Signals and Systems

Tutorial Sheet 6 – Fourier Transform

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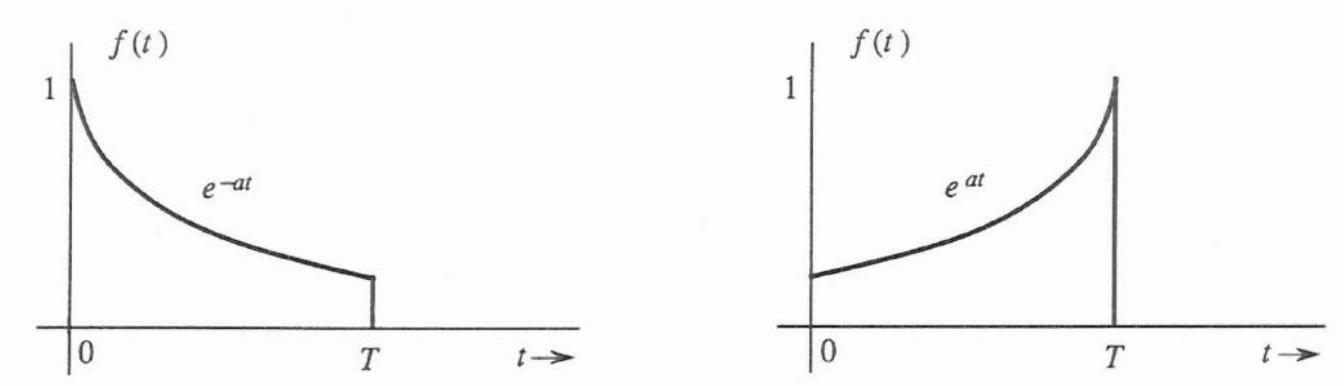
Problem 1

Derive the Fourier transform of the signals $f(t)$ shown in Figures below.

(a) $F(\omega) = \int_0^T e^{-at} e^{-j\omega t} dt = \int_0^T e^{-(a+j\omega)t} dt = \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_0^T =$
$\frac{-1}{(a+j\omega)} \left(e^{-(a+j\omega)T} - e^{-(a+j\omega)0} \right) = \frac{1}{(a+j\omega)} \left(1 - e^{-(a+j\omega)T}\right)$

(b) $F(\omega) = \int_0^T e^{at} e^{-j\omega t} dt = \int_0^T e^{(a-j\omega)t} dt = \frac{1}{(a-j\omega)} e^{(a-j\omega)t} \Big|_0^T =$
$\frac{1}{(a-j\omega)} \left(e^{(a-j\omega)T} - e^{(a-j\omega)0} \right) = \frac{1}{(a-j\omega)} \left(e^{(a-j\omega)T} - 1\right)$



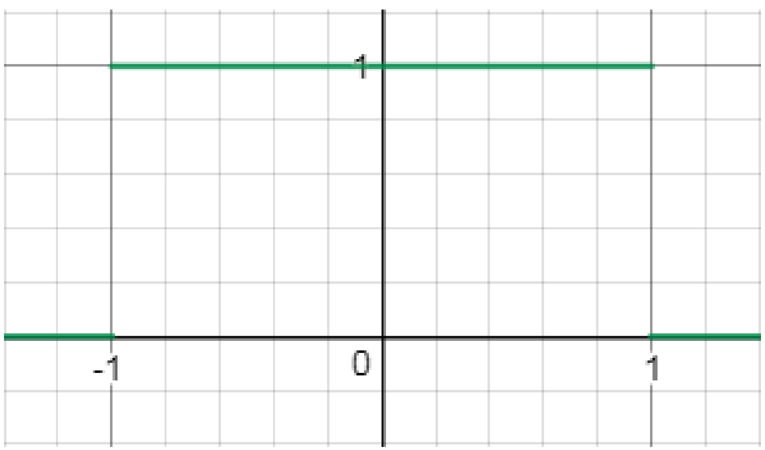
(a) (b)

Problem 2 (a), (b)

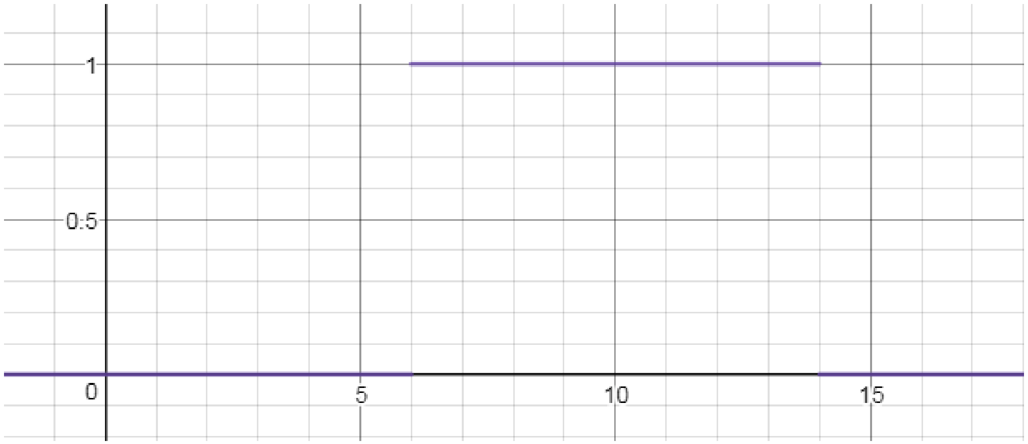
Sketch the following functions:

(a) $\text{rect}(\frac{t}{2})$

(b) $\text{rect}(\frac{t-10}{8})$



(a)



(b)

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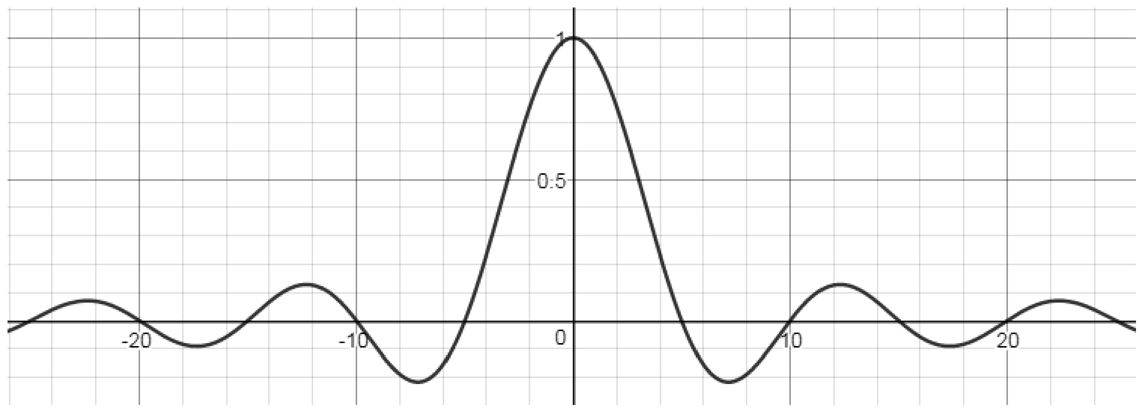
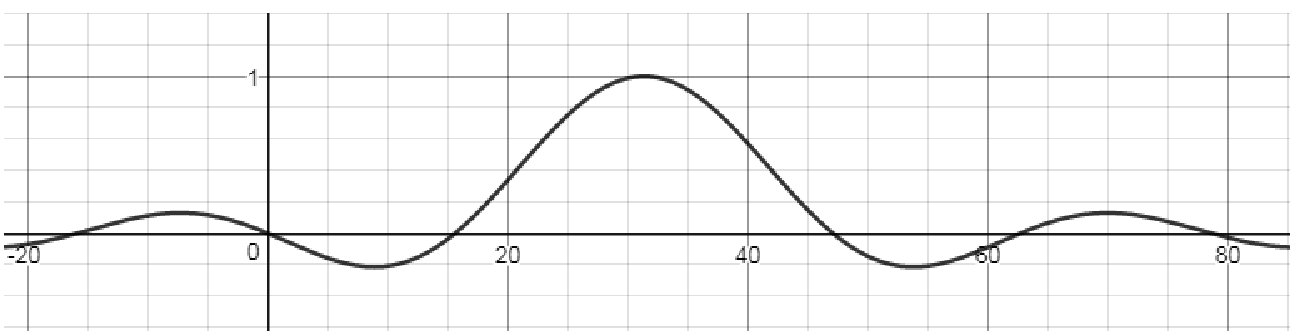
Problem 2 (c), (d)

(c) $\operatorname{sinc}\left(\frac{\pi\omega}{5}\right)$



(d) $\operatorname{sinc}\left(\frac{\omega-10\pi}{5}\right)$



Duality property

- If $x(t) \Leftrightarrow X(\omega)$ then $X(t) \Leftrightarrow 2\pi x(-\omega)$

Proof

From the definition of the inverse Fourier transform we get:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Therefore,

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega$$

Swapping t with ω and using the definition of forward Fourier transform we have:

$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

Problem 3 (a)

Apply the duality property to the appropriate function and show that:

$$\frac{1}{2}[\delta(t)+\frac{j}{\pi t}] \Leftrightarrow u(\omega)$$

- We proved previously that the following time-frequency relationship holds:

$$u(t) \Leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$$

- By applying the **duality property** we obtain: $\pi\delta(t)+\frac{1}{jt} \Leftrightarrow 2\pi u(-\omega)$ (1)
- By applying the **scaling property** $x(at) \Leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

  with $a=-1$ in (1) we obtain: $\pi\delta(-t)-\frac{1}{jt} \Leftrightarrow 2\pi u(\omega)$ (2)
- Relation (2) can also be written as $\pi\delta(-t)-\frac{(-j)}{j(-j)t}=\pi\delta(-t)+\frac{j}{t} \Leftrightarrow 2\pi u(\omega)$
- Knowing that $\delta(-t)=\delta(t)$ and by dividing both sides of the last relationship with 2π we obtain:

$$\frac{1}{2}[\delta(t)+\frac{j}{\pi t}] \Leftrightarrow u(\omega)$$

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Problem 3 (b)

Apply the duality property to the appropriate function and show that:

$$\frac{1}{t} \Leftrightarrow -j\pi\,\text{sign}(\,\omega)$$

- We proved previously that the following time-frequency relationship holds:

$$\mathcal{F}[\text{sign}(t)] = \frac{2}{j\omega}$$

- By applying the duality property we obtain: $\frac{2}{jt} \Leftrightarrow 2\pi\text{sign}(-\omega)$ (1)
- Since $\text{sign}(-\omega) = -\text{sign}(\omega)$, relation (1) can also be written as:

$$\frac{2}{jt} \Leftrightarrow -2\pi\text{sign}(\omega)$$

- By multiplying both sides of the last relationship with $j/2$ we obtain:

$$\frac{1}{t} \Leftrightarrow -j\pi\,\text{sign}(\,\omega)$$

Problem 3 (c)

Apply the duality property to the appropriate function and show that:

$$\delta(t+T)-\delta(t-T) \Leftrightarrow 2j\sin(T\omega)$$

- The following time-frequency relationship holds:

$$\sin\omega_0 t \Leftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

- By applying the duality property we obtain:

$$j\pi[\delta(t+\omega_0)-\delta(t-\omega_0)] \Leftrightarrow 2\pi\sin(-\omega_0\omega).$$

- We set $\omega_0 = T$ and we obtain:

$$j\pi[\delta(t+T)-\delta(t-T)] \Leftrightarrow 2\pi\sin(-T\omega).$$

- By multiplying both sides of the last relationship with $-j/\pi$ and using the fact that the $\mathbf{sin}$ function is odd we obtain:

$$\delta(t+T)-\delta(t-T) \Leftrightarrow 2j\sin(T\omega)$$

Problem 4

The Fourier transform of the triangular pulse $f(t)$ shown in Figure (a) below is **given** to be:

$$F(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1)$$

Use this information and the time-shifting and time-scaling properties to find the Fourier transforms of the signals $f_1(t)$ to $f_5(t)$ shown in Figures (b)-(f).

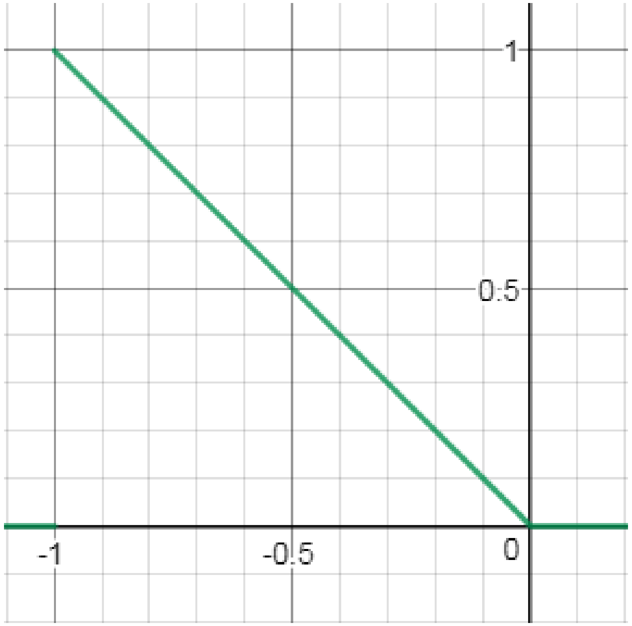

Figure (a)

Problem 4 cont.

- The signal $f_1(t)$ is shown in Figure (b) below.
- From Figure (b) we obtain $f_1(t) = f(-t)$.
- By applying the scaling property $x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ with $a = -1$ we obtain: $x(-t) \Leftrightarrow X(-\omega)$.
- Therefore, $F_1(\omega) = F(-\omega) = \frac{1}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1)$

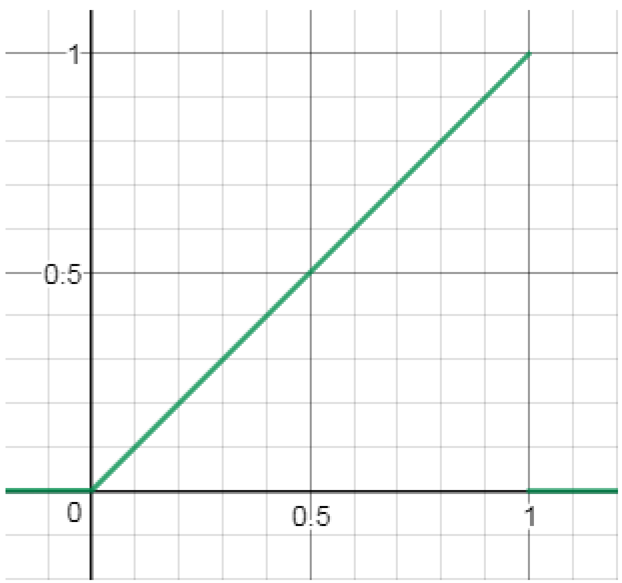

Figure (b)

Problem 4 cont.

From Figure (c) we obtain $f_2(t) = f(t-1) + f_1(t-1)$ and therefore,

$$F_2(\omega) = e^{-j\omega}F(\omega) + e^{-j\omega}F_1(\omega) = e^{-j\omega}F(\omega) + e^{-j\omega}F(-\omega)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + \frac{e^{-j\omega}}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1 + e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left[2\cos\omega + \left(-j\omega e^{j\omega}\right) + \left(-j\omega e^{j\omega}\right)^* - 2\right]$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(2\cos\omega + 2\mathrm{Re}\{-j\omega e^{j\omega}\} - 2\right)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(2\cos\omega + 2\omega\sin\omega - 2\right)$$

$$= \frac{2e^{-j\omega}}{\omega^2}\left(\cos\omega + \omega\sin\omega - 1\right)$$

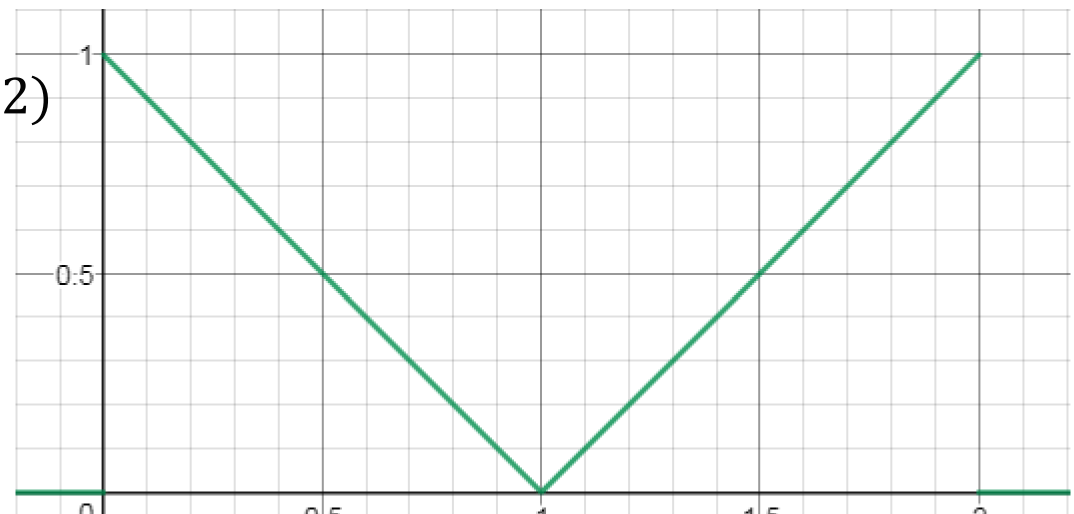

Figure (c)

Problem 4 cont.

From Figure (d) we obtain $f_3(t) = f(t-1) + f_1(t+1)$ and therefore,

$$F_3(\omega) = e^{-j\omega}F(\omega) + e^{j\omega}F_1(\omega) = e^{-j\omega}F(\omega) + e^{j\omega}F(-\omega)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + \frac{e^{j\omega}}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{1}{\omega^2}\left(1 - j\omega - e^{-j\omega}\right) + \frac{1}{\omega^2}\left(1 + j\omega - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}(2 - 2\cos\omega) = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right) = \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

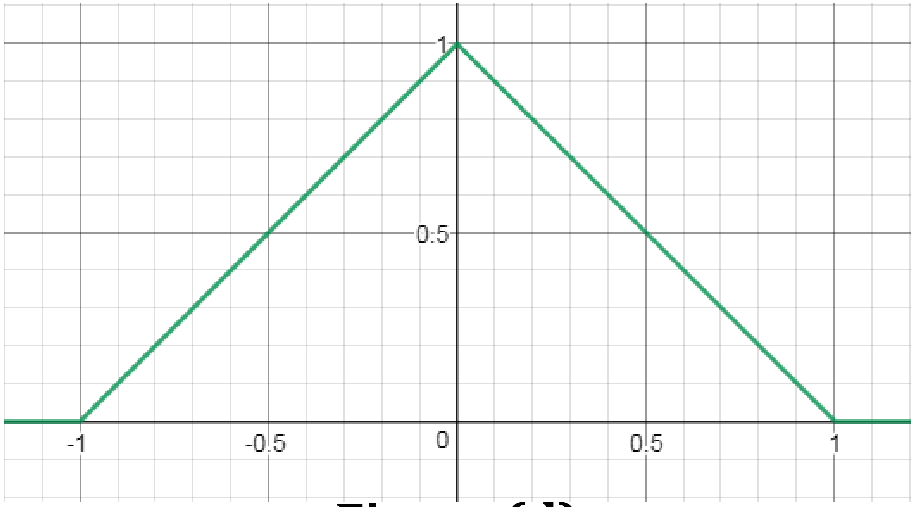

Figure (d)

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Problem 4 cont.

From Figure (e) we obtain $f_4(t) = f(t - 1/2) + f_1(t + 1/2)$ and therefore,

$$F_4(\omega) = e^{-j\omega/2}F(\omega) + e^{j\omega/2}F_1(\omega) = e^{-j\omega/2}F(\omega) + e^{j\omega/2}F(-\omega)$$

$$= \frac{e^{-j\omega/2}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + \frac{e^{j\omega/2}}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1)$$

$$= \frac{1}{\omega^2}\left(e^{j\omega/2} - j\omega e^{j\omega/2} - e^{-j\omega/2}\right) + \frac{1}{\omega^2}\left(e^{-j\omega/2} + j\omega e^{-j\omega/2} - e^{j\omega/2}\right)$$

$$= \frac{1}{\omega^2}\left[\left(-j\omega e^{j\omega/2}\right) + \left(-j\omega e^{j\omega/2}\right)^*\right]$$

$$= \frac{1}{\omega^2}\left(2\mathrm{Re}\left\{-j\omega e^{\frac{j\omega}{2}}\right\}\right)$$

$$= \frac{2}{\omega^2}\omega\sin(\omega/2) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) = \mathrm{sinc}(\omega/2)$$

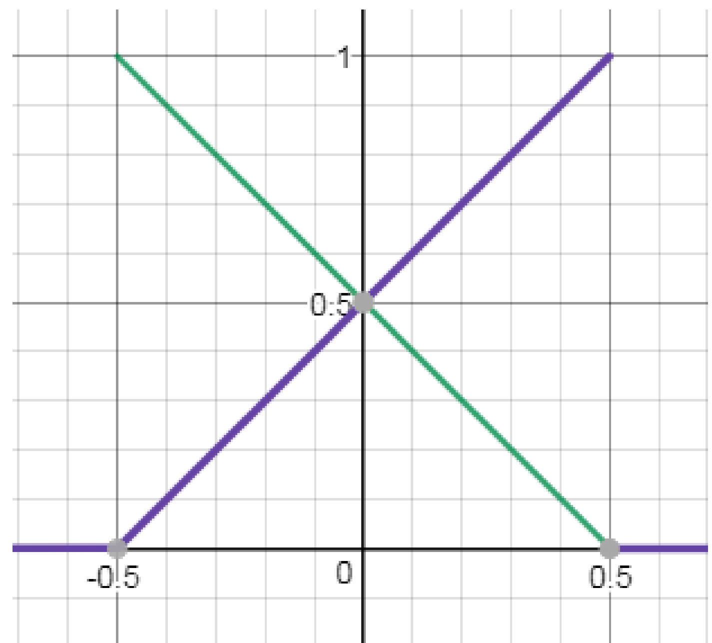

Figure (e)

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Problem 4 cont.

From Figure (f) we observe that we can obtain $f_5(t)$ after the following operations on $f(t)$.

- Expand $f(t)$ by a factor of 2.
  By applying the scaling property with $a = 2$ we obtain $f(t/2) \Leftrightarrow 2F(2\omega)$,
  $f(t/2) \Leftrightarrow 2F(2\omega)$ or $f(t/2) \Leftrightarrow \frac{1}{2\omega^2}(e^{j2\omega} - j2\omega e^{j2\omega} - 1)$
- Shift to the right $f(t/2)$ by 2 units of time.
  $$f((t-2)/2) \Leftrightarrow 2e^{-j2\omega}F(2\omega) = \frac{1}{2\omega^2}(1 - j2\omega - e^{-j2\omega})$$
- Multiply the result by 1.5.
  $f_5(t) = 1.5f((t-2)/2) \Leftrightarrow \frac{3}{4\omega^2}(1 - j2\omega - e^{-j2\omega})$
- Figure is shown in the next slide.

Problem 4 cont.

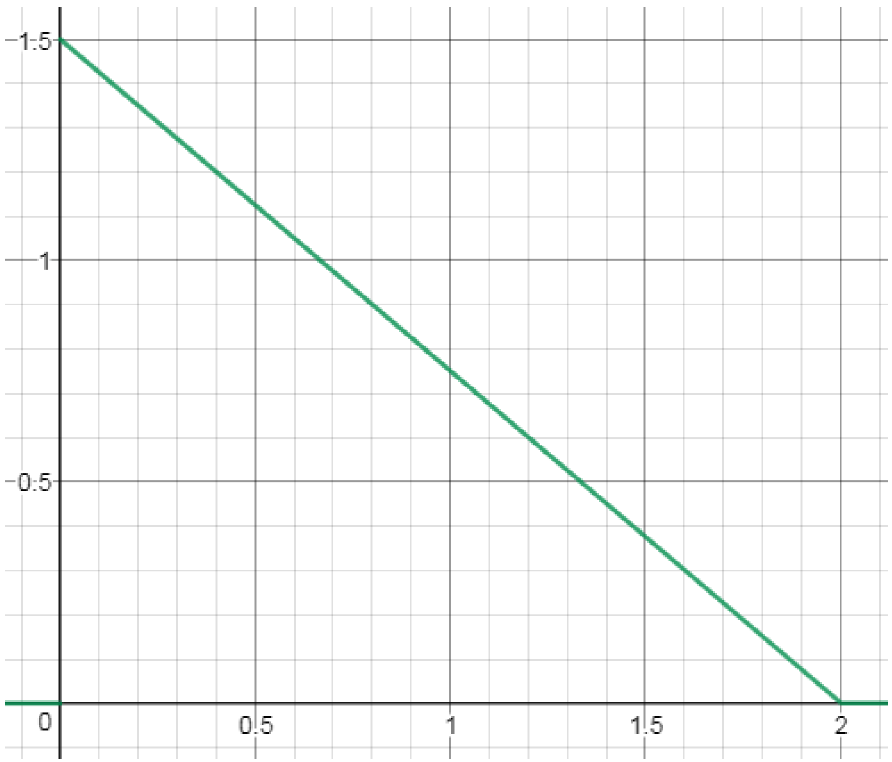


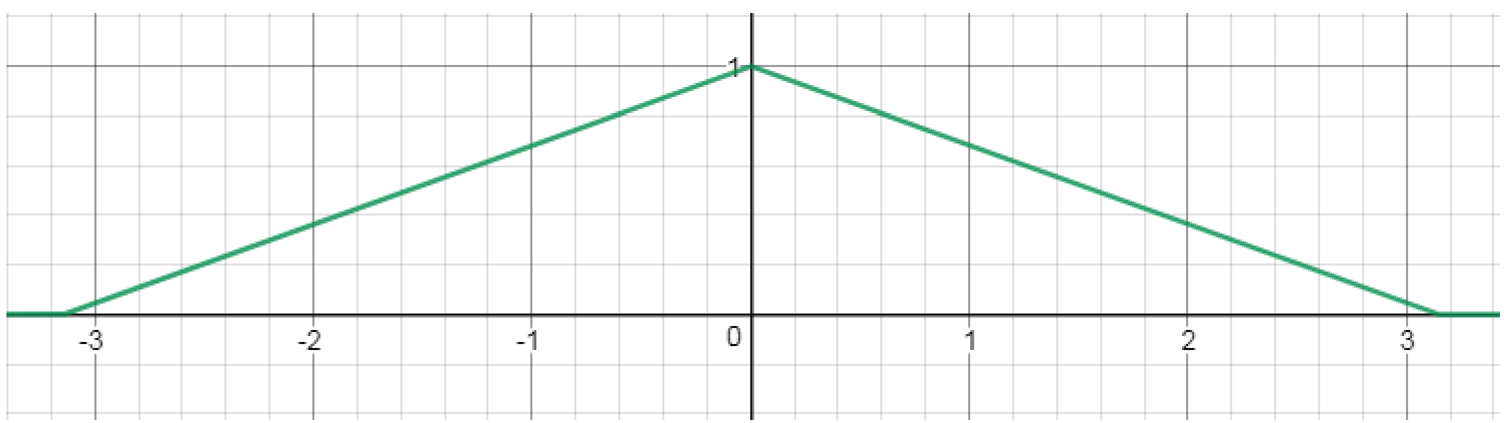
Figure (f)

Problem 5

The signals in Figures (a)-(c) are modulated signals with carrier $\cos(10t)$. Find the Fourier transforms of these signals using appropriate properties of the Fourier transform and the FT table given in Lectures. Sketch the amplitude and phase spectra for (a) and (b).

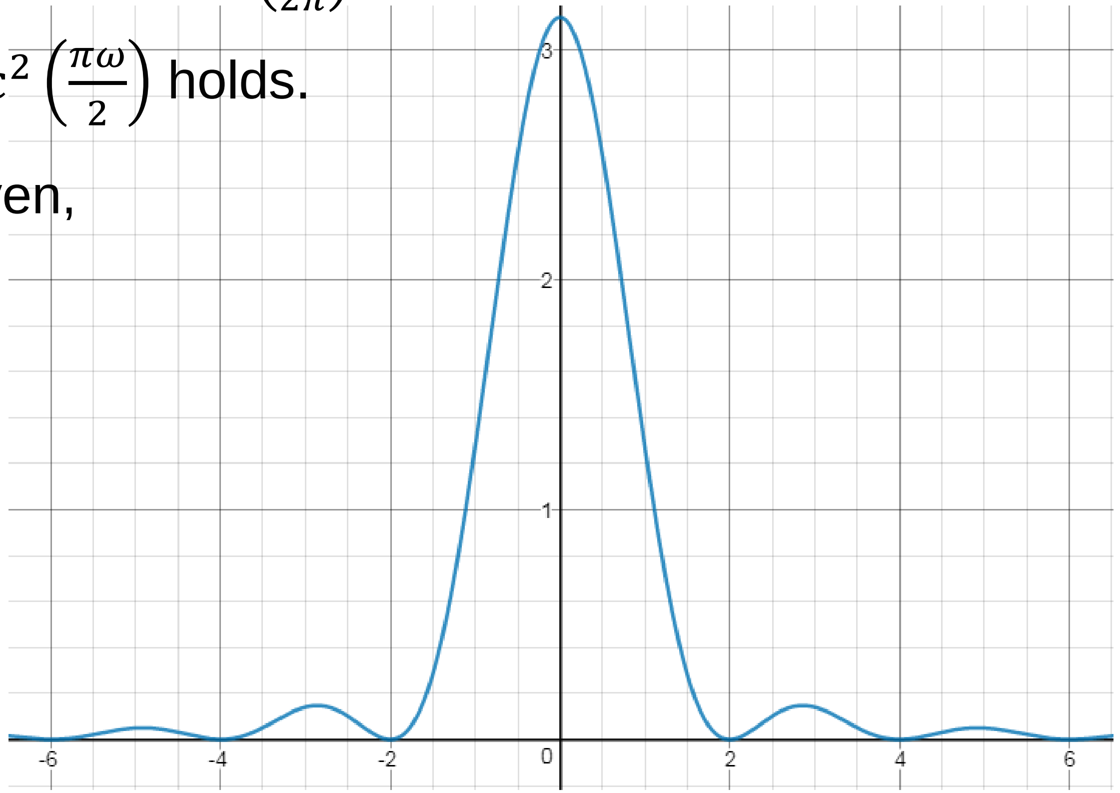
- A unit triangle function $\Delta(x)$ is defined as:

$$\Delta(x) = \begin{cases} 0 & |x| \geq \dfrac{1}{2} \\ 1 - 2|x| & |x| < \dfrac{1}{2} \end{cases}$$



Problem 5 (a)

- It can be shown that $\Delta\left(\frac{t}{\tau}\right) \Leftrightarrow \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4}\right)$
- In this problem we have the function $\Delta\left(\frac{t}{2\pi}\right)$ depicted in the previous slide.
- The relation $\Delta\left(\frac{t}{2\pi}\right) \Leftrightarrow \pi\operatorname{sinc}^2\left(\frac{\pi\omega}{2}\right)$ holds.
- Since the signal $\Delta\left(\frac{t}{2\pi}\right)$ is even, the Fourier transform is real **(prove it)** and therefore, the phase is zero.
- The amplitude is shown on the right.



Problem 5 (a) cont.

The modulated signal $\Delta\left(\frac{t}{2\pi}\right)\cos(10t)$ is depicted below.

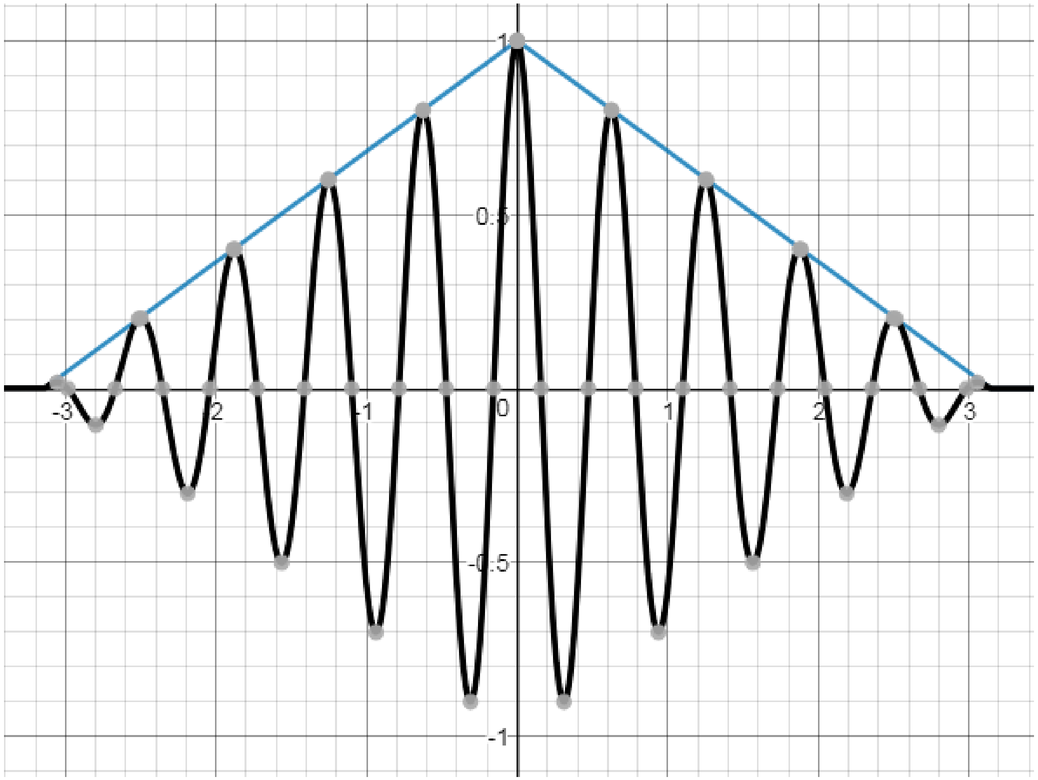
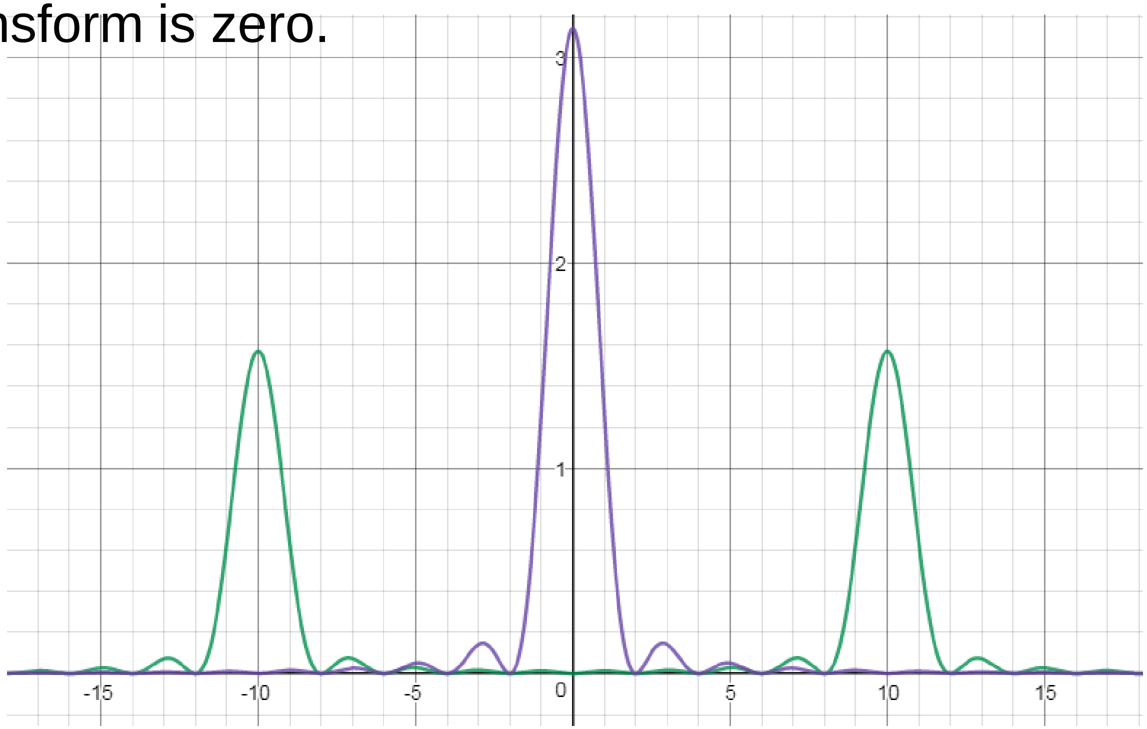


Figure (a)

Problem 5 (a) cont.

- The relationship $\Delta\left(\frac{t}{2\pi}\right)\cos(10t) \Leftrightarrow \frac{\pi}{2}\left[\operatorname{sinc}^2\left(\frac{\pi(\omega-10)}{2}\right) + \operatorname{sinc}^2\left(\frac{\pi(\omega+10)}{2}\right)\right]$ holds.
- The Fourier transform of the modulated signal is depicted in the Figure below with green.
- As previously, it is the same as the amplitude, since the phase of the Fourier transform is zero.



Problem 5 (b)

- Figure (b) is a shift of Figure (a) by 2π.

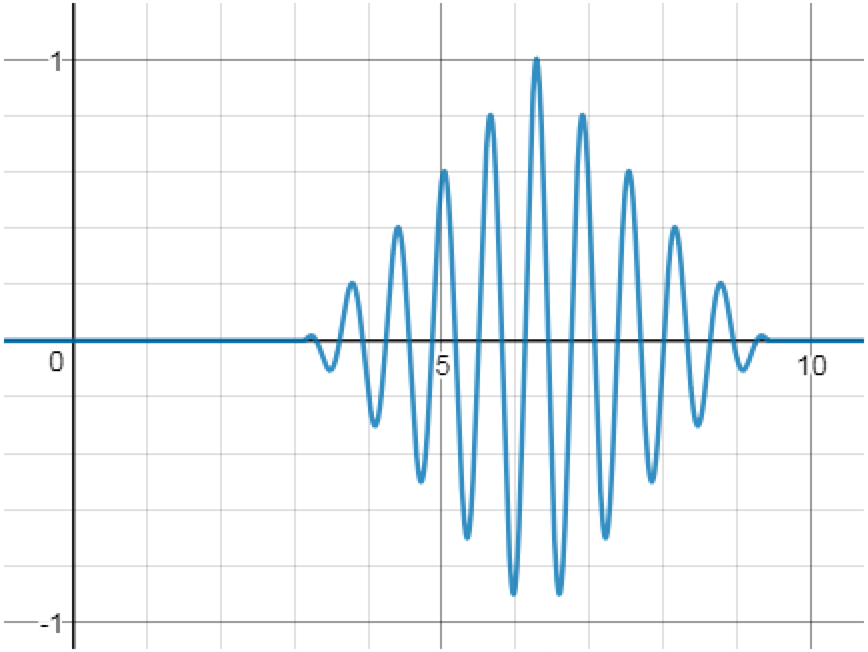


Figure (b)

- $\Delta\left(\frac{t-2\pi}{2\pi}\right)\cos(10(t-2\pi)) \Leftrightarrow \frac{\pi}{2}[\mathrm{sinc}^2\left(\frac{\pi(\omega-10)}{2}\right)+\mathrm{sinc}^2\left(\frac{\pi(\omega+10)}{2}\right)]e^{-j2\pi\omega}$
- The amplitude of the Fourier transform remains the same but a linear phase $-2\pi\omega$ is introduced.

Problem 5 (c)

- Figure (c) depicts a rectangular pulse $\mathrm{rect}\left(\frac{t-2\pi}{2\pi}\right)\cos(10(t-2\pi))$
- It can be shown that

$$\mathrm{rect}\left(\frac{t}{2\pi}\right) \Leftrightarrow 2\pi\mathrm{sinc}(\pi\omega)$$

$$\mathrm{rect}\left(\frac{t}{2\pi}\right)\cos(10t) \Leftrightarrow \frac{1}{2}2\pi(\mathrm{sinc}[\pi(\omega-10)]+\mathrm{sinc}[\pi(\omega-10)])$$

$$\mathrm{rect}\left(\frac{t}{2\pi}\right)\cos(10t) \Leftrightarrow \pi(\mathrm{sinc}[\pi(\omega-10)]+\mathrm{sinc}[\pi(\omega-10)])$$

- The given signal is the same as the above signal but delayed by 2π. This delay introduces the phase $e^{-j2\pi\omega}$.

$$\mathrm{rect}\left(\frac{t-2\pi}{2\pi}\right)\cos(10(t-2\pi)) \Leftrightarrow \pi(\mathrm{sinc}[\pi(\omega-10)]+\mathrm{sinc}[\pi(\omega-10)])\; e^{-j2\pi\omega}$$

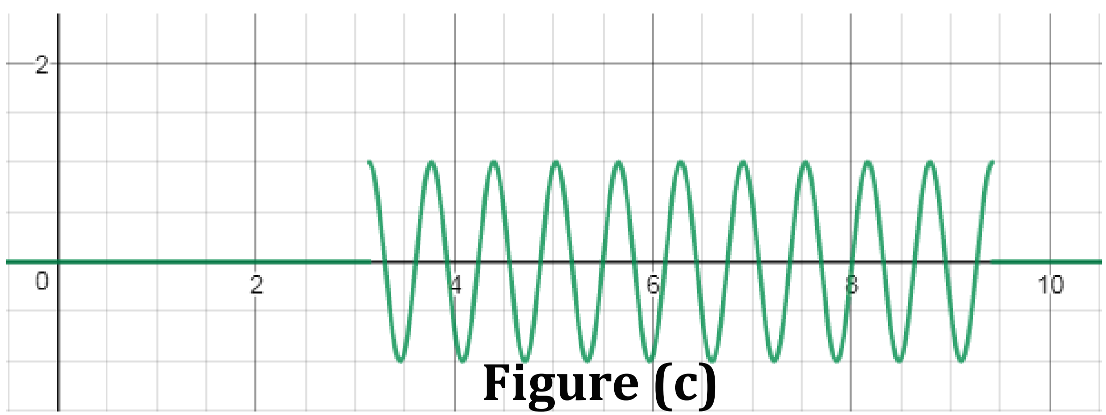


Figure (c)