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Digital Image Processing

Image Transforms

Karhunen-Loeve Transform

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Eigenvalues and eigenvectors

- In this lecture we will introduce a new transform, the so called Karhunen-Loeve Transform or KLT.

- The concepts of eigenvalues and eigenvectors are important for understanding the KLT.

- If C is a matrix of dimension $n \times n$, then a scalar λ is called an eigenvalue of C if there is a non-zero vector $\underline{e}$ in R^n such that:

$$C\underline{e} = \lambda\underline{e}$$

- The vector $\underline{e}$ is called an eigenvector of matix C corresponding to the eigenvalue λ.

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Definition of a population of vectors

- Consider a population of random vectors of the following form:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

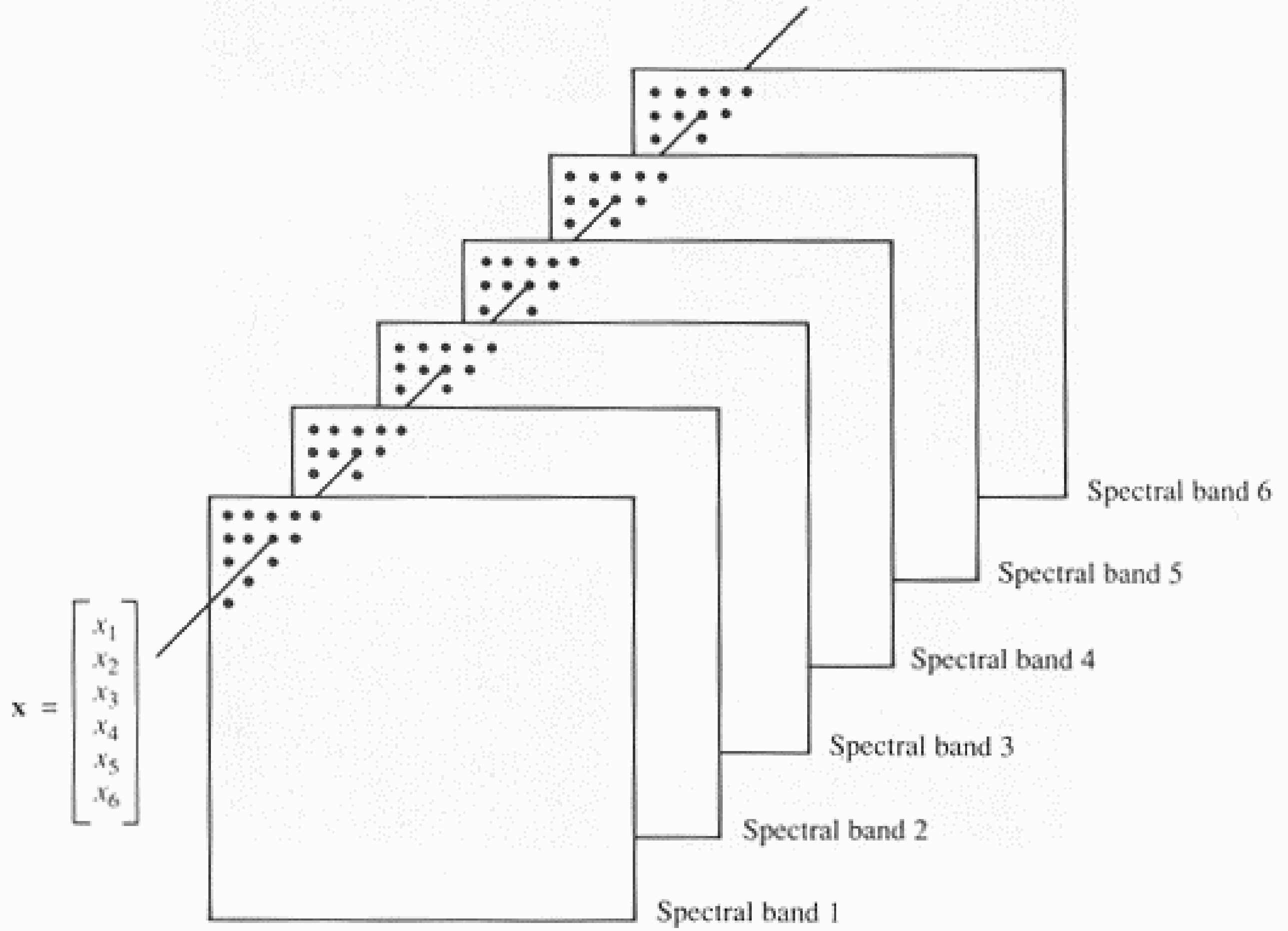
- The quantity x_i may represent the value (grey level) of an image i. We have n images, all of equal size $M \times N$. Each of the above vectors refers to the exact same location across the n images (look at the next slide).

- Therefore, it is more accurate to write

$$\underline{x}_{(k,l)} = \begin{bmatrix} x_1(k,l) \\ x_2(k,l) \\ \vdots \\ x_n(k,l) \end{bmatrix}$$

with $k \in [0 \ldots M-1]$ and $l \in [0 \ldots N-1]$.

Depiction of previous scenario with $n = 6$



Formation of a vector from corresponding pixels in six images.

Mean of the population

- The mean vectors of the population are defined as:

$$\underline{m}_{\underline{x}_{(k,l)}} = E\{\underline{x}_{(k,l)}\} = [m_{1,(k,l)} \quad m_{2,(k,l)} \quad \cdots \quad m_{n,(k,l)}]^T$$

- As you can see, we assume that the mean of each pixel (k, l) in each image i is different.
- In that case we would require a large number of realizations of each image i in order to calculate the means $m_{i,(k,l)}$.
- However, if we assume that each image signal is ergodic we can calculate a single mean value for all pixels from a single realization using the entire collection of pixels of this particular image. In that case:

$$m_{i,(k,l)} = m_i = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} x_{i,(k,l)}$$

- $\underline{m}_{\underline{x}_{(k,l)}} = \underline{m}_{\underline{x}} = [m_1 \quad m_2 \quad \cdots \quad m_n]^T$

Covariance of the population

- The covariance matrix of the population is defined as:

$$C_{\underline{x}} = E\{(\underline{x} - \underline{m}_{\underline{x}})(\underline{x} - \underline{m}_{\underline{x}})^T\}$$

  $C_{\underline{x}}$ is of dimension $n \times n$.
- Let us recall that:
  - In probability theory and statistics, a **covariance matrix** (known also as **dispersion matrix** or **variance-covariance matrix**) is a matrix whose element in the (i, j) position is the covariance between the i −th and j −th elements of a random vector (a vector whose elements are random variables).
  - Because the covariance of the i −th random variable with itself is simply that random variable's variance, each element on the principle diagonal of the covariance matrix is the variance of one of the random variables.
- **Every covariance matrix is symmetric and positive semi-definite**.
- Recall that for every symmetric matrix of dimension $n \times n$ we can always find a set of n orthonormal eigenvectors.

Karhunen-Loeve Transform

- Let A be a matrix whose rows are formed from the eigenvectors of the covariance matrix $C_{\underline{x}}$ of the population of vectors $\underline{x}$.
- The eigenvectors are ordered so that the first row of A is the eigenvector corresponding to the largest eigenvalue of $C_{\underline{x}}$ and the last row is the eigenvector corresponding to the smallest eigenvalue of $C_{\underline{x}}$.
- We define the following transform:

$$\underline{y} = A(\underline{x} - \underline{m}_{\underline{x}})$$

- It is called the **Karhunen-Loeve Transform** (**KLT**).
- This transform takes a vector $\underline{x}$ and converts it into a vector $\underline{y}$.
- We will see shortly that the new population of vectors $\underline{y}$ possesses a couple of very useful properties.

Mean and covariance of the new population

- You can demonstrate very easily that

$$\underline{m}_{\underline{y}} = E\{\underline{y}\} = 0$$

Proof

$$E\{\underline{y}\} = E\{A(\underline{x} - \underline{m}_{\underline{x}})\} = AE\{\underline{x} - \underline{m}_{\underline{x}}\} = A(E\{\underline{x}\} - E\{\underline{m}_{\underline{x}}\})$$

$$E\{\underline{m}_{\underline{x}}\} = \underline{m}_{\underline{x}},\ E\{\underline{x}\} = \underline{m}_{\underline{x}}$$

$$\underline{m}_{\underline{y}} = E\{\underline{y}\} = A(E\{\underline{x}\} - E\{\underline{m}_{\underline{x}}\}) = A(0-0) = 0$$

- Let us find the covariance matrix of the population $\underline{y}$. You can demonstrate very easily that $C_{\underline{y}} = AC_{\underline{x}}A^T$.

Proof

$$C_{\underline{y}} = E\left\{\left(\underline{y} - \underline{m}_{\underline{y}}\right)\left(\underline{y} - \underline{m}_{\underline{y}}\right)^T\right\} = E\left\{\underline{y}\underline{y}^T\right\} = E\left\{A(\underline{x} - \underline{m}_{\underline{x}})\left[A(\underline{x} - \underline{m}_{\underline{x}})\right]^T\right\}$$

$$= E\left\{A(\underline{x} - \underline{m}_{\underline{x}})(\underline{x} - \underline{m}_{\underline{x}})^T A^T\right\} = A\,E\left\{(\underline{x} - \underline{m}_{\underline{x}})(\underline{x} - \underline{m}_{\underline{x}})^T\right\} A^T.$$

Therefore, $C_{\underline{y}} = AC_{\underline{x}}A^T$ and is of dimension $n \times n$ as $C_{\underline{x}}$.

Covariance of the new population

- Let us further analyze the relationship $C_{\underline{y}} = AC_{\underline{x}}A^T$.
- Suppose that the eigenvectors of matrix $C_{\underline{x}}$ are the column vectors $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$.
- We assume that the eigenvectors of the covariance matrix $C_{\underline{x}}$ of dimension $n \times n$ form an orthonormal set in the n −dimensional space.
- From the definition of matrix A we know that

$$A = \begin{bmatrix} \underline{e}_1{}^T \\ \underline{e}_2{}^T \\ \vdots \\ \underline{e}_n{}^T \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \underline{e}_1{}^T \\ \underline{e}_2{}^T \\ \vdots \\ \underline{e}_n{}^T \end{bmatrix}^T = [\underline{e}_1 \quad \underline{e}_2 \quad \cdots \quad \underline{e}_n]$$

- $C_{\underline{x}}A^T = C_{\underline{x}}[\underline{e}_1 \quad \underline{e}_2 \quad \cdots \quad \underline{e}_n] = [C_{\underline{x}}\underline{e}_1 \quad C_{\underline{x}}\underline{e}_2 \quad \cdots \quad C_{\underline{x}}\underline{e}_n] = [\lambda_1\underline{e}_1 \quad \lambda_2\underline{e}_2 \quad \cdots \quad \lambda_n\underline{e}_n]$

Covariance of the new population cont.

- From the previous analysis we have

$$C_{\underline{y}} = AC_{\underline{x}}\, A^T = \begin{bmatrix} \underline{e}_1{}^T \\ \underline{e}_2{}^T \\ \vdots \\ \underline{e}_n{}^T \end{bmatrix} [\lambda_1 \underline{e}_1 \quad \lambda_2 \underline{e}_2 \quad \ldots \quad \lambda_n \underline{e}_n]$$

- $\underline{e}_i{}^T\, \underline{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ since the set of column vectors $\underline{e}_j, i = 1, \ldots, n$ consists a set of orthonormal eigenvectors of the covariance matrix $C_{\underline{x}}$.

- From the above analysis it is straightforward that:

$$C_{\underline{y}} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

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Observations regarding the new population

- In this transformation we started with vectors $\underline{x}$ and we created vectors $\underline{y}$.
- Since both original and transformed vectors consist of the values of n images at a specific location, we see that starting from n images we create n new images by assembling properly all the vectors of population $\underline{y}$.
- Looking at the form of $C_{\underline{y}}$ we immediately see that the new images are decorrelated to each other.
- Furthermore, since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ we see that as the index of a new image increases, its variance decreases. Recall that from the form of $C_{\underline{y}}$ we see that the variance of image i in the new set is λ_i.
- It is possible that a couple of the last images in the new set have very small variance. They are almost constant images.
- In general, a signal with small variance is a slowly varying signal. This type of signal does not offer any information.
- We can discard the images which have low variance in order to achieve data compression.

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Inverse Karhunen-Loeve transform

- To reconstruct the original vectors $\underline{x}$ from the transformed vectors $\underline{y}$ we note that $A^{-1} = A^T$.
- Therefore, starting from the forward transform $\underline{y} = A(\underline{x} - \underline{m}_{\underline{x}})$ we obtain the inverse Karhunen-Loeve transform as follows:

$$\underline{y} = A(\underline{x} - \underline{m}_{\underline{x}}) \Rightarrow A^T \underline{y} = (\underline{x} - \underline{m}_{\underline{x}}) \Rightarrow \underline{x} = A^T \underline{y} + \underline{m}_{\underline{x}}$$

- We now form a “cropped” matrix A_K of size $K \times n$ using only the K eigenvectors of $C_{\underline{x}}$ which correspond to the K largest eigenvalues. The vectors of the new population are now of size $K \times 1$ and are denoted by $\underline{y}_K$.
- By using the inverse transform in the later case we obtain an approximation of the original vectors and therefore the original images, as follows:

$$\underline{\hat{x}} = {A_K}^T \underline{y}_K + \underline{m}_{\underline{x}}$$

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Mean squared error of approximate reconstruction

- It can be proven that the Mean Square Error (MSE) between the perfect reconstruction $\underline{x}$ and the approximate reconstruction $\hat{\underline{x}}$ is given by the expression:

$$e_{MSE} = \left\|\underline{x} - \hat{\underline{x}}\right\|^2 = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{K} \lambda_j = \sum_{j=K+1}^{n} \lambda_j$$

We see that the error is the sum of the eigenvalues whose eigenvectors we ignored in the vector reconstruction.

Drawbacks of the KL Transform

Despite its favourable theoretical properties, the KLT is not used in often practice for the following reasons.

- Its basis functions depend on the covariance matrix of the image, and hence they have to recomputed and transmitted for every image. It is, therefore, what we call data dependent.
- Perfect decorrelation is not possible, since images can rarely be modelled as realisations of ergodic fields.
- There are no fast computational algorithms for its implementation.
- Multiple realizations of an image are required. This is not always possible to achieve before hand.

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Example of the KLT: Original images

6 spectral images from an airborne Scanner.

Channel	Wavelength band (microns)
1	0.40–0.44
2	0.62–0.66
3	0.66–0.72
4	0.80–1.00
5	1.00–1.40
6	2.00–2.60

Channel 1

Channel 2

Channel 3

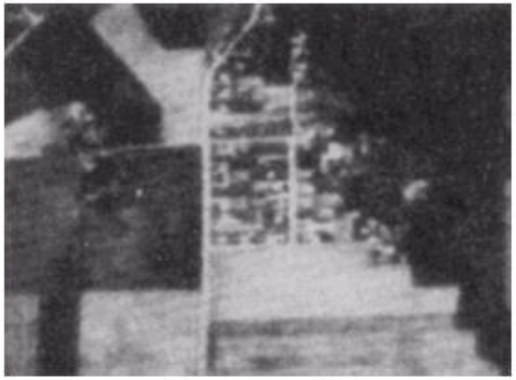
Channel 4

Channel 5

Channel 6

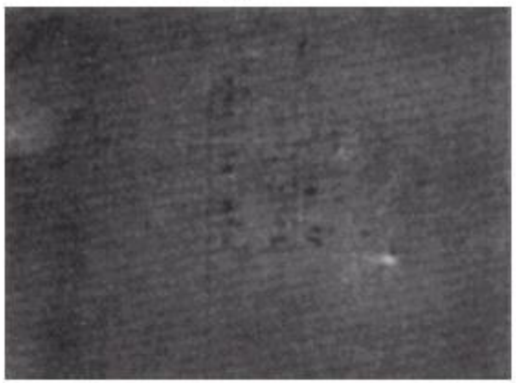
(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

Example: Principal Components



Component 1

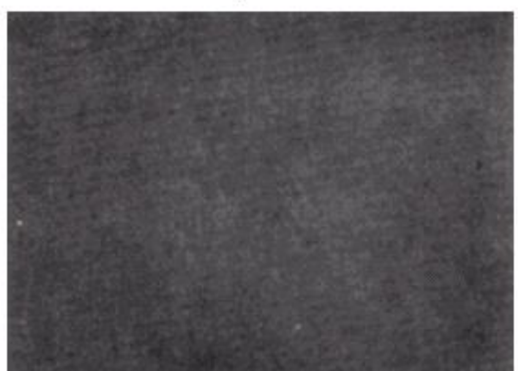
Component 2



Component 3

Component 4

Component 5



Component 6

Component	λ
1	*3210*
2	*931.4*
3	*118.5*
4	*83.88*
5	*64.00*
6	*13.40*

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

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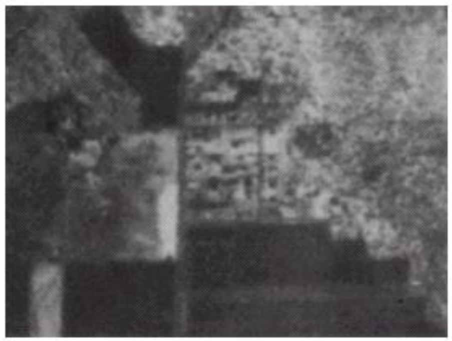
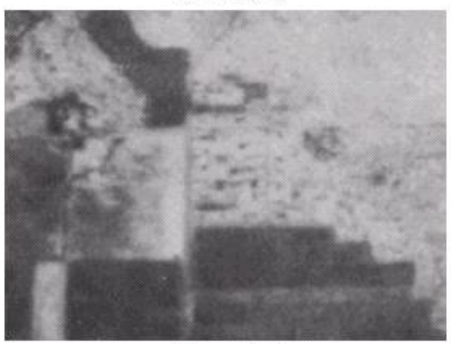
Example: Principal Components (cont.)


Channel 1


Channel 2

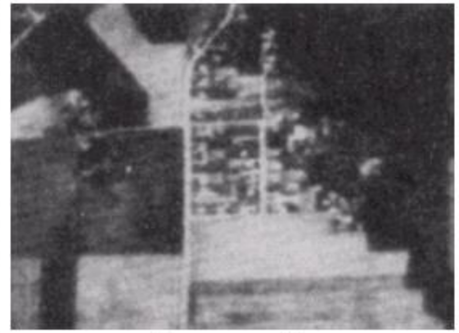

Channel 3


Channel 4

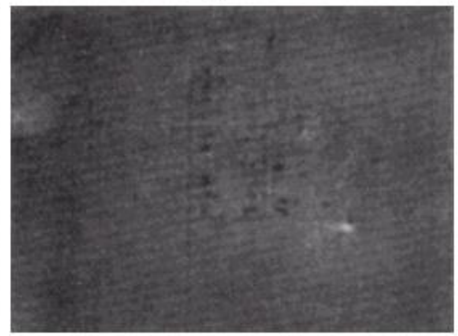

Channel 5

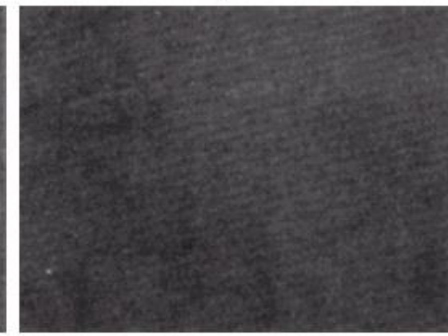

Channel 6

Original images (channels)


Component 1


Component 2


Component 3


Component 4


Component 5


Component 6

Six principal components after KL transform

Example: Original Images (left) and Principal Components (right)