continuously varying analogue signal and outputs the following signal $x_{DA}(t)$.

$$x_{DA}(t) = \begin{cases} x[n] & \frac{n}{f_s} - \frac{T_s}{2} < t < \frac{n}{f_s} + \frac{T_s}{2} \\ x[n]/2 & t = \frac{n}{f_s} \pm \frac{T_s}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x[n] & nT_s - \frac{T_s}{2} < t < nT_s + \frac{T_s}{2} \\ x[n]/2 & t = nT_s \pm \frac{T_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Reconstruction of the continuous time using the **sample and hold** operation cont.

- The figure below facilitates the understanding of **sample and hold** operation.
- The grey continuous curve depicts the continuous signal $x(t)$.
- The red arrows depict the locations of the discrete (sampled) signal $x[n]$.
- The green continuous curve depicts the signal $x_{DA}(t)$.



Problem: Reconstruction of the continuous time using the sample and hold operation cont.

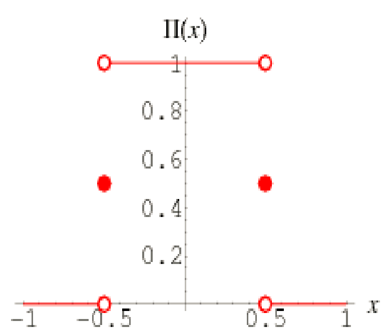
In the **sample and hold** operation the signal is written as:

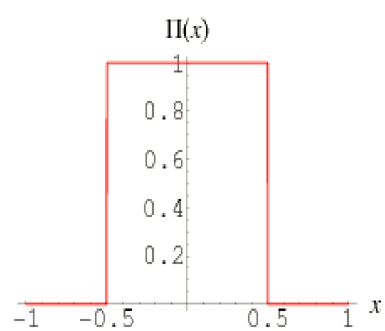
$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t - \frac{n}{f_s}}{T_s}\right)$$

with $\Pi(t)$ the very well known **unit gate** or **rectangle** function:

$$\Pi(t) = \mathrm{rect}(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Note that the values at $t = \pm 0.5$ do not have any impact in the Fourier Transform of $\Pi(t)$ and alternative definitions of $\Pi(t)$ have $\mathrm{rect}(\pm 0.5)$ to be 0, 1 or undefined.





Problem: Reconstruction of the continuous time using the sample and hold operation cont.

Taking into consideration that:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t-\frac{n}{f_s}}{T_s}\right)$$

it is straightforward to show that:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\left[\delta(t-nT_s) * \Pi\left(\frac{t}{T_s}\right)\right]$$

$$x_{DA}(t) = (\sum_{n=-\infty}^{\infty} [x(nT_s)\delta(t-nT_s)]) * \Pi\left(\frac{t}{T_s}\right)$$

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Problem: Reconstruction of the continuous time using the sample and hold operation cont.

$$x_{DA}(t) = \left(\sum_{n=-\infty}^{\infty}[x(nT_s)\delta(t-nT_s)]\right) * \Pi\left(\frac{t}{T_s}\right)$$

- The Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$ is $\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})$.

- The Fourier transform of the function $\Pi\left(\frac{t}{T_s}\right)$ can be easily found using the definition of the Fourier transform.

- The Fourier transform of $x_{DA}(t)$ is the product of the two Fourier transforms described above (remember that convolution in time becomes multiplication in frequency).

Problem: Reconstruction of the continuous time using the sample and hold operation cont.

$$x_{DA}(t) = \left(\sum_{n=-\infty}^{\infty}[x(nT_s)\delta(t-nT_s)]\right) * \Pi\left(\frac{t}{T_s}\right)$$

- We see that

$$X_{DA}(\omega) = \left(\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})\right) \cdot \left(T_s \mathrm{sinc}\left(\frac{\omega T_s}{2}\right)\right)$$

$$= \left(\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})\right) \cdot \mathrm{sinc}\left(\frac{\omega T_s}{2}\right)$$

- In order to recover the original spectrum $X(\omega)$ we can remove the replications $X(\omega + k\frac{2\pi}{T_s})$ by passing $X_{DA}(\omega)$ through a lowpass filter.

- Note that the term $\mathrm{sinc}\left(\frac{\omega T_s}{2}\right)$ must be removed from $X_{DA}(\omega)$.

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Example on a discrete system's transfer function

- Consider a LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n-2] - \frac{5}{2}y[n-1] + y[n] = x[n]$$

  Determine the impulse response and its $z-$transform in the following three cases:

  - The system is causal.
  - The system is stable.
  - The system is neither stable nor causal.

Recall: Find the z −transform of $x[n] = \gamma^n u[n]$

- Find the z −transform of the causal signal $\gamma^n u[n]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

- We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, |x| < 1$$

- Therefore,

$$X[z] = \frac{1}{1-\frac{\gamma}{z}}, \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{z}{z-\gamma}, |z| > |\gamma|$$

- We notice that the z −transform exists for certain values of z. These values form the so called Region-Of-Convergence (ROC) of the transform.

Recall: Find the z −transform of $x[n] = -\gamma^n u[-n-1]$

- Find the z −transform of the anticausal signal $-\gamma^n u[-n-1]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} -\gamma^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n$$

$$= -\frac{z}{\gamma}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right)\left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right]$$

- Therefore,

$$X[z] = -\left(\frac{z}{\gamma}\right)\frac{1}{1-\frac{z}{\gamma}}, \ \left|\frac{z}{\gamma}\right| < 1$$

$$= \frac{z}{z-\gamma}, \ |z| < |\gamma|$$

- We notice that the z −transform exists for certain values of z, which consist the complement of the ROC of the function $\gamma^n u[n]$ with respect to the z −plane.

Example cont.

- Consider a LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n-2] - \frac{5}{2}y[n-1] + y[n] = x[n]$$

By taking the z −transform in both sides of the above equation, we obtain:

$$\left(z^{-2} - \frac{5}{2}z^{-1} + 1\right)Y(z) = X(z) \Rightarrow$$

$$\frac{Y(z)}{X(z)} = \frac{1}{\left(z^{-2}-\frac{5}{2}z^{-1}+1\right)} = \frac{1}{(z^{-1}-2)(z^{-1}-\frac{1}{2})} = \frac{\frac{2}{3}}{(z^{-1}-2)} - \frac{\frac{2}{3}}{(z^{-1}-\frac{1}{2})}$$

Example cont.

- The impulse response that corresponds to the term $\frac{\frac{2}{3}}{(z^{-1}-2)} = (-\frac{1}{3})\frac{z}{z-\frac{1}{2}}$ can be:
  - $h_1[n] = \left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)^n u[n], |z| > |1/2|$
  - $h_2[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1], |z| < |1/2|$
- The impulse response that corresponds to the term $\frac{-\frac{2}{3}}{(z^{-1}-\frac{1}{2})} = \frac{4}{3}\frac{z}{z-2}$ can be:
  - $h_3[n] = \frac{4}{3}2^n u[n], |z| > |2|$
  - $h_4[n] = (-\frac{4}{3})2^n u[-n-1], |z| < |2|$

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Example cont.

- In order for the system to be stable:

$$h[n] = \left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1],\ |z| > |1/2| \cap |z| < |2|$$

- In order for the system to be causal:

$$h[n] = \left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}2^n u[n],\ |z| > |1/2| \cap |z| > |2| = |z| > |2|$$

The ROC doesn’t include the unit circle.

- In order for the system to be neither stable nor causal the remaining two combinations should be considered.

$$h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1] + \frac{4}{3}2^n u[n],\ |z| < |1/2| \cap |z| > |2| = \emptyset$$

$$h[n] = \frac{1}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{4}{3}2^n u[-n-1],\ |z| < |1/2| \cap |z| < |2|$$

The ROC doesn’t include the unit circle.