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DSP & Digital Filters

Lecture 1 z-Transform

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Continuous-time signals

- Recall that in order to describe a continuous-time signal $x(t)$ in frequency domain we use:
  - ❑ The Continuous-Time Fourier Transform (or Fourier Transform):

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - ❑ The Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- The above transforms and their basic properties are considered known in this course.
- If you have doubts please consult any book on Signals and Systems.

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Discrete-time signals
The z-transform derived from the Laplace transform

- Consider a discrete-time signal $x(t)$ sampled every T seconds.

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + x_2\delta(t-2T) + x_3\delta(t-3T) + \cdots$$

- Recall that in the Laplace domain we have:

$$\mathcal{L}\{\delta(t)\} = 1$$
$$\mathcal{L}\{\delta(t-T)\} = e^{-sT}$$

- Therefore, the Laplace transform of $x(t)$ is:

$$X(s) = x_0 + x_1e^{-sT} + x_2e^{-s2T} + x_3e^{-s3T} + \cdots$$

- Now define $z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$.
- Finally, define

$$X[z] = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \cdots$$

z^{-1}: the sampling period delay operator

- From the Laplace time-shift property, we know that an additional term $z = e^{sT}$ in the Laplace domain, corresponds to time-advance by T seconds (T is the sampling period) of the original function in time.
- Accordingly, $z^{-1} = e^{-sT}$ corresponds to a time-delay of one sampling period.
- As a result, all sampled data (and discrete-time systems) can be expressed in terms of the variable z.
- More formally, the **unilateral $z-$transform** of a causal sampled sequence:

$$x[n] = \{x[0], x[1], x[2], x[3], \dots\}$$

  is given by:

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \cdots = \sum_{n=0}^{\infty} x[n] z^{-n}, \; x_n = x[n]$$

- The **bilateral $z-$transform** for any sampled sequence is:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Example: Find the z –transform of $x[n] = \gamma^n u[n]$

- Find the z –transform of the **causal** signal $\gamma^n u[n]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

- We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, |x| < 1$$

- Therefore,

$$X[z] = \frac{1}{1-\frac{\gamma}{z}}, \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{z}{z-\gamma}, |z| > |\gamma|$$

- We notice that the z –transform exists for certain values of z. These values form the so called Region-of-Convergence (ROC) of the transform.

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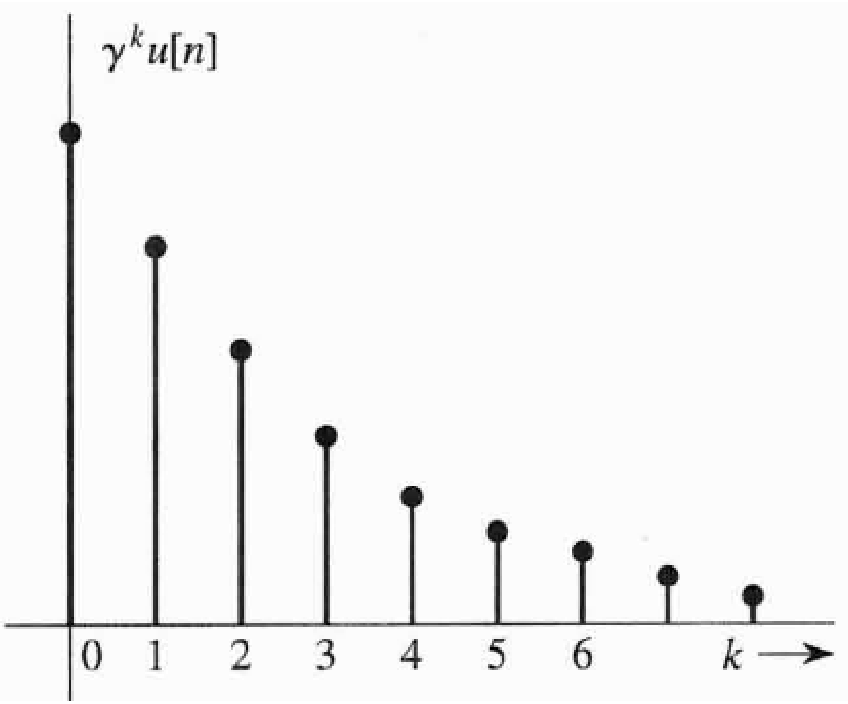
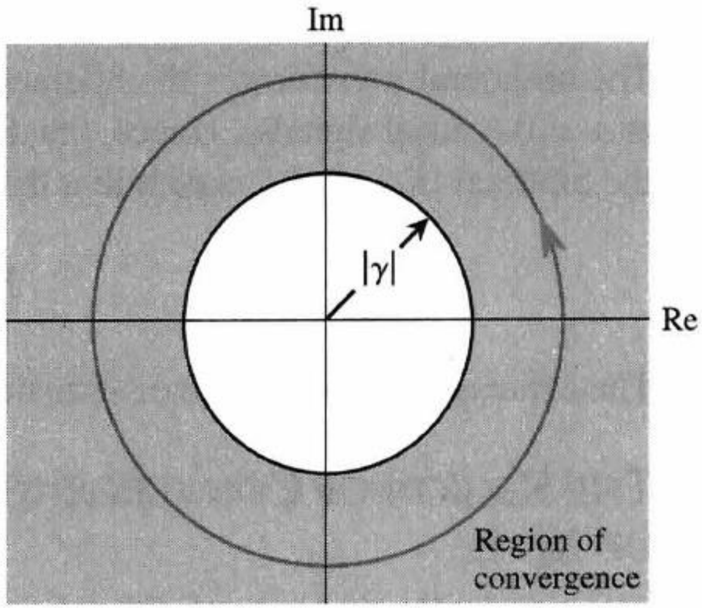
Example: Find the z –transform of $x[n] = \gamma^n u[n]$ cont.

- Observe that a simple rational equation in z-domain corresponds to an infinite sequence of samples in time-domain.
- The figures below depict the signal in time (left) for $|\gamma| < 1$ and the ROC, shown with the shaded area, within the z –plane.





Generic form of a causal signal

- Consider the causal signal $x[n] = \sum_{i=1}^{K} \gamma_i^n u[n]$ with $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$.
- In that case the ROC is the intersection of the ROCs of the individual terms, i.e., the intersection of the sets $|z| > |\gamma_i|$ i.e., ROC: $|z| > |\gamma_{\max}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$ with poles γ_i.
- The above analysis yields the following properties regarding the ROC:

PROPERTY:

If $x[n]$ is a causal signal, the ROC of its z −transform is $|z| > |\gamma_{\max}|$ with $\gamma_{\max}$ the maximum magnitude pole of the z −transform.

- ❑ **In the general case of $x[n]$ being a right-sided signal (RSS) the ROC is as above but might not include ∞ (think why).**

PROPERTY:

No pole can exist in ROC.

Generic form of a causal signal cont.

- The signal $x[n] = \sum_{i=1}^{K} \gamma_i^n u[n]$ is bounded only if $|\gamma_i| < 1\ \forall i$ or $|\gamma_{\max}| < 1$.
- In that case the ROC includes a circle with radius equal to 1. This is known as the **unit circle**.
- The above observation yields the following property:

PROPERTY:

If the ROC of $X(z)$ includes the unit circle in z −plane, then the signal in time is bounded and its Discrete Time Fourier Transform exists.

- In case that $\gamma^n u[n]$ is part of a causal system's impulse response, we see that the condition $|\gamma| < 1$ must hold. This is because, since $\lim_{n\to\infty} (\gamma)^n = \infty$, for $|\gamma| > 1$, the system will be unstable in that case.

- Therefore, in causal systems, stability requires that the ROC of the system's transfer function includes the unit circle.

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Example: Find the z −transform of $x[n] = -\gamma^n u[-n-1]$

- Find the z −transform of the anti-causal signal $-\gamma^n u[-n-1]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} -\gamma^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n$$
$$= -\frac{z}{\gamma}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right)\left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right]$$

- Therefore,

$$X[z] = -\left(\frac{z}{\gamma}\right)\frac{1}{1-\frac{z}{\gamma}},\ \left|\frac{z}{\gamma}\right| < 1$$
$$= \frac{z}{z-\gamma},\ |z| < |\gamma|$$

- We notice that the z −transform exists for certain values of z, which consist the complement of the ROC of the function $\gamma^n u[n]$ with respect to the z −plane.

Generic form of an anti-causal signal

- Consider the anti-causal signal $x[n] = \sum_{i=1}^{K} -\gamma_i^n u[-n-1]$ with z −transform $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$.
- In that case the ROC is the intersection of the sets $|z| < |\gamma_i|$, i.e., ROC: $|z| < |\gamma_{\min}|$
- In case that $x[n]$ is the impulse response of a system, the transfer function of the system is the rational function $X(z) = \sum_{i=1}^{K} \frac{z}{z-\gamma_i}$ with poles γ_i.
- The above analysis yield the following property regarding ROCs:

PROPERTY:

If $x[n]$ is an anti-causal signal, the ROC of its z −transform is $|z| < |\gamma_{\min}|$ with $\gamma_{\min}$ the minimum magnitude pole of the z −transform.

- ❑ **In the general case of $x[n]$ being a left-sided signal (LSS) the ROC is as above but might not include 0 (think why).**

Summary of previous examples

- We proved that the following two functions:
  - The causal function $\gamma^n u[n]$ and
  - the anti-causal function $-\gamma^n u[-n-1]$ have:
    - The same analytical expression for their z −transforms.
    - Complementary ROCs. More specifically, the union of their ROCS forms the entire z −plane.

- The above observations verify that the analytical expression alone is not sufficient to define the z −transform of a signal. The ROC is also required.

Two-sided signals

- **Example:** Find the z –transform of the two-sided signal:

  $x[n] = 2^n u[n] - 4^n u[-n-1]$

  Based on the previous analysis we have:

  $$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 2 \cap |z| < 4 \text{ or ROC: } 2 < |z| < 4$$

- **Example:** Find the z –transform of the two-sided signal:

  $x[n] = 4^n u[n] - 2^n u[-n-1]$

  Based on the previous analysis we have:

  $$X[z] = \frac{z}{z-2} + \frac{z}{z-4}, \text{ ROC: } |z| > 4 \cap |z| < 2 \text{ or ROC: } \emptyset$$

PROPERTY:

If $x[n]$ is two-sided signal then the ROC of its z –transform is of the form:

- ❑ $\gamma_1 < |z| < \gamma_2$ **with** γ_1, γ_2 **poles of the system or**
- ❑ $\emptyset$

Example: Find the z –transform of $\delta[n]$ and $u[n]$

- By definition $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$.

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$$

- By definition $u[n] = 1$ for $n \geq 0$.

$$X[z] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}}, \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}, |z| > 1$$

Example: Find the z −transform of $\cos\beta n u[n]$

- We write $\cos\beta n = \frac{1}{2}\left(e^{j\beta n} + e^{-j\beta n}\right)$.
- From previous analysis we showed that:

$$\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma},\ |z| > |\gamma|$$

- Hence,

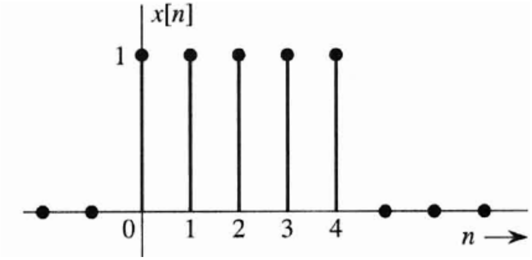
$$e^{\pm j\beta n}u[n] \Leftrightarrow \frac{z}{z-e^{\pm j\beta}},\ |z| > \left|e^{\pm j\beta}\right| = 1$$

- Therefore,

$$X[z] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right] = \frac{z(z-\cos\beta)}{z^2-2z\cos\beta+1},\ |z| > 1$$

z –transform of 5 impulses

- Find the z –transform of the signal depicted in the figure.



- By definition:

$$X[z] = 1 + \frac{1}{z} + \frac{1`}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \sum_{k=0}^{4} (z^{-1})^k = \frac{1 - (z^{-1})^5}{1 - z^{-1}} = \frac{z}{z-1}(1 - z^{-5})$$

Inverse z −transform

- As with other transforms, inverse z −transform is used to derive $x[n]$ from $X[z]$, and is formally defined as:

$$x[n] = \frac{1}{2\pi j}\oint X[z]z^{n-1}dz$$

- Here the symbol $\oint$ indicates an integration in counter-clockwise direction around a circle within the ROC and $z = Re^{j\theta}$.
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse z −transform.
- One such technique is to use a z −transform pairs Table shown in the last two slides with partial fraction expansion.

Inverse z −transform: Proof

Proof:

$$\frac{1}{2\pi j}\oint X[z]z^{n-1}dz = \frac{1}{2\pi j}\oint\left(\sum_{m=-\infty}^{\infty} x[m]z^{-m}\right)z^{n-1}dz$$

$$= \sum_{m=-\infty}^{\infty} x[m]\frac{1}{2\pi j}\oint z^{n-m-1}dz = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m] = x[n]$$

- For the above we used the Cauchy's theorem:

  $\frac{1}{2\pi j}\oint z^{k-1}dz = \delta[k]$ for $z = Re^{j\theta}$ anti-clockwise.

  $\frac{dz}{d\theta} = jRe^{j\theta} \Rightarrow \frac{1}{2\pi j}\oint z^{k-1}dz = \frac{1}{2\pi j}\int_{\theta=0}^{2\pi} R^{k-1}e^{j(k-1)\theta}\,jRe^{j\theta}d\theta =$
  $\frac{R^k}{2\pi}\int_{\theta=0}^{2\pi} e^{jk\theta}\,d\theta = R^k\,\delta[k]$

  $$[\frac{R^k}{2\pi}\int_{\theta=0}^{2\pi} e^{jk\theta}\,d\theta = \begin{cases} 0 & k \neq 0 \\ \frac{R^k}{2\pi}2\pi = R^k & k = 0 \end{cases}]$$

Find the inverse z –transform in the case of real unique poles

- Find the inverse z –transform of $X[z] = \frac{8z-19}{(z-2)(Z-3)}$

Solution

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(Z-3)} = \frac{(-\frac{19}{6})}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

By using the simple transforms that we derived previously we get:

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}2^n + \frac{5}{3}3^n\right]u[n]$$

Find the inverse z –transform in the case of real repeated poles

- Find the inverse z –transform of $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Solution

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- We use the so called **<u>covering method</u>** to find k and a_0

$$k = \left.\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\right|_{z=1} = -3$$

$$a_0 = \left.\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

The shaded areas above indicate that they are excluded from the entire function when the specific value of z is applied.

Find the inverse z –transform in the case of real repeated poles cont.

- Find the inverse z –transform of $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Solution

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} + \frac{-2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- To find a_2 we multiply both sides of the above equation with z and let $z \to \infty$.

$$0 = -3 - 0 + 0 + a_2 \Rightarrow a_2 = 3$$

- To find a_1 let $z \to 0$.

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \Rightarrow a_1 = -1$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{(z-2)} \Rightarrow$$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

Find the inverse z −transform in the case of real repeated poles cont.

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

- We use the following properties:
  - $\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}$
  - $\frac{n(n-1)(n-2)\ldots(n-m+1)}{\gamma^m m!}\gamma^n u[n] \Leftrightarrow \frac{z}{(z-\gamma)^{m+1}}$

$[-\frac{2z}{(z-2)^3} = (-2)\frac{z}{(z-2)^{2+1}} \Leftrightarrow (-2)\,\frac{n(n-1)}{2^2 2!}\gamma^n u[n] = -2\,\frac{n(n-1)}{8}\cdot 2^n u[n]$

- Therefore,

$$x[n] = [-3\cdot 1^n - 2\,\frac{n(n-1)}{8}\cdot 2^n - \frac{n}{2}\cdot 2^n + 3\cdot 2^n]u[n]$$

$$= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]u[n]$$

Find the inverse z –transform in the case of complex poles

- Find the inverse z –transform of $X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

Solution

$$X[z] = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Whenever we encounter a complex pole we need to use a special partial fraction method called **quadratic factors method**.

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

We multiply both sides with z and let $z \to \infty$:

$$0 = 2 + A \Rightarrow A = -2$$

Therefore,

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

Find the inverse z —transform in the case of complex poles cont.

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B we let $z = 0$:

$$\frac{-34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \Rightarrow X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

- We use the following property:

$r|\gamma|^n \cos(\beta n + \theta)\, u[n] \Leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$ with $A = -2$, $B = 16$, $a = -3$, $|\gamma| = 5$.

$r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = \sqrt{\frac{4\cdot 25+256-2\cdot(-2)\cdot(-3)\cdot 16}{25-9}} = 3.2$, $\beta = \cos^{-1}\frac{-a}{|\gamma|} = 0.927 rad$,

$\theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -2.246 rad$.

Therefore, $x[n] = [2 + 3.2\cos(0.927n - 2.246)]u[n]$

z —transform Table

No.	$x[n]$	$X[z]$
1	$\delta[n-n]$	z^{-k}
2	$u[n]$	$\dfrac{z}{z-1}$
3	$nu[n]$	$\dfrac{z}{(z-1)^2}$
4	$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$

z —transform Table

No.	$x[n]$	$X[z]$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos \beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert \cos \beta)}{z^2-(2\lvert\gamma\rvert\cos \beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin \beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert \sin \beta}{z^2-(2\lvert\gamma\rvert\cos \beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos (\beta n+\theta)u[n]$	$\dfrac{rz[z\cos \theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos \beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos (\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos (\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}} \qquad \beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert} \qquad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$