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Signals and Systems

Class 8 Sampling

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Problem 1

By applying the Parseval's theorem, show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(kx)dx = \frac{\pi}{k}$$

Solution

Consider the function $f(t) = \text{sinc}(kt)$. In that case we know that the Fourier transform is:

$$F(\omega) = \frac{\pi}{k}\text{rect}(\frac{\omega}{2k})$$

$\int_{-\infty}^{\infty} \text{sinc}^2(kx)dx$ is the energy of $f(t)$.

Based on Parseval's Theorem we have that:

$$\int_{-\infty}^{\infty} \text{sinc}^2(kx)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{k^2}\left[\text{rect}(\frac{\omega}{2k})\right]^2 d\omega = \frac{\pi}{2k^2}2k = \frac{\pi}{k}$$

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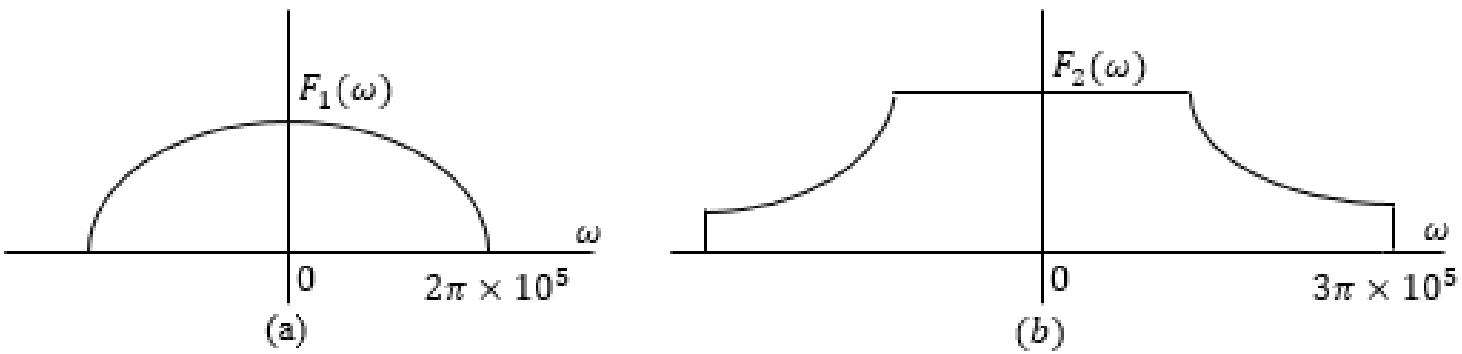
Problem 2 (a)

Figures (a) and (b) show the Fourier spectra of signals $f_1(t)$ and $f_2(t)$. Determine the Nyquist sampling rates for the following signals. (**Hint:** Use convolution in frequency and the width property of the convolution.)

(a) $f_1(t)$

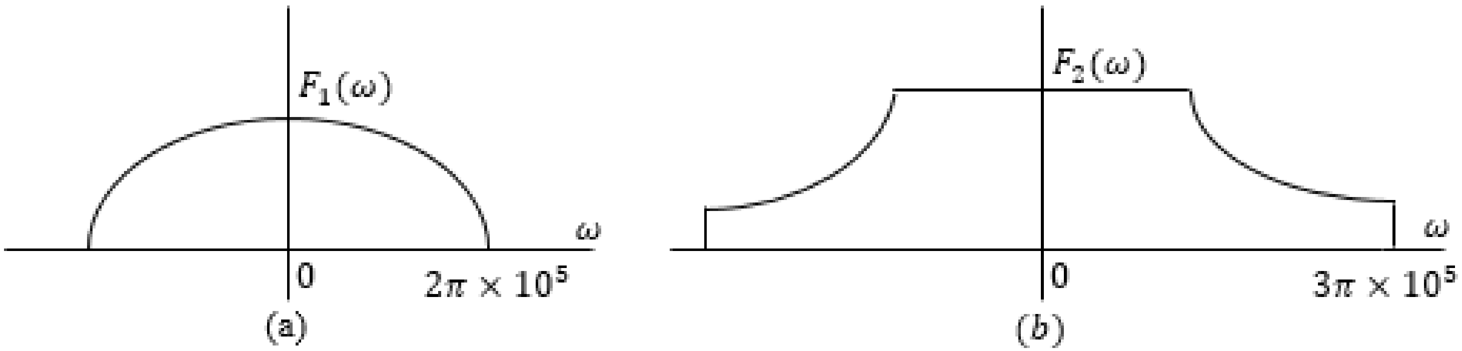


Solution

The bandwidth of $f_1(t)$ is 100000Hz. Therefore, the Nyquist rate is 200000Hz or 200kHz.

Problem 2 (b)

(b) $f_2(t)$

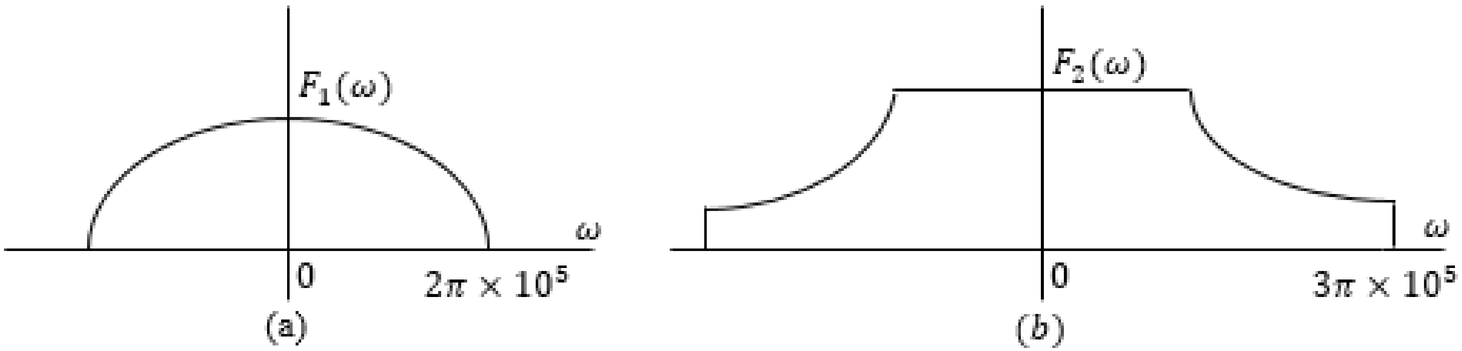


Solution

The bandwidth of $f_2(t)$ is 150kHz. Therefore, the Nyquist rate is 300kHz.

Problem 2 (c)

(c) $f_1{}^2(t)$



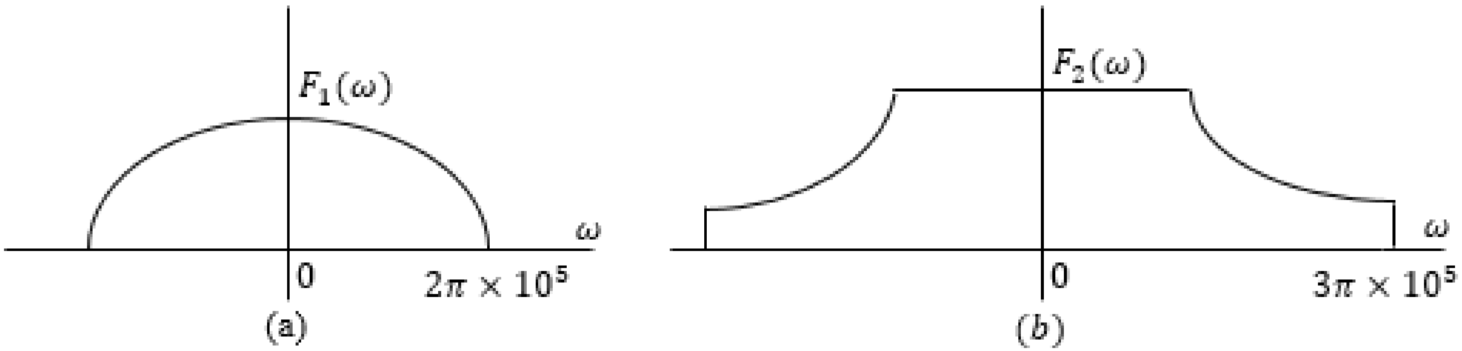
Solution

$$\mathcal{F}\{f_1{}^2(t)\} = \frac{1}{2\pi} F_1(\omega) * F_1(\omega)$$

Due to the above relationship in the frequency domain, we see that the bandwidth of $f_1{}^2(t)$ is twice the bandwidth of $f_1(t)$, i.e., it is 200kHz. Therefore, the Nyquist rate is 400kHz.

Problem 2 (d)

(d) $f_2{}^3(t)$



Solution

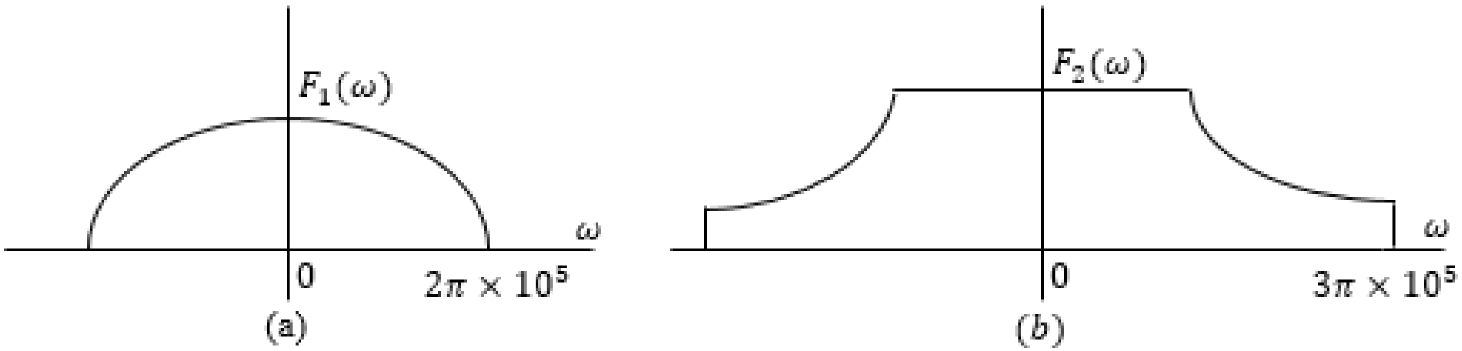
$$\mathcal{F}\{f_2{}^3(t)\} = \mathcal{F}\{f_2{}^2(t) f_2(t)\} = \frac{1}{2\pi}\mathcal{F}\{f_2{}^2(t)\} * F_2(\omega)$$

$$\frac{1}{2\pi}\left(\frac{1}{2\pi}F_2(\omega) * F_2(\omega)\right) * F_2(\omega) = \frac{1}{4\pi^2}F_2(\omega) * F_2(\omega) * F_2(\omega)$$

The bandwidth of $f_2{}^3(t)$ is three times the bandwidth of $f_2(t)$, i.e., it is 450kHz. Therefore, the Nyquist rate is 900kHz.

Problem 2 (e)

(e) $f_1(t)f_2(t)$



Solution

$$\mathcal{F}\{f_1(t)f_2(t)\} = \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

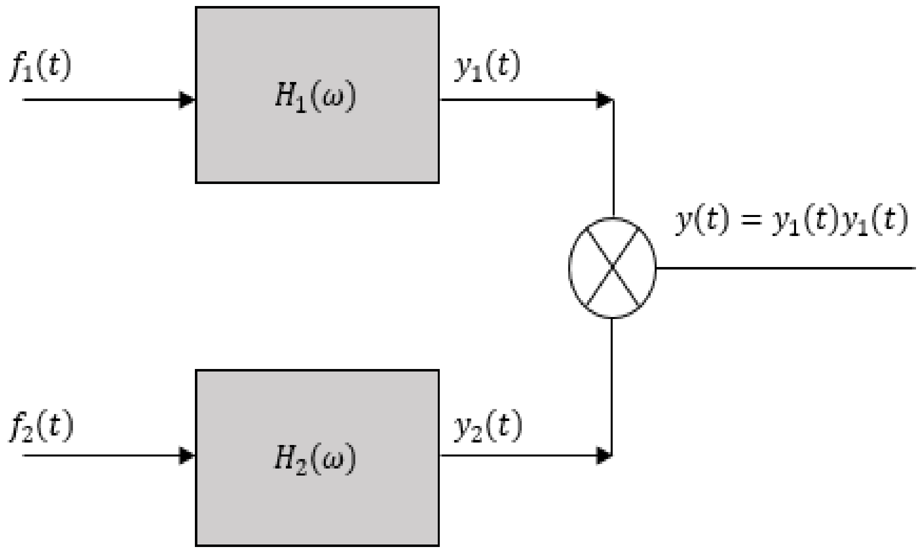
The bandwidth of $f_1(t)f_2(t)$ is the sum of the individual bandwidths of $f_1(t)$ and $f_2(t)$, i.e., it is 250kHz. Therefore, the Nyquist rate is 500kHz.

Problem 3

Signals $f_1(t) = 10^4\Pi(10^4 t)$ and $f_2(t) = \delta(t)$ are applied at the inputs of the ideal lowpass filters $H_1(t) = \Pi\left(\frac{\omega}{40000\pi}\right)$ and $H_2(t) = \Pi\left(\frac{\omega}{20000\pi}\right)$ respectively.

$$\Pi(t) = \text{rect}(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

The outputs $y_1(t)$ and $y_2(t)$ of these filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$ as shown in the figure below.



Problem 3 (a)

(a) Sketch $F_1(\omega)$ and $F_2(\omega)$.

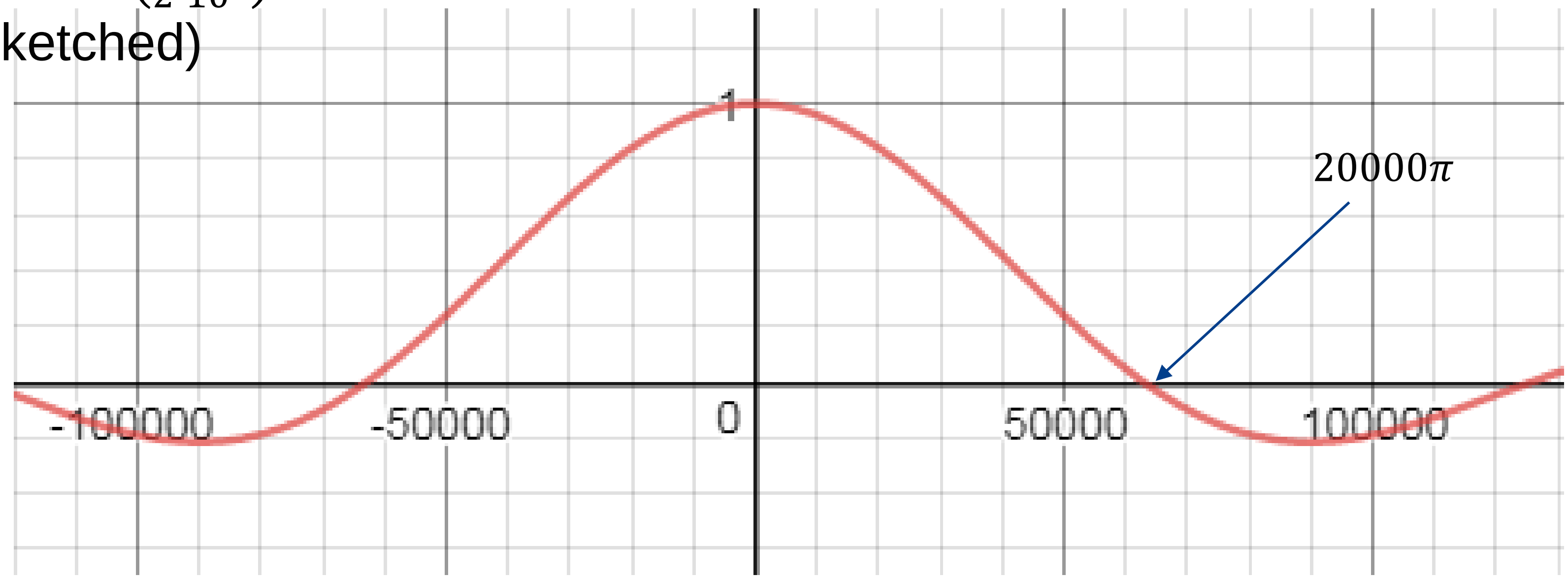
Solution

$f_1(t) = 10^4\Pi(10^4 t)$

We know that $\mathrm{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau \mathrm{sinc}(\frac{\omega\tau}{2})$. Therefore,

$$F_1(\omega) = \mathcal{F}\{10^4\Pi(10^4 t)\} = \mathcal{F}\left\{10^4\Pi\left(\frac{t}{\frac{1}{10^4}}\right)\right\} = 10^4\frac{1}{10^4}\mathrm{sinc}\left(\frac{\omega\frac{1}{10^4}}{2}\right)$$

$= \mathrm{sinc}\left(\frac{\omega}{2\cdot 10^4}\right)$ (shown in the figure below). Furthermore, $F_2(\omega) = 1$ (not sketched)

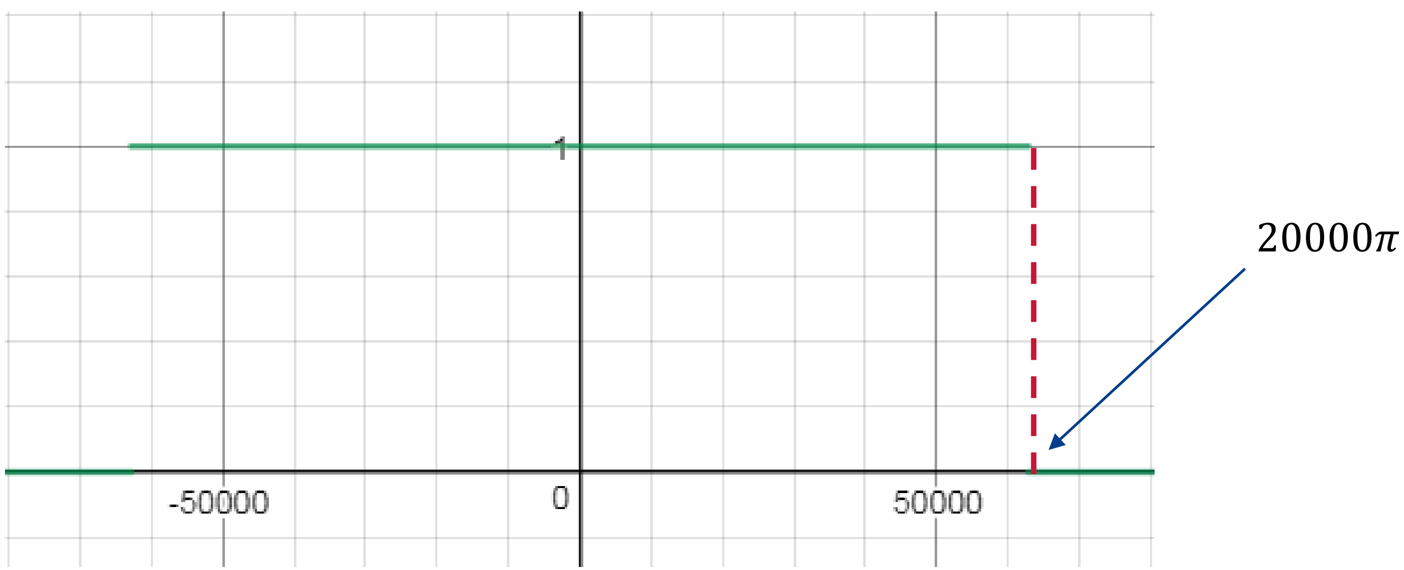


Problem 3 (b)

(b) Sketch $H_1(\omega)$ and $H_2(\omega)$.

Solution

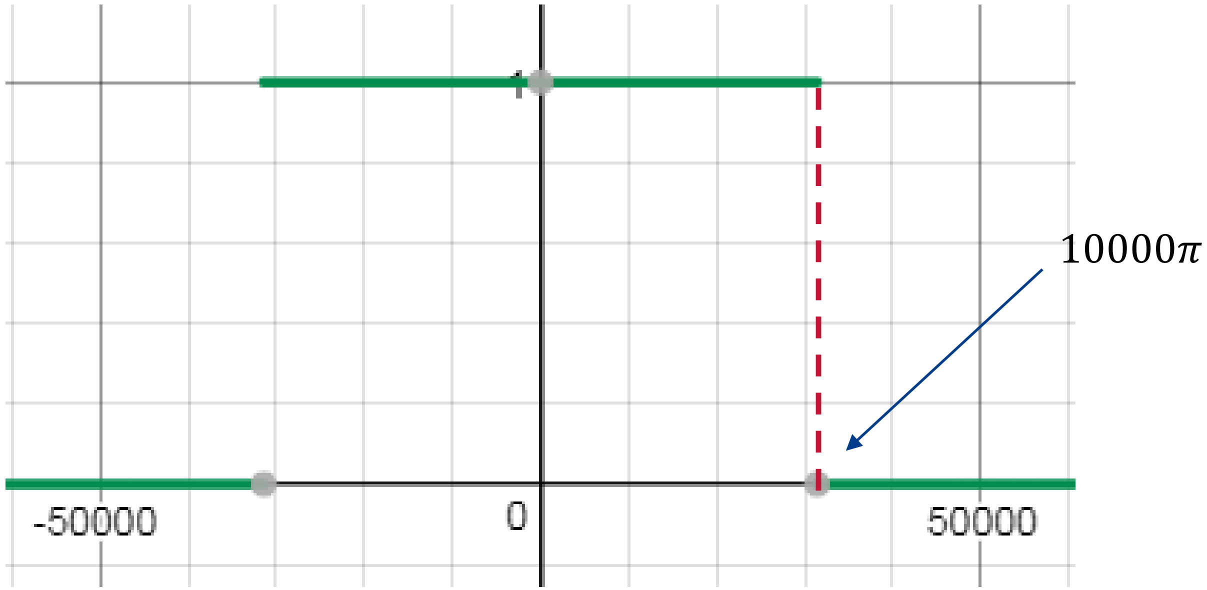
$$H_1(\omega) = \Pi\left(\frac{\omega}{40000\pi}\right) = \begin{cases} 1 & \left|\frac{\omega}{40000\pi}\right| < 0.5 \Rightarrow |\omega| < 20000\pi \\ 0.5 & |\omega| = 20000\pi \\ 0 & \text{otherwise} \end{cases}$$



Problem 3 (b) cont.

(b) Sketch $H_1(\omega)$ and $H_2(\omega)$.

$$H_2(\omega) = \Pi\left(\frac{\omega}{20000\pi}\right) = \begin{cases} 1 & \left|\frac{\omega}{20000\pi}\right| < 0.5 \Rightarrow |\omega| < 10000\pi \\ 0.5 & |\omega| = 10000\pi \\ 0 & \text{otherwise} \end{cases}$$



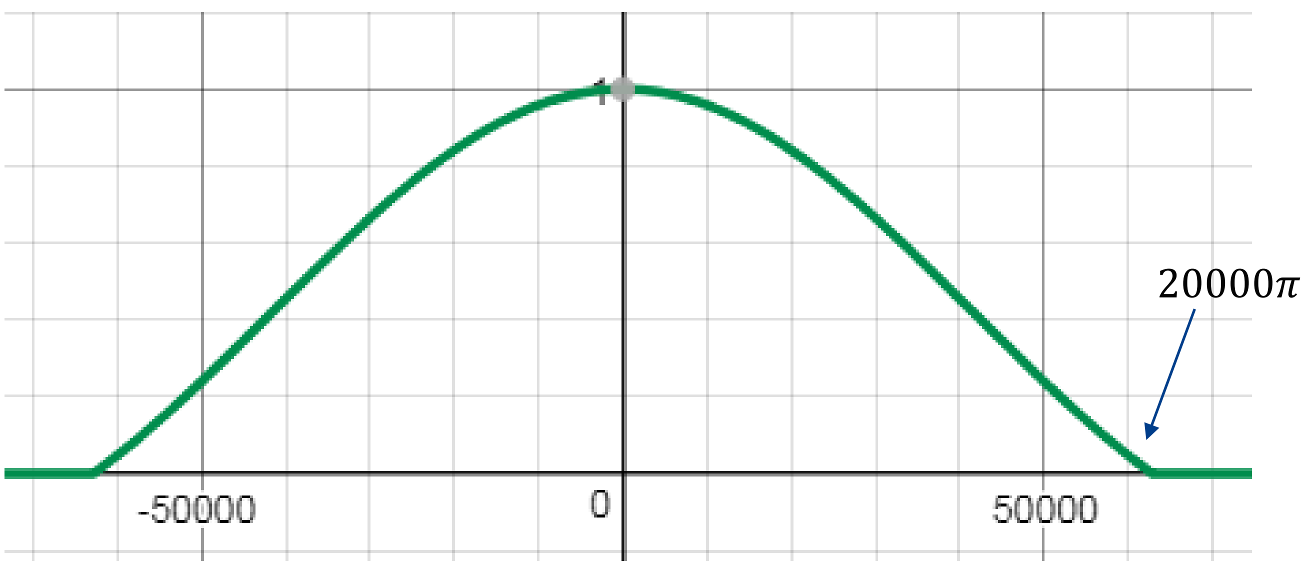
Problem 3 (c)

(c) Sketch $Y_1(\omega)$ and $Y_2(\omega)$.

$$Y_1(\omega) = F_1(\omega)H_1(t) = \text{sinc}\left(\frac{\omega}{2 \cdot 10^4}\right)\Pi\left(\frac{\omega}{20000\pi}\right)$$

(shown in the figure below)

$$Y_2(\omega) = F_2(\omega)H_2(t) = 1 \cdot \Pi\left(\frac{\omega}{10000\pi}\right) = H_2(\omega)$$ (sketched previously)



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Problem 3 (d)

(d) Find the Nyquist sampling rate of $y_1(t)$, $y_2(t)$ and $y(t) = y_1(t) \cdot y_2(t)$

The maximum frequency of $Y_1(\omega)$ in Hz is $\frac{20000\pi}{2\pi}$ =10000Hz.

The maximum frequency of $Y_2(\omega)$ in Hz is $\frac{10000\pi}{2\pi}$ =5000Hz.

Multiplication $y(t) = y_1(t) \cdot y_2(t)$ results in the convolution $Y(\omega) = Y_1(\omega) * Y_2(\omega)$ which has maximum frequency (bandwidth) 10000Hz+5000Hz=15000Hz15kHz. Therefore, the Nyquist frequency is 30kHz.

Problem 4

For the signal $x(t) = e^{-at}u(t)$, determine the bandwidth of an anti-aliasing filter if the essential bandwidth of the signal contains 99% of the signal energy.

Solution

$$E_x = \int_{-\infty}^{\infty} (e^{-at}u(t))^2\, dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{-2a} e^{-2at}|_0^{\infty} = \frac{1}{2a}$$

$$X(\omega) = \frac{1}{j\omega + a}$$

$$E_x = \frac{1}{\pi}\int_0^{\infty} X(\omega)X^*(\omega)\, d\omega = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2\, d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega^2+a^2} d\omega$$

$$= \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^{\infty} = \frac{1}{2a}$$

We are looking for an $\omega = \omega_0$ such as $\frac{1}{\pi}\int_0^{\omega_0} \frac{1}{\omega^2+a^2} d\omega = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_0^{\omega_0} =$ $\frac{1}{\pi a}\tan^{-1}\frac{\omega_0}{a} = \frac{0.99}{2a} \Rightarrow \tan^{-1}\frac{\omega_0}{a} = \frac{0.99\pi}{2} \Rightarrow \omega_0 = 63.66a \text{ rads}/s$

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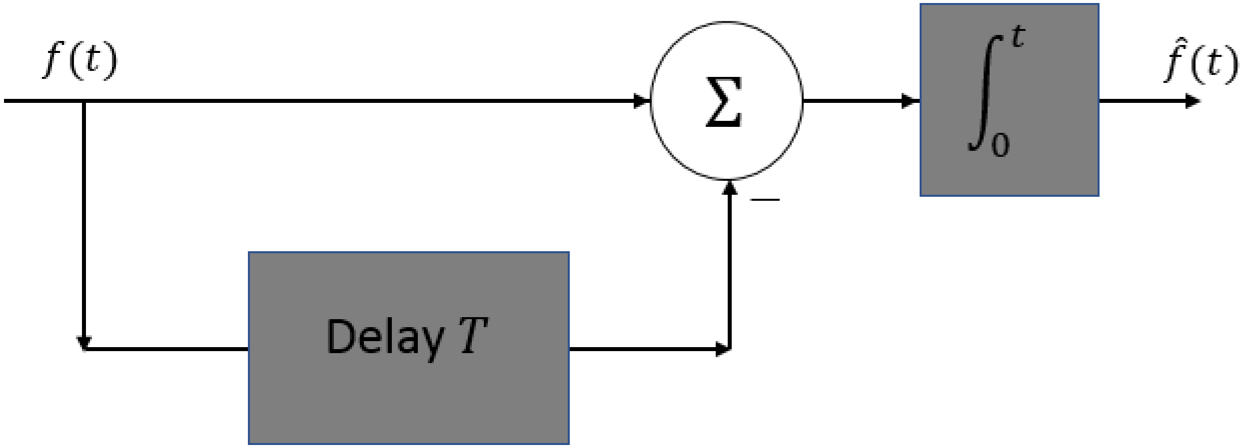
Problem 5

A zero-order hold circuit shown in the figure below is often used to reconstruct a signal $f(t)$ from its samples.

(a) Find the unit impulse response of this circuit.

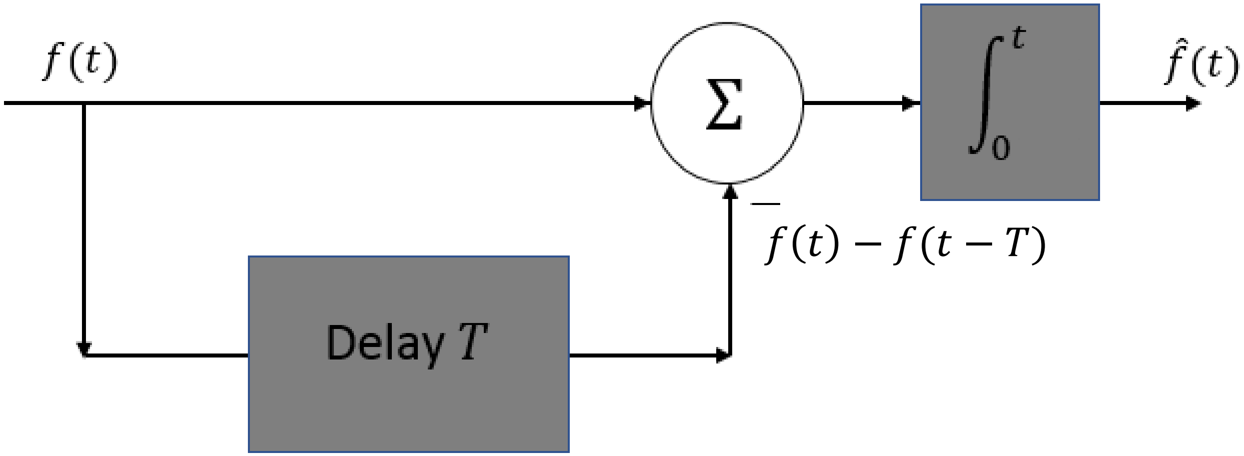
(b) Find the transfer function $H(\omega)$, and sketch $|H(\omega)|$.

(c) Sketch the output of this circuit for an input $f[n]$ which is the sampled version of $f(t)$, where $f(t)$, is ¼ cycle of a sinewave. The sampling period is T.



Problem 5 (a)

(a) Find the unit impulse response of this circuit.



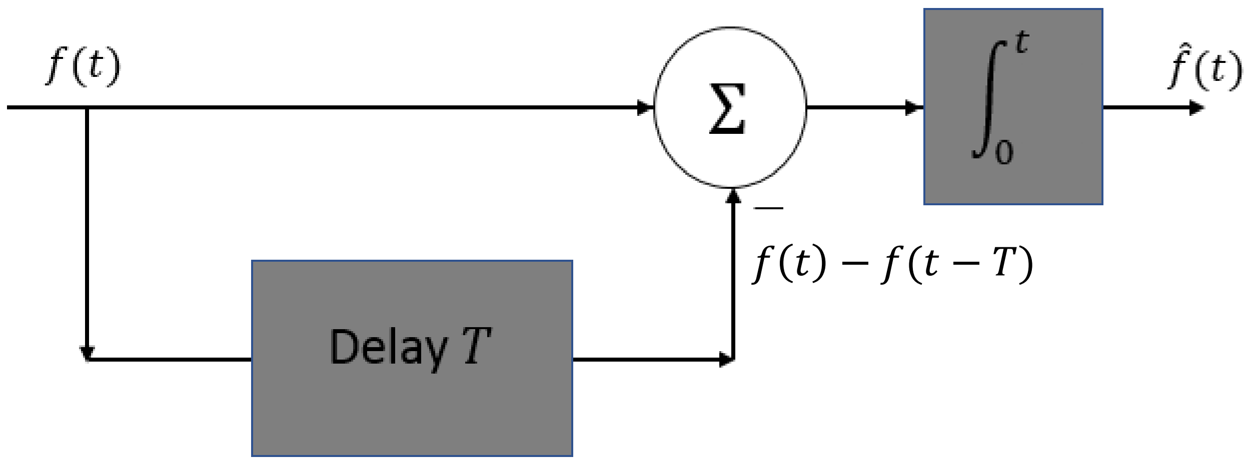
Solution

The impulse response is the output of the system when the input is the Dirac function. Therefore,

$$h(t) = \int_0^t \big(\delta(\tau) - \delta(\tau - T)\big)d\tau = u(t) - u(t-T) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

Problem 5 (b)

(b) Find the transfer function $H(\omega)$, and sketch $|H(\omega)|$.



Solution

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\text{rect}\left(\frac{t-\frac{T}{2}}{T}\right)\right\} = T\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$

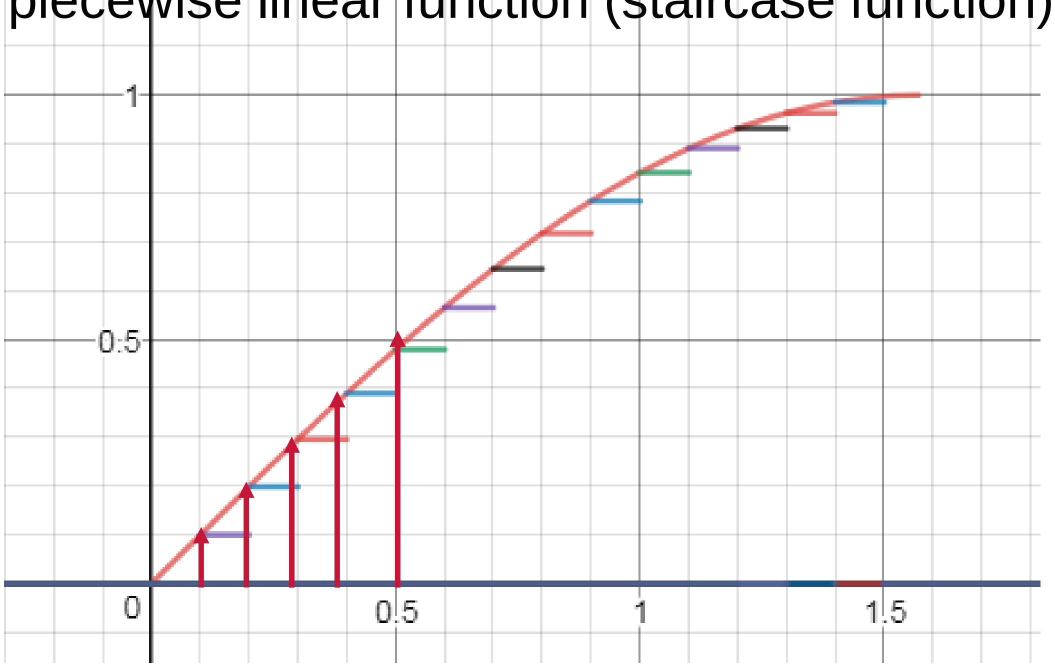
$$|H(\omega)| = T\left|\text{sinc}\left(\frac{\omega T}{2}\right)\right|$$

Problem 5 (b)

(c) Sketch the output of this circuit for an input $f[n]$ which is the sampled version of $f(t)$ which is ¼ cycle of a sinewave. The sampling period is T.

Solution

The impulse response is a rectangular pulse of width T. When a sampled signal is applied to this, the samples are convolved with this pulse. The result is the piecewise linear function (staircase function) shown below.



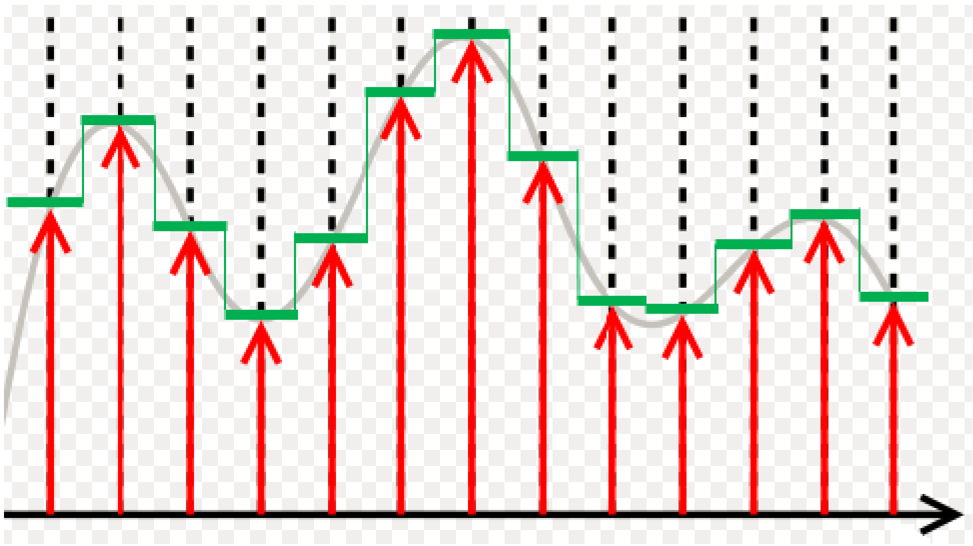
Problem 6: Reconstruction of the continuous time using the sample and hold operation

- As already mentioned, in order to reconstruct the original signal $x(t)$ in theory, we can use an ideal lowpass filter on the sampled spectrum.
- In practice, there are various circuits/devices which facilitate reconstruction of the original signal from its sampled version.
- Consider a continuous-time, band-limited signal $x(t)$, limited to bandwidth $|\omega| \leq 2\pi \times B$ rad/sec. We sample $x(t)$ uniformly with sampling frequency $f_s = 1/T_s$ to obtain the discrete-time signal $x[n]$.
- A possible technique to reconstruct the continuous-time signal from its samples is the so called **sample and hold** circuit. This is an analogue device which samples the value of a continuously varying analogue signal and outputs the following signal $x_{DA}(t)$.

$$x_{DA}(t) = \begin{cases} x[n] & nT_s - \frac{T_s}{2} < t < nT_s + \frac{T_s}{2} \\ x[n]/2 & t = nT_s \pm \frac{T_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Problem 6: Reconstruction of the continuous time using the sample and hold operation cont.

- The figure below facilitates the understanding of **sample and hold** operation.
- The grey continuous curve depicts the continuous signal $x(t)$.
- The red arrows depict the locations of the discrete (sampled) signal $x[n]$.
- The green continuous curve depicts the signal $x_{DA}(t)$.



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Problem 6: Reconstruction of the continuous time using the **sample and hold** operation cont.

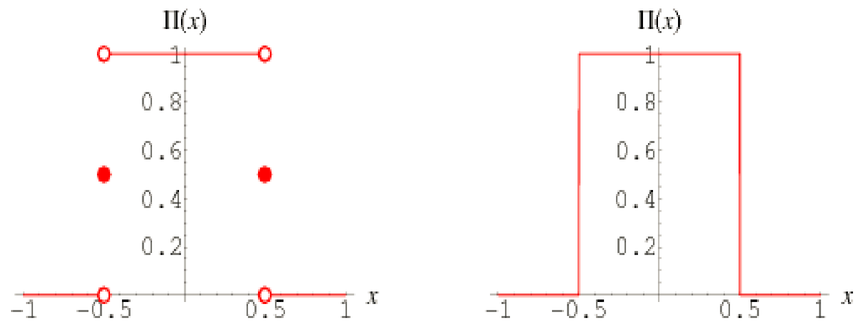
- In the **sample and hold** operation the signal is written as:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t-nT_s}{T_s}\right)$$

with $\Pi(t)$ the well known **unit gate** or **rectangle** function:

$$\Pi(t) = \mathrm{rect}(t) = \begin{cases} 1 & |t| < 0.5 \\ 0.5 & |t| = 0.5 \\ 0 & \text{otherwise} \end{cases}$$

- Note that the values at $t = \pm 0.5$ do not have any impact in the Fourier Transform of $\Pi(t)$ and alternative definitions of $\Pi(t)$ have $\mathrm{rect}(\pm 0.5)$ to be 0, 1 or undefined.



Problem 6: Reconstruction of the continuous time using the sample and hold operation cont.

- Consider that the continuous-time, band-limited signal $x(t)$, is limited to bandwidth $|\omega| \leq 2\pi \times 10^3$rad/sec.
- We sample $x(t)$ uniformly with sampling frequency $f_s = 1/T_s = 5 \times 10^3 Hz$ to obtain the discrete-time signal $x[n] = x(nT_s)$.

  We see that $T_s = 0.2 \times 10^{-3}$.
- Taking into consideration that:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\Pi\left(\frac{t - \frac{n}{f_s}}{0.2 \times 10^{-3}}\right)$$

- It is straightforward to show that:

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\left[\delta(t - nT_s) * \Pi\left(\frac{t}{0.2 \times 10^{-3}}\right)\right]$$

$$x_{DA}(t) = (\sum_{n=-\infty}^{\infty} [x(nT_s)\delta(t - nT_s)]) * \Pi\left(\frac{t}{0.2 \times 10^{-3}}\right)$$

Problem 6: Reconstruction of the continuous time using the sample and hold operation cont.

$$x_{DA}(t) = \left(\sum_{n=-\infty}^{\infty}[x(nT_s)\delta(t-nT_s)]\right) * \Pi\left(\frac{t}{T_s}\right)$$

- The Fourier transform of the function $\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$ is $\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})$.

- The Fourier transform of the function $\Pi\left(\frac{t}{T_s}\right)$ can be easily found using the definition of the Fourier transform.

- The Fourier transform of $x_{DA}(t)$ is the product of the two Fourier transforms described above (remember that convolution in time becomes multiplication in frequency).

Problem 6: Reconstruction of the continuous time using the sample and hold operation cont.

$$x_{DA}(t) = \left(\sum_{n=-\infty}^{\infty}[x(nT_s)\delta(t-nT_s)]\right) * \Pi\left(\frac{t}{T_s}\right)$$

- We can show that

$$X_{DA}(\omega) = \left(\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})\right)\cdot\left(T_s\text{sinc}\left(\frac{\omega T_s}{2}\right)\right)$$

$$= \left(\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})\right)\cdot\text{sinc}\left(\frac{\omega T_s}{2}\right)$$

- In order to recover the original spectrum $X(\omega)$ we must remove the replications $X(\omega + k\frac{2\pi}{T_s})$ by passing $X_{DA}(\omega)$ through a lowpass filter.

- Note that the term $\text{sinc}\left(\frac{\omega T_s}{2}\right)$ must also be removed from $X_{DA}(\omega)$.

Problem 6: Reconstruction of the continuous time using the sample and hold operation cont.

- We now wish to pass $X_{DA}(\omega)$ through a system $H(\omega)$ and get $X(\omega)$ at the output.
- We are looking for a function $H(\omega)$ that satisfies the relation

$$X(\omega) = H(\omega)X_{DA}(\omega)$$

- In order to remove the replications $X\left(\omega + k\frac{2\pi}{T_s}\right), k \neq 0$, $H(\omega)$ must be zero for $|\omega| > 2\pi \times 10^3$rad/sec.
- Recall that:

$X_{DA}(\omega) = \left(\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T_s})\right) \cdot \text{sinc}\left(\frac{\omega T_s}{2}\right)$

$T_s = 0.2 \times 10^{-3}$

- Therefore,

$$H(\omega) = \frac{\Pi\left(\frac{|\omega|}{4\pi\times 10^3}\right)}{\text{sinc}(10^{-4}\omega)}$$