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Signals and Systems

Tutorial Sheet 3 - Convolution

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Problem 1 (i)

Find the convolution:

(i) $y(t) = u(t) * u(t)$, $u(t)$ is the unit step function

We will use the definition:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$$

METHOD

In all questions we will find the range of values of τ for which both functions inside the integral are non zero. Remember that one of the functions is reversed and shifted.

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_{0}^{t} u(\tau)u(t-\tau)d\tau = \int_{0}^{t} d\tau = t$, $t \geq 0$.

Hence, $y(t) = tu(t)$ (easy to plot).

Problem 1 (ii)

Find the convolution:

(ii) $y(t) = e^{-at}u(t) * e^{-bt}u(t)$

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau$$

$$= \int_0^t e^{-a\tau}e^{-b(t-\tau)}d\tau = e^{-bt}\int_0^t e^{-a\tau}e^{b\tau}d\tau = e^{-bt}\int_0^t e^{-(a-b)\tau}d\tau$$

$$= \frac{e^{-bt}}{-(a-b)}e^{-(a-b)\tau}\Big|_0^t = \frac{e^{-bt}}{-(a-b)}\left(e^{-(a-b)t} - 1\right) = \frac{1}{-(a-b)}\left(e^{-at} - e^{-bt}\right), t \geq 0$$

Hence,

$$y(t) = \frac{e^{-at}-e^{-bt}}{b-a}u(t)$$

Problem 1 (iii)

Find the convolution:

(iii) $y(t) = tu(t) * u(t)$

$$y(t) = \int_{-\infty}^{\infty} \tau u(\tau) u(t-\tau) d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$$y(t) = \int_{-\infty}^{\infty} \tau u(\tau) u(t-\tau) d\tau = \int_{0}^{t} \tau u(\tau) u(t-\tau) d\tau$$

$$= \int_{0}^{t} \tau d\tau = \frac{\tau^2}{2}\bigg|_{0}^{t} = \frac{t^2}{2}, t \geq 0$$

Hence,

$$y(t) = \frac{t^2}{2} u(t)$$

Problem 2 (i)

Find the convolution:

(i) $y(t) = (\sin(t)u(t)) * u(t)$

$$y(t) = \int_{-\infty}^{\infty} \sin(\tau)u(\tau)u(t-\tau)d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$y(t) = \int_{-\infty}^{\infty} \sin(\tau)u(\tau)u(t-\tau)d\tau = \int_{0}^{t} \sin(\tau)u(\tau)u(t-\tau)d\tau$

$= \int_{0}^{t} \sin(\tau)d\tau = -\cos(\tau)|_{0}^{t} = -(\cos(t) - \cos(0)) = 1 - \cos(t), t \geq 0$

Hence,

$y(t) = (1 - \cos(t))u(t)$

Problem 2 (ii)

Find the convolution:

(ii) $y(t) = (\cos(t)u(t)) * u(t)$

$$y(t) = \int_{-\infty}^{\infty} \cos(\tau)u(\tau)u(t-\tau)d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$$y(t) = \int_{-\infty}^{\infty} \cos(\tau)u(\tau)u(t-\tau)d\tau = \int_{0}^{t} \cos(\tau)u(\tau)u(t-\tau)d\tau$$

$$= \int_{0}^{t} \cos(\tau)d\tau = \sin(\tau)|_{0}^{t} = \sin(t) - \sin(0) = \sin(t), t \geq 0$$

Hence,

$$y(t) = \sin(t)u(t)$$

Problem 3 (a)

(a) The unit impulse response of an LTI system is $h(t) = e^{-t}u(t)$. Find this system's zero-state response $y(t)$ if the input is $f(t) = u(t)$.

$$y(t) = e^{-t}u(t) * u(t)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t - \tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau = \int_0^t e^{-\tau}u(\tau)u(t-\tau)d\tau$$

$$= \int_0^t e^{-\tau}d\tau = -e^{-\tau}|_0^t = -(e^{-t} - 1) = 1 - e^{-t}, t \geq 0$$

Hence,

$$y(t) = (1 - e^{-t})u(t)$$

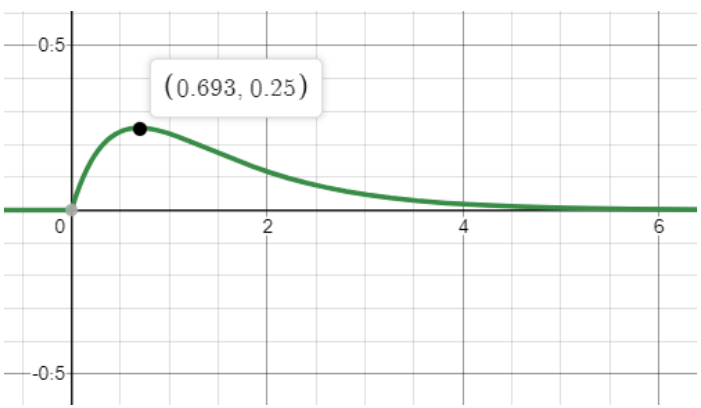
Problem 3 (b)

(b) The unit impulse response of an LTI system is $h(t) = e^{-t}u(t)$. Find this system's zero-state response $y(t)$ if the input is $f(t) = e^{-2t}u(t)$.

$y(t) = e^{-t}u(t) * e^{-2t}u(t)$
For this question we can refer to Question 1(ii) with $a = 1, b = 2$.
We see immediately that:

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

The convolution is shown at the bottom figure with maximum of 0.25 at $0.693 = \ln(2)$.



Problem 3 (c)

(c) The unit impulse response of an LTI system is $h(t) = e^{-t}u(t)$. Use Integration Tables to find this system's zero-state response $y(t)$ if the input is $f(t) = \sin(3t)u(t)$.

$y(t) = e^{-t}u(t) * \sin(3t)u(t)$

$$y(t) = \int_{-\infty}^{\infty} \sin(3\tau)\, u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$u(\tau) \neq 0$ if $\tau \geq 0$ (1)

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ (2)

Therefore, from (1) and (2) we form the condition $0 \leq \tau \leq t$. This condition makes sense if $t \geq 0$.

$y(t) = \int_{-\infty}^{\infty} \sin(3\tau)\, u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_0^t \sin(3\tau)e^{-(t-\tau)}d\tau$

$$= e^{-t}\int_0^t \sin(3\tau)e^{\tau}d\tau$$

The solution continues on the next slide. To find the expression for the integral you can use the site

https://en.wikipedia.org/wiki/List_of_integrals_of_exponential_functions

Problem 3 (c) cont.

From Tables of integrals involving exponential and trigonometric functions (see link at the end of previous slide) we have:

$$\int \sin(b\tau)e^{a\tau}d\tau = \frac{e^{a\tau}}{a^2+b^2}\left(a\sin(b\tau) - b\cos(b\tau)\right) = \frac{e^{a\tau}}{\sqrt{a^2+b^2}}\sin(b\tau - \phi)$$

where $\cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}$

For $a = 1, b = 3$ we have $\cos(\phi) = \frac{1}{\sqrt{10}} \Rightarrow \phi = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) = 71.565^{o}$

Therefore, $\int \sin(3\tau)e^{\tau}d\tau = \frac{e^{\tau}}{\sqrt{10}}\sin(3\tau - 71.565^{o})$.

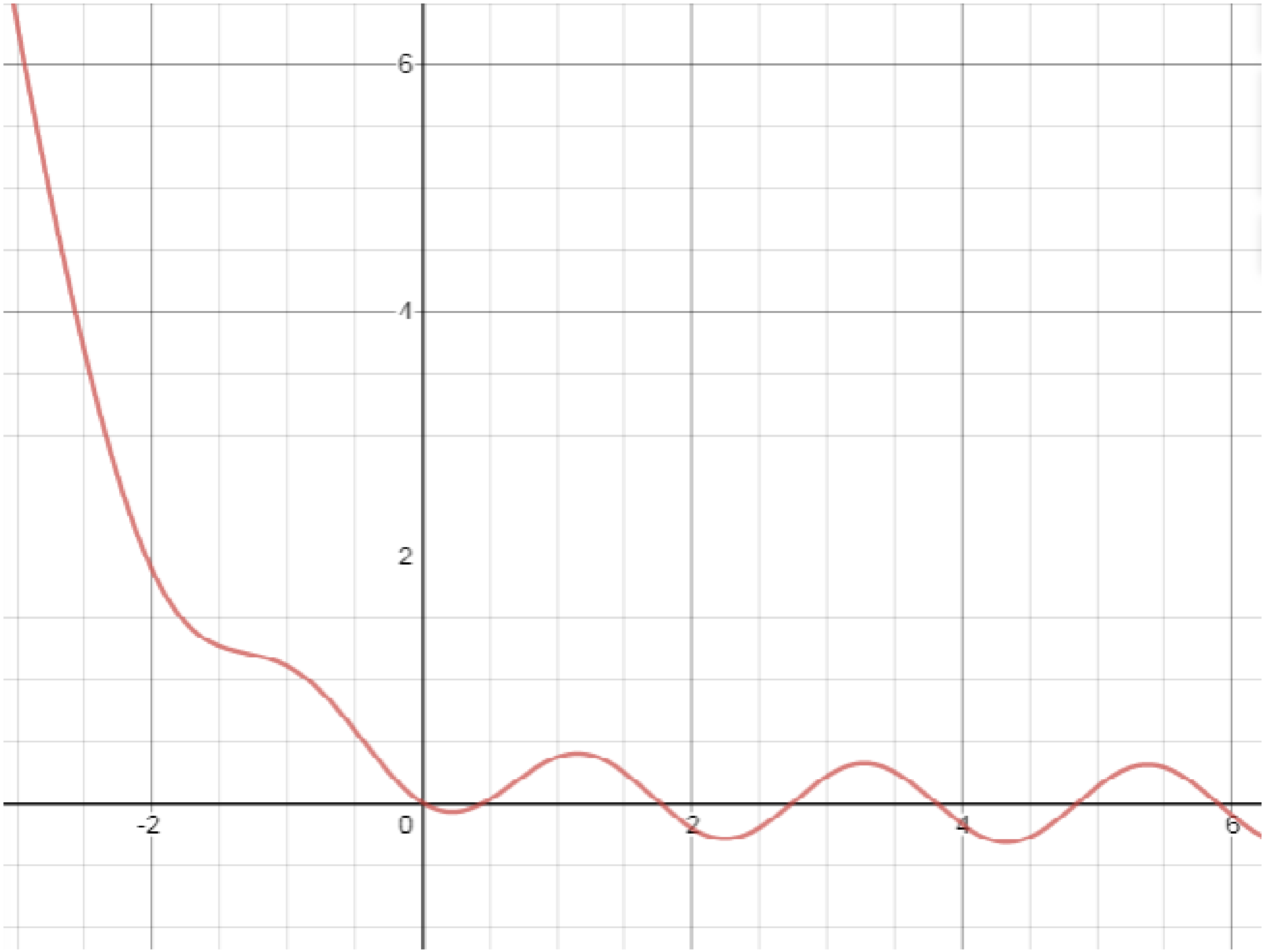
$$y(t) = e^{-t}\int_0^t \sin(3\tau)e^{\tau}d\tau = e^{-t}\left(\frac{e^{t}}{\sqrt{10}}\sin(3t - 71.565^{o}) - \frac{1}{\sqrt{10}}\sin(-71.565^{o})\right)$$

$$= \frac{1}{\sqrt{10}}\sin(3t - 71.565^{o}) + \frac{e^{-t}}{\sqrt{10}}\sin(71.565^{o})$$

$$= \frac{1}{\sqrt{10}}\sin(3t - 71.565^{o}) + \frac{0.9486e^{-t}}{\sqrt{10}} = -\frac{1}{\sqrt{10}}\cos\left(3t - 71.565^{o} + \frac{\pi}{2}\right) + \frac{0.9486e^{-t}}{\sqrt{10}}$$

$$= -\frac{1}{\sqrt{10}}\cos(3t - 71.565^{o} + 90^{o}) + \frac{0.9486e^{-t}}{\sqrt{10}} = -\frac{1}{\sqrt{10}}\cos(3t + 18.435^{o}) + \frac{0.9486e^{-t}}{\sqrt{10}}$$

Problem 3 (c) cont.



Problem 4

By applying the shift property of convolution, find the system's response $y(t)$ (i.e. zero-state response) given that $h(t) = e^{-t}u(t)$ and that the input $f(t)$ is as shown in figure below.

Solution

We observe that the input is $f(t) = u(t) - u(t-1)$.

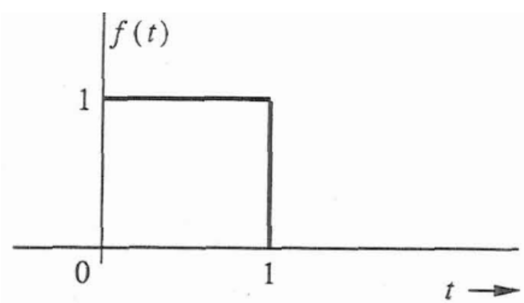
$y(t) = e^{-t}u(t) * \big(u(t) - u(t-1)\big) = e^{-t}u(t) * u(t) - e^{-t}u(t) * u(t-1)$

In Problem 3(a) we proved that $e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)$.

Therefore, from the shift property of convolution (**See Slide 15 Lecture 4**)

$e^{-t}u(t) * u(t-1) = (1 - e^{-(t-1)})u(t-1)$.

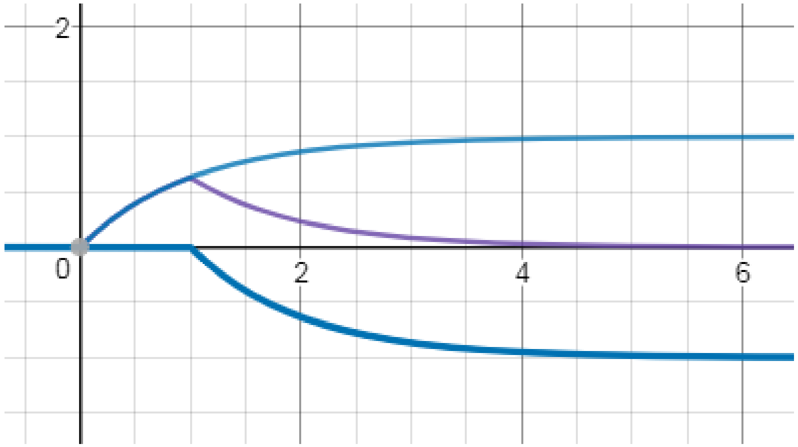
$y(t) = e^{-t}u(t) * \big(u(t) - u(t-1)\big) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$.



Problem 4 cont.

$$y(t) = e^{-t}u(t) * \big(u(t) - u(t-1)\big) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1).$$

- The function $(1 - e^{-t})u(t)$ is the positive blue curve shown in figure below.
- The function $-(1 - e^{-(t-1)})u(t-1)$ is the negative blue curve shown in figure below.
- The required function $y(t)$ is the purple and green curve shown in figures below.



Problem 5 (a)

A first-order allpass filter impulse response is given by

$$h(t) = -\delta(t) + 2e^{-t}u(t)$$

(a) Find the zero-state response of this filter for the input $e^{t}u(-t)$.

$y(t) = (-\delta(t) + 2e^{-t}u(t)) * e^{t}u(-t) = -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t) = -e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t)$

Let's focus on $2e^{-t}u(t) * e^{t}u(-t)$

$$2e^{-t}u(t) * e^{t}u(-t) = \int_{-\infty}^{\infty} e^{\tau}u(-\tau)2e^{-(t-\tau)}u(t-\tau)d\tau$$

$u(-\tau) \neq 0$ if $-\tau \geq 0 \Rightarrow \tau \leq 0$ **(1)**

$u(t-\tau) \neq 0$ if $t-\tau \geq 0 \Rightarrow \tau \leq t$ **(2)**

For $t \geq 0$ the intersection of conditions (1) and (2) is

$(\tau \leq 0) \cap (\tau \leq t) = (\tau \leq 0)$

$$2e^{-t}u(t) * e^{t}u(-t) = \int_{-\infty}^{0} e^{\tau}u(-\tau)2e^{-(t-\tau)}u(t-\tau)d\tau = \int_{-\infty}^{0} e^{\tau}2e^{-(t-\tau)}d\tau$$

$$= e^{-t}\int_{-\infty}^{0} 2e^{2\tau}d\tau = e^{-t}\, e^{2\tau}\Big|_{-\infty}^{0} = e^{-t}$$

Problem 5 cont.

For $t < 0$ the intersection of conditions (1) and (2) is
$(\tau \leq 0) \cap (\tau \leq t) = (\tau \leq t)$

$$2e^{-t}u(t) * e^{t}u(-t) = \int_{-\infty}^{t} e^{\tau}u(-\tau)2e^{-(t-\tau)}u(t-\tau)d\tau = \int_{-\infty}^{t} e^{\tau}2e^{-(t-\tau)}d\tau$$

$$= e^{-t}\int_{-\infty}^{t} 2e^{2\tau}d\tau = e^{-t}\, e^{2\tau}\Big|_{-\infty}^{t} = e^{t}$$

Therefore,

$$2e^{-t}u(t) * e^{t}u(-t) = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

$$-\delta(t) * e^{t}u(-t) = -e^{t}u(-t)$$

$$-\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t) = \begin{cases} -e^{t}u(-t)+e^{t} & t < 0 \\ -e^{t}u(-t)+e^{-t} & t \geq 0 \end{cases}$$

$$= \begin{cases} -e^{t}+e^{t} = 0 & t < 0 \\ 0 + e^{-t} = e^{-t} & t \geq 0 \end{cases}$$

Hence, $y(t) = e^{-t}u(t)$.

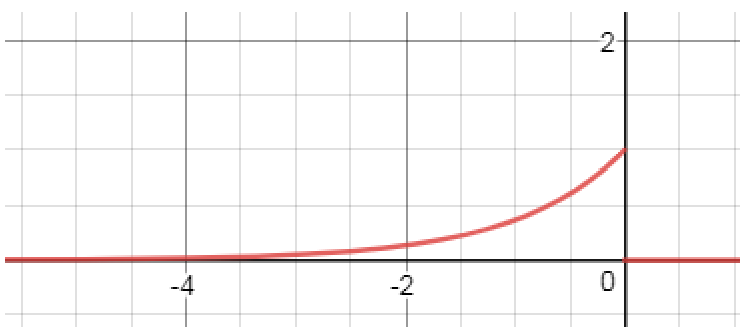
Problem 5 cont.

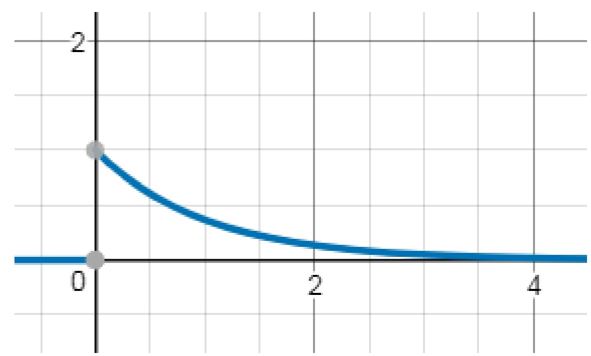
A first-order allpass filter impulse response is given by

$$h(t) = -\delta(t) + 2e^{-t}u(t)$$

(b) Sketch the input and the corresponding zero-state response.

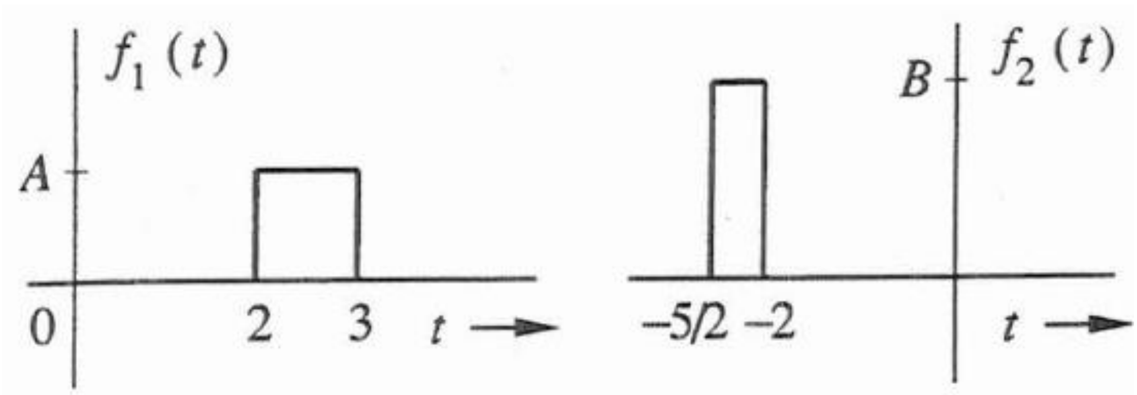
- The input $e^t u(-t)$ is the red curve shown in figure below left.
- The zero-state response $y(t) = e^{-t}u(t)$ is the blue curve shown in figure below right.





Problem 6 (a)

(a) Find and sketch the convolution:
$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$



$f_1(t) = Au(t-2) - Au(t-3)$
$f_2(t) = Bu(-t-2) - Bu\left(-t-\frac{5}{2}\right)$

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Problem 6 (a) cont.

(a) Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

$f_1(t) = Au(t-2) - Au(t-3)$ and $f_2(t) = Bu(-t-2) - Bu\left(-t-\frac{5}{2}\right)$

$f_1(t) = \begin{cases} A & 2 \le t \le 3 \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(t) = \begin{cases} B & -2.5 \le t \le -2 \\ 0 & \text{elsewhere} \end{cases}$

$f_1(\tau) \neq 0 \Rightarrow 2 \le \tau \le 3 \qquad (1)$

$f_2(t-\tau) \neq 0 \Rightarrow -2.5 \le t-\tau \le -2 \Rightarrow 2 \le \tau - t \le 2.5$

$\Rightarrow t+2 \le \tau \le t+2.5 \qquad (2)$

- **Condition (1) forms Interval 1 shown in blue line below with fixed bounds 2 and 3.**
- **Condition (2) forms Interval 2 shown in red line below with moving bounds $t+2$ and $t+2.5$. Interval 2 is narrower than Interval 1.**
- **The green line below represents the variable of integration τ.**

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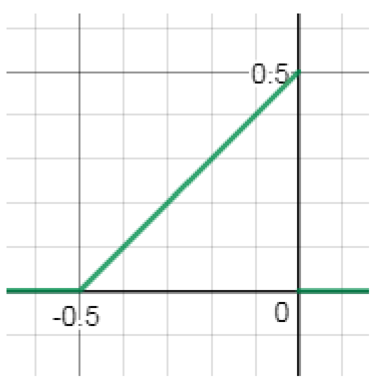
Problem 6 (a) cont.

Scenario I:

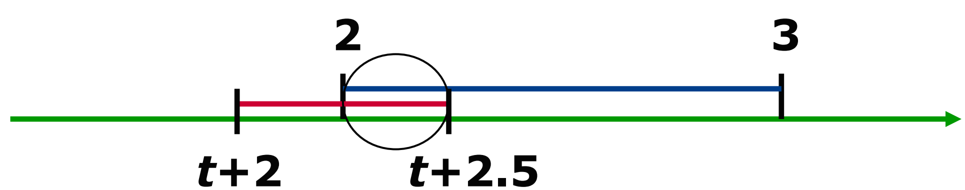
- **Interval 2** overlaps with **Interval 1** from the left.
- This means that the upper bound of **Interval 2** lies within **Interval 1** and the lower bound of **Interval 2** is outside **Interval 1**.
  - The above implies that $t + 2.5 \geq 2 \Rightarrow t \geq -0.5$ and $t + 2 \leq 2 \Rightarrow t \leq 0$ and therefore, by combining the above conditions we obtain $-0.5 \leq t \leq 0$.
- The overlapping area is from 2 to $t + 2.5$. It is highlighted with a circle in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario I.

$$f_1(t) * f_2(t) = \int_2^{t+2.5} f_1(\tau) f_2(t-\tau) d\tau = AB \int_2^{t+2.5} d\tau = AB(t+0.5),$$

$-0.5 \leq t \leq 0$



Note that the amplitude $AB = 1$ in the figure left.

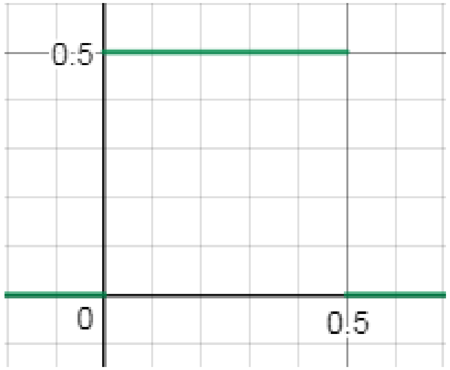


Problem 6 (a) cont.

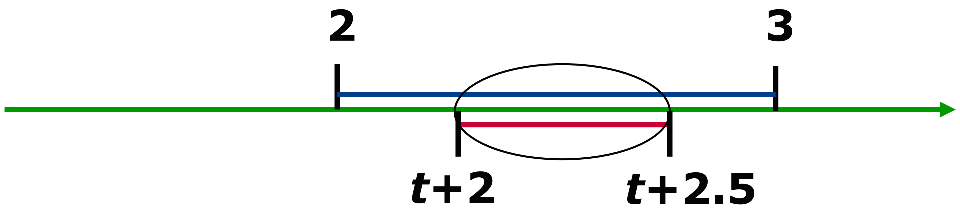
Scenario II:

- **Interval 2** lies within **Interval 1**.
- This means that both the upper and lower bounds of **Interval 2** lie within **Interval 1**.
  The above implies that $t+2 \geq 2 \Rightarrow t \geq 0$ and $t+2.5 \leq 3 \Rightarrow t \leq 0.5$, and therefore, by combining the above conditions we obtain $0 \leq t \leq 0.5$
- The overlapping area is from $t+2$ to $t+2.5$. It is highlighted with an oval in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario II.

$$f_1(t) * f_2(t) = \int_{t+2}^{t+2.5} f_1(\tau) f_2(t-\tau) d\tau = \int_{t+2}^{t+2.5} AB d\tau = \frac{AB}{2}, 0 \leq t \leq 0.5$$



(shown left with $AB = 1$)

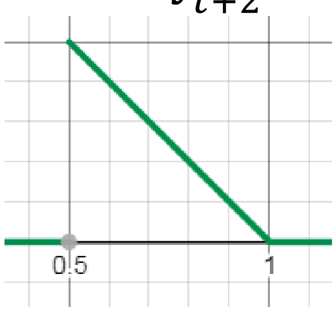


Problem 6 (a) cont.

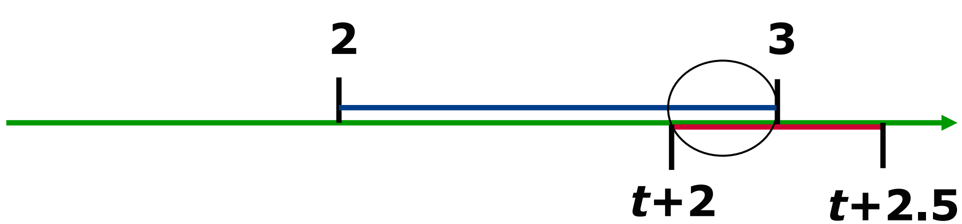
Scenario III:

- **Interval 2** overlaps with **Interval 1** from the right.
- This means that the lower bound of **Interval 2** lies within **Interval 1** and the upper bound of **Interval 2** is outside **Interval 1**.
- The above implies that $t+2 \leq 3 \Rightarrow t \leq 1$ and $t+2.5 \geq 3 \Rightarrow t \geq 0.5$ and therefore, by combining the two conditions we obtain $0.5 \leq t \leq 1$.
- The overlapping area is from $t+2$ to 3. It is highlighted with a circle in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario III.

$$f_1(t) * f_2(t) = \int_{t+2}^{3} f_1(\tau) f_2(t-\tau) d\tau = \int_{t+2}^{3} AB d\tau = AB(1-t), 0.5 \leq t \leq 1$$

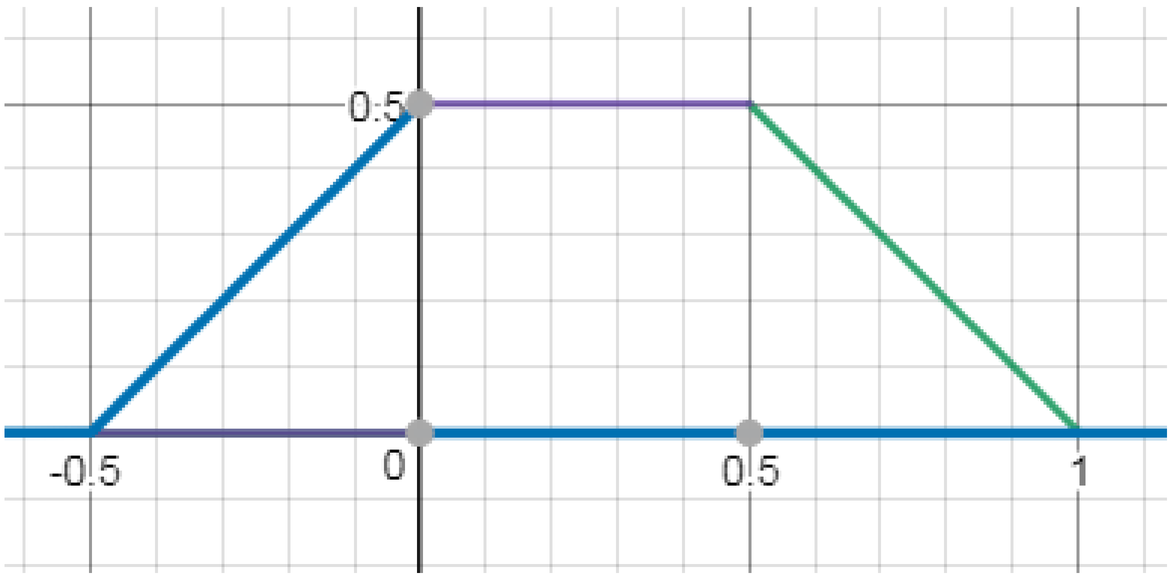


(shown left with $AB = 1$)



Problem 6 (a) cont.

By combining Scenarios I, II and III above, we obtain the result of convolution for the entire range of time as depicted in the figure below.



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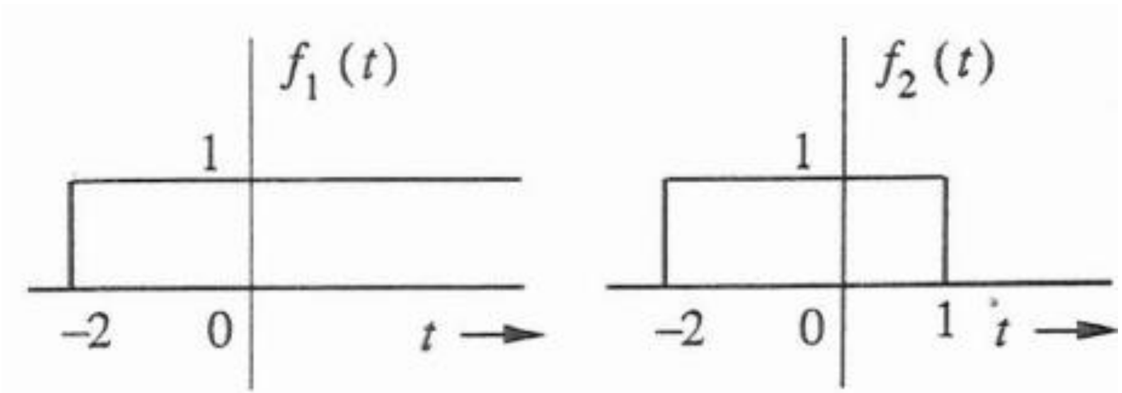
Problem 6 (b)

(b) Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$



$f_1(t) = u(t+2)$

$f_2(t) = u(t+2) - u(t-1)$

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Problem 6 (b)

(b) Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

$f_1(t) = u(t+2)$ and $f_2(t) = u(t+2) - u(t-1)$

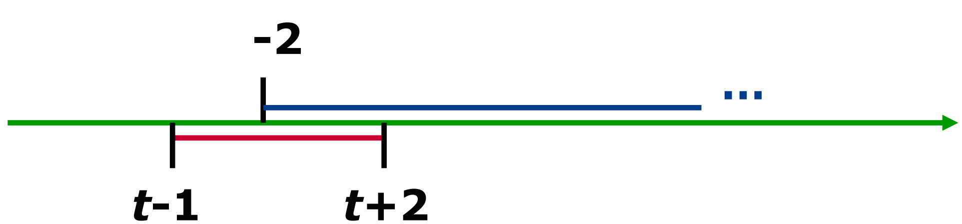
$f_1(t) = \begin{cases} 1 & t+2 \geq 0 \Rightarrow t \geq -2 \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(t) = \begin{cases} 1 & -2 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

$f_1(\tau) \neq 0 \Rightarrow -2 \leq \tau$ (1)

$f_2(t-\tau) \neq 0 \Rightarrow -2 \leq t-\tau \leq 1 \Rightarrow -1 \leq \tau - t \leq 2$

$\Rightarrow t-1 \leq \tau \leq t+2$ (2)

- **Condition (1) forms Interval 1 shown in blue line below with a fixed lower bound -2. It has infinite length since its upper bound is $+\infty$.**
- **Condition (2) forms Interval 2 shown in red line below with moving bounds $t-1$ and $t+2$.**
- **The green line below represents the variable of integration τ.**

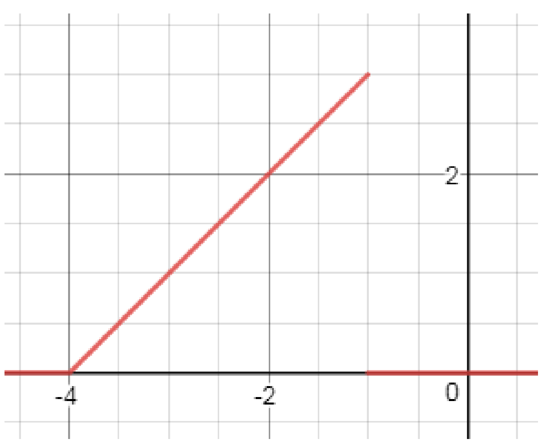


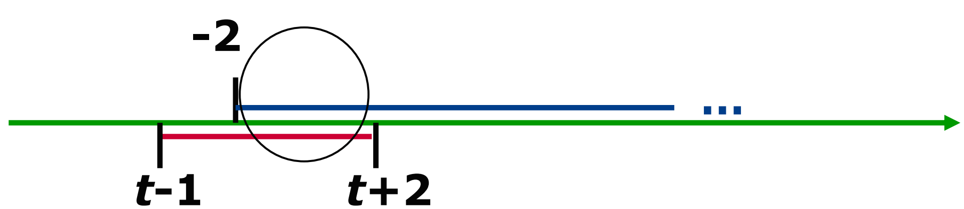
Problem 6 (b) cont.

Scenario I:

- **Interval 2** overlaps with **Interval 1** from the left.
- This means that the upper bound of **Interval 2** lies within **Interval 1** and the lower bound of **Interval 2** is outside **Interval 1**.
  - The above implies that $t+2 \geq -2 \Rightarrow t \geq -4$ and $t-1 \leq -2 \Rightarrow t \leq -1$ and therefore, by combining the above conditions we obtain $-4 \leq t \leq -1$.
- The overlapping area is from -2 to $t+2$. It is highlighted with a circle.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario I.

$$f_1(t) * f_2(t) = \int_{-2}^{t+2} f_1(\tau) f_2(t-\tau) d\tau = \int_{-2}^{t+2} d\tau = t+4, \; -4 \leq t \leq -1.$$



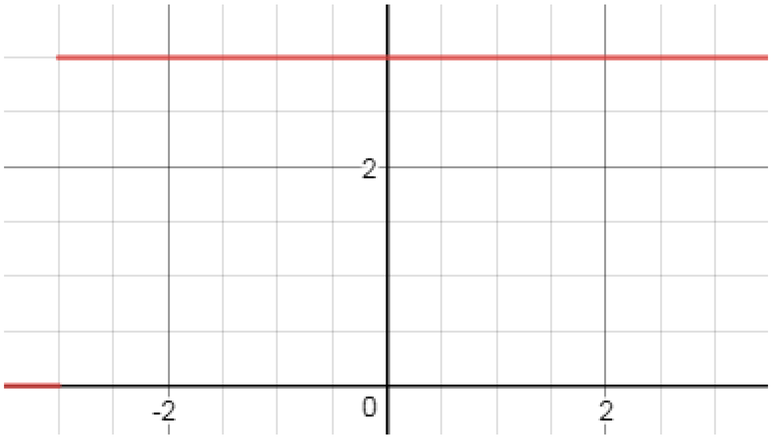


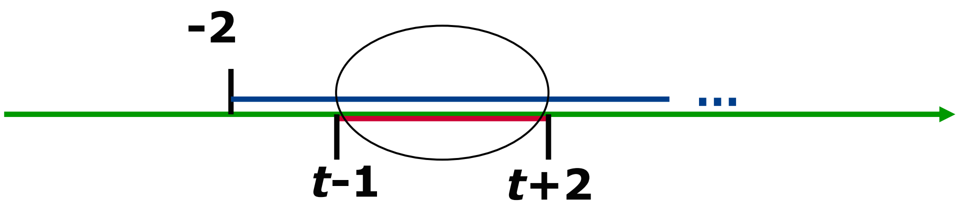
Problem 6 (b) cont.

Scenario II:

- **Interval 2** lies within **Interval 1**.
- This means that the lower bound of **Interval 2** lies within **Interval 1**.
  - The above implies that $t-1 \geq -2 \Rightarrow t \geq -1$.
- The overlapping area is from $t-1$ to $t+2$. It is highlighted with an oval.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario II.

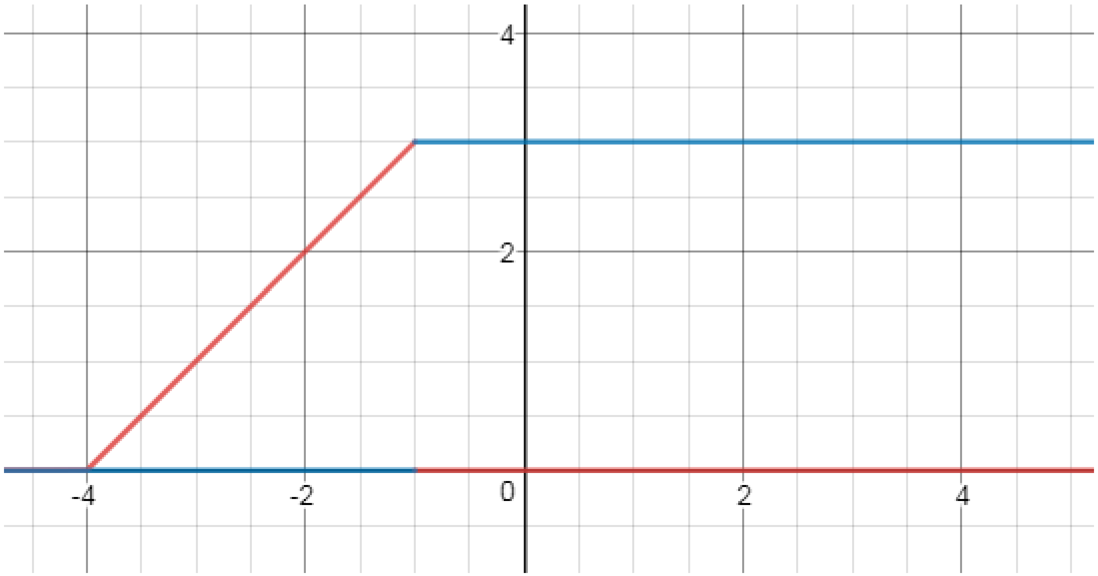
$$f_1(t) * f_2(t) = \int_{t-1}^{t+2} f_1(\tau) f_2(t-\tau)d\tau = \int_{t-1}^{t+2} d\tau = 3,\ -1 \leq t.$$





Problem 6 (b) cont.

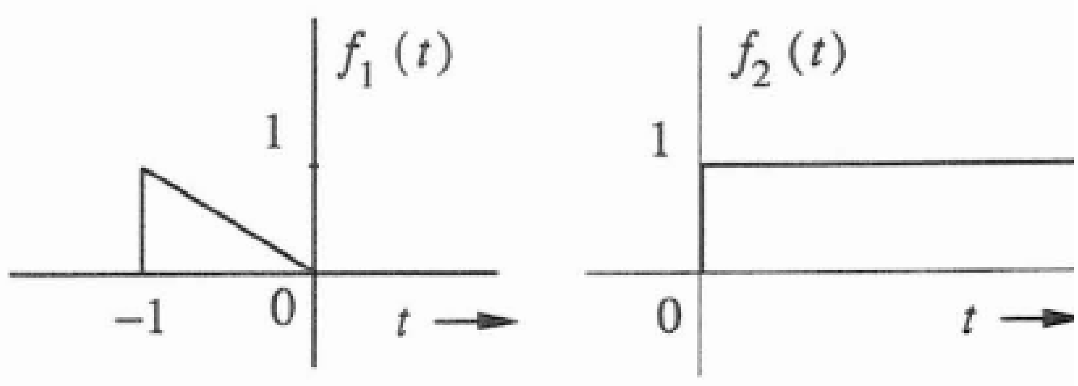
By combining Scenarios I and II above, we obtain the result of convolution for the entire range of time as depicted in the figure below.



Problem 6 (c)

(c) Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$



$f_1(t) = -t\big(u(-t) - u(-t-1)\big)$

$f_2(t) = u(t)$

Problem 6 (c) cont.

(c) Find and sketch the convolution:

$$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

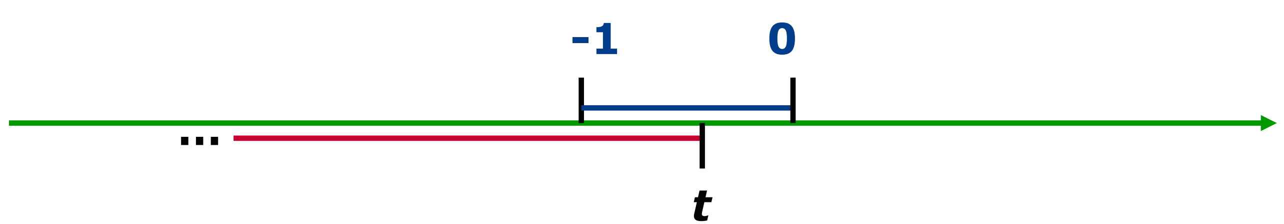
$f_1(t) = -t(u(-t) - u(-t-1))$ and $f_2(t) = u(t)$

$f_1(t) = \begin{cases} -t & -1 \le t \le 0 \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(t) = \begin{cases} 1 & t \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

$f_1(\tau) \neq 0 \Rightarrow -1 \le \tau \le 0$ (1)

$f_2(t-\tau) \neq 0 \Rightarrow 0 \le t - \tau \Rightarrow \tau \le t$ (2)

- **Condition (1) forms Interval 1 shown in blue line below with fixed bounds -1 and 0.**
- **Condition (2) forms Interval 2 shown in red line below with a moving upper bound t. It is of infinite length since its lower bound is $-\infty$.**
- **The green line below represents the variable of integration τ.**

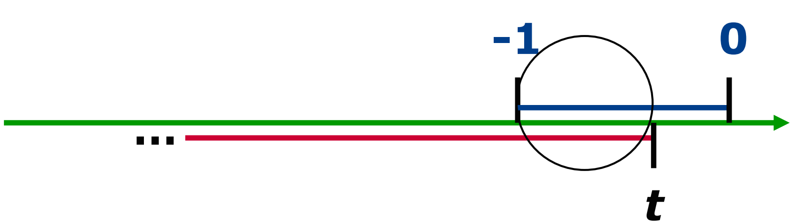


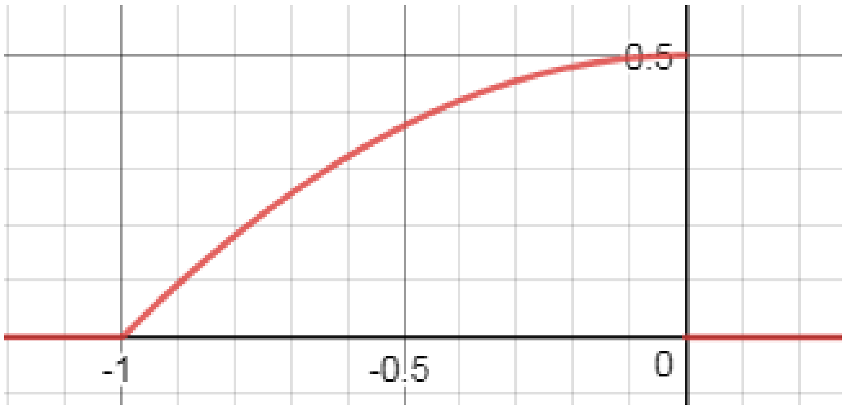
Problem 6 (c)

Scenario I:

- **Interval 2** overlaps with **Interval 1** from the left.
- This means that the upper bound of **Interval 2** lies within **Interval 1**.
  - The above implies that $t \geq -1$ and $t \leq 0$ and therefore, by combining the above conditions we obtain $-1 \leq t \leq 0$.
- The overlapping area is from -1 to t. It is highlighted with a circle.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario I.

$$f_1(t) * f_2(t) = \int_{-1}^{t} -\tau \, d\tau = -\frac{\tau^2}{2}\Big|_{-1}^{t} = -\frac{t^2}{2} + \frac{1}{2}, \ -1 \leq t \leq 0.$$





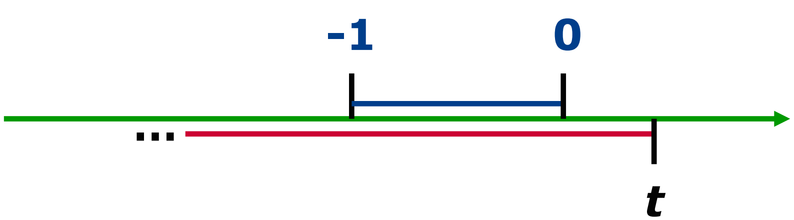
Problem 6 (c) cont.

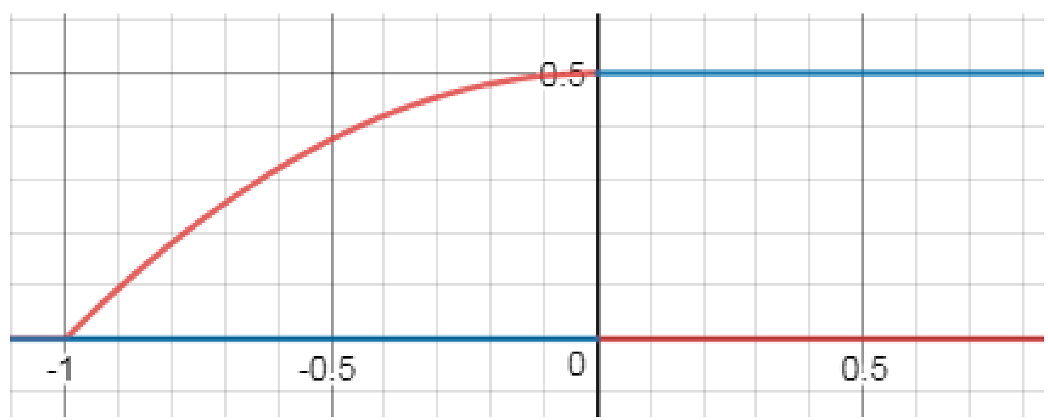
Scenario II:

- **Interval 1** lies within **Interval 2**.
  - The above implies that $t \geq 0$.
- The overlapping area is from -1 to 0.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario II.

$$f_1(t) * f_2(t) = \int_{-1}^{0} -\tau \, d\tau = -\frac{\tau^2}{2}\Big|_{-1}^{0} = \frac{1}{2}, \; 0 \leq t.$$

By combining Scenarios I and II above, we obtain the result of convolution for the entire range of time as depicted in the figure below.

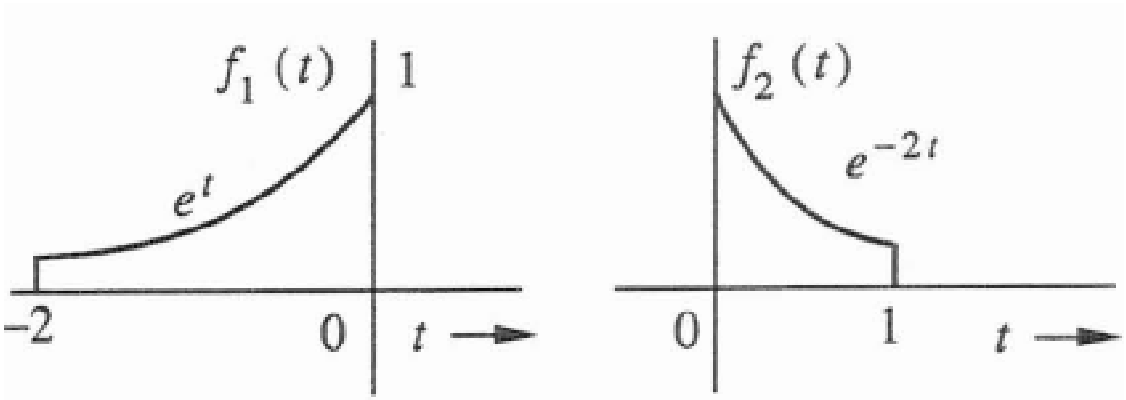




Problem 6 (d)

(d) Find and sketch the convolution:

$$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$



$$f_1(t) = e^t\big(u(-t) - u(-t-2)\big)$$
$$f_2(t) = e^{-2t}\ (u(t) - u(t-1))$$

Problem 6 (d) cont.

(d) Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

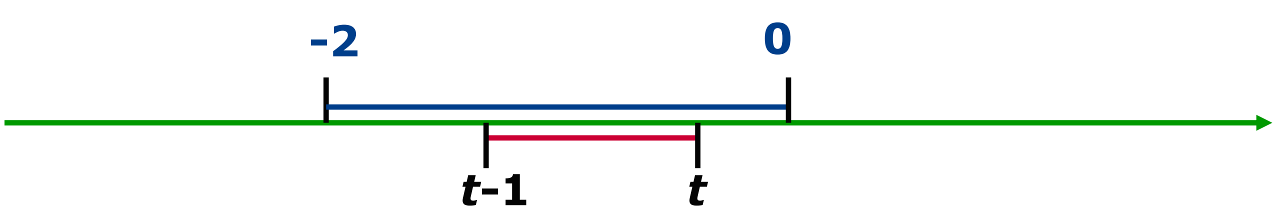
$f_1(t) = e^{t}(u(-t) - u(-t-2))$ and $f_2(t) = e^{-2t}\,(u(t) - u(t-1))$

$f_1(t) = \begin{cases} e^{t} & -2 \leq t \leq 0 \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(t) = \begin{cases} e^{-2t} & 0 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

$f_1(\tau) \neq 0 \Rightarrow -2 \leq \tau \leq 0$ (1)

$f_2(t-\tau) \neq 0 \Rightarrow 0 \leq t-\tau \leq 1 \Rightarrow -1 \leq \tau - t \leq 0 \Rightarrow t-1 \leq \tau \leq t$ (2)

- Condition (1) forms Interval 1 shown in blue line below with fixed bounds -2 and 0.
- Condition (2) forms Interval 2 shown in red line below with moving bounds $t-1$ and t.
- The green line below represents the variable of integration τ.



Problem 6 (d) cont.

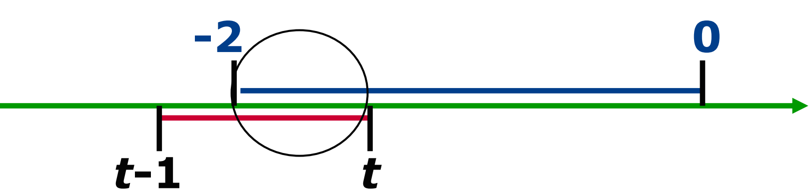
Scenario I:

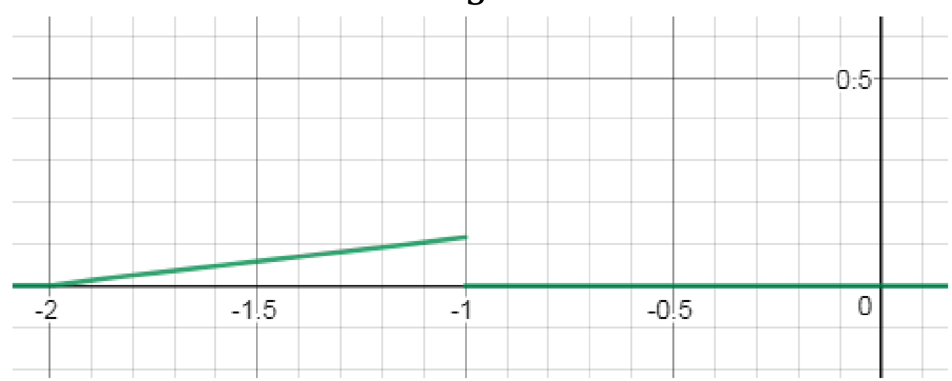
- **Interval 2** overlaps with **Interval 1** from the left.
- This means that the upper bound of **Interval 2** lies within **Interval 1** and the lower bound of **Interval 2** is outside **Interval 1**.
  - The above implies that $t \geq -2$ and $t-1 \leq -2 \Rightarrow t \leq -1$ and therefore, by combining the above conditions we obtain $-2 \leq t \leq -1$.
- The overlapping area is from -2 to t. It is highlighted with a circle in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario I.

$$f_1(t) * f_2(t) = \int_{-2}^{t} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{-2}^{t} e^{3\tau} d\tau = e^{-2t} \frac{1}{3}(e^{3t} - e^{-6})$$

$$= \frac{1}{3}(e^{t} - e^{-6} e^{-2t}), -2 \leq t \leq -1$$

(shown with the green curve right)





Problem 6 (d) cont.

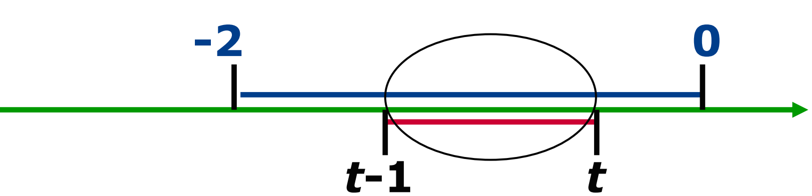
Scenario II:

- **Interval 2** lies within **Interval 1**.
  - The above implies that $t-1 \geq -2 \Rightarrow t \geq -1$ and $t \leq 0$ and therefore, by combining the above conditions we obtain $-1 \leq t \leq 0$.
- The overlapping area is from $t-1$ to t. It is highlighted with an oval in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario II.

$$f_1(t) * f_2(t) = \int_{t-1}^{t} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{t-1}^{t} e^{3\tau} d\tau = e^{-2t} \frac{1}{3}\left(e^{3t} - e^{(3t-3)}\right)$$

$$= \frac{1}{3}\left(e^{t} - e^{t-3}\right), -1 \leq t \leq 0$$

(shown with the red curve right)



Problem 6 (d) cont.

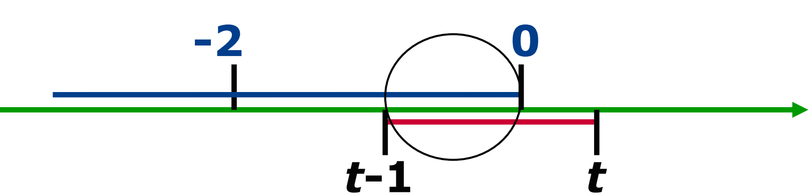
Scenario III:

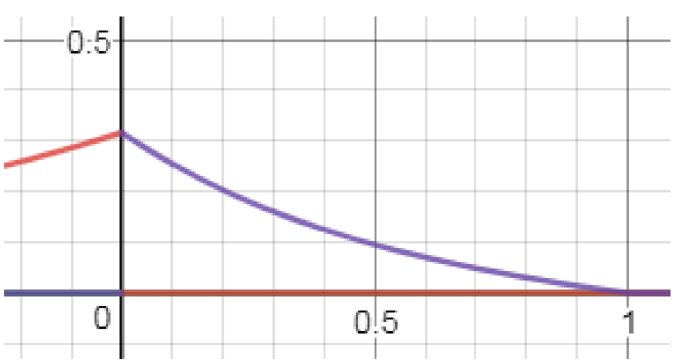
- **Interval 2** overlaps with **Interval 1** from the right.
- This means that the lower bound of **Interval 2** lies within **Interval 1** and the upper bound of **Interval 2** is outside **Interval 1**.
  - The above implies that $t - 1 \leq 0 \Rightarrow t \leq 1$ and $t \geq 0$ and therefore, by combining the above conditions we obtain $0 \leq t \leq 1$.
- The overlapping area is from $t - 1$ to 0. It is highlighted with a circle in the figure below.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario III.

$$f_1(t) * f_2(t) = \int_{t-1}^{0} e^{\tau} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{t-1}^{0} e^{3\tau} d\tau = e^{-2t} \frac{1}{3}(1 - e^{(3t-3)})$$

$$= \frac{1}{3}(e^{-2t} - e^{t-3}), 0 \leq t \leq 1$$

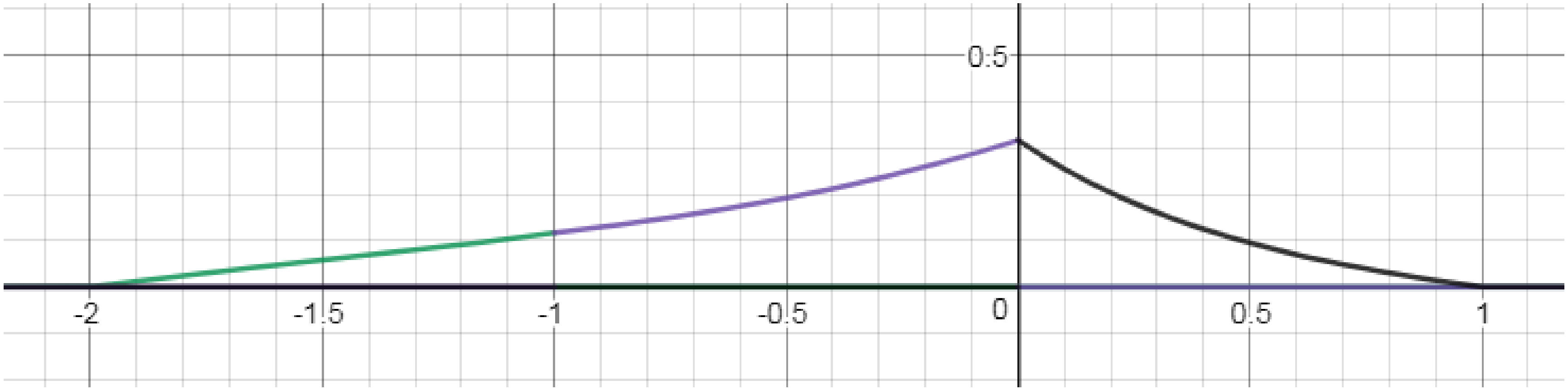
(shown in the purple curve right)





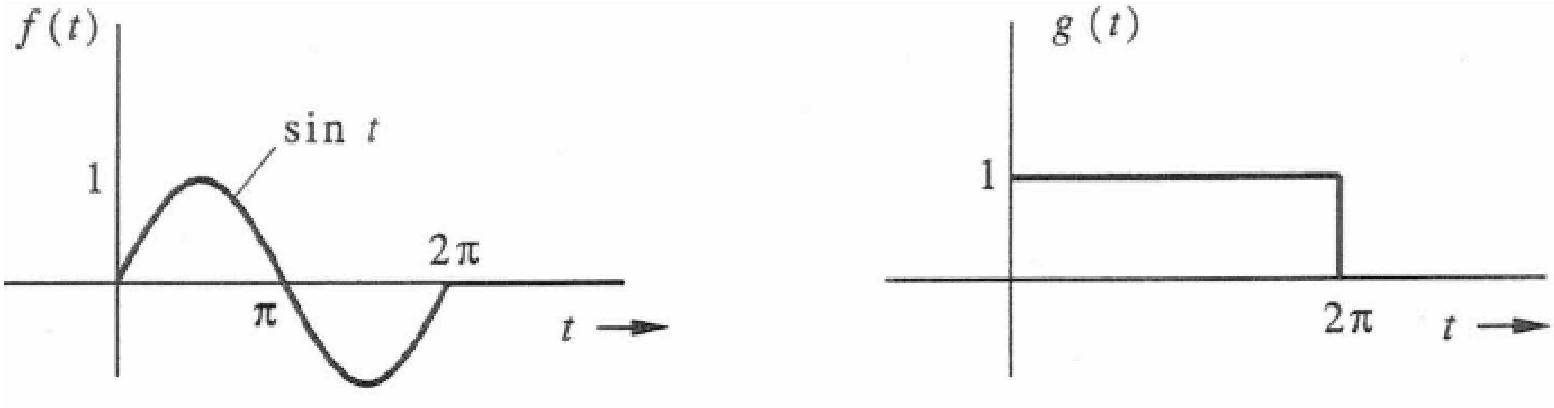
Problem 6 (d) cont.

By combining Scenarios I, II and III above, we obtain the result of convolution for the entire range of time as depicted in the figure below.



Problem 7

Find and sketch $c(t) = f(t) * g(t)$ for the pair of functions shown below.



$$f(t) = \sin(t)\left(u(t) - u(t - 2\pi)\right)$$
$$g(t) = u(t) - u(t - 2\pi)$$

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Problem 7 cont.

Find and sketch the convolution:

$y(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

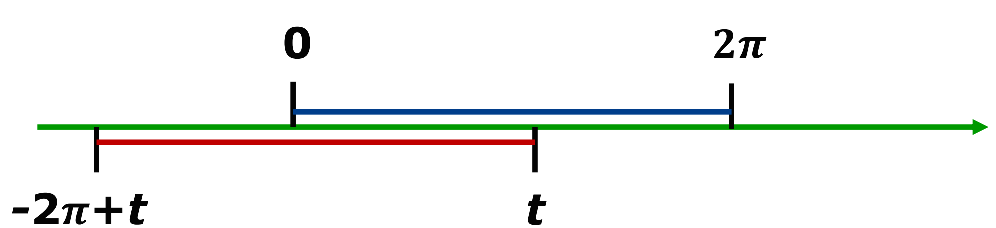
$f(t) = \sin(t)\big(u(t) - u(t-2\pi)\big)$ and $g(t) = u(t) - u(t-2\pi)$

$f_1(t) = \begin{cases} \sin(t) & 0 \le t \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$ and $f_2(t) = \begin{cases} 1 & 0 \le t \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$

$f_1(\tau) \ne 0 \Rightarrow 0 \le \tau \le 2\pi$ (1)

$f_2(t-\tau) \ne 0 \Rightarrow 0 \le t-\tau \le 2\pi \Rightarrow -2\pi \le \tau - t \le 0 \Rightarrow -2\pi + t \le \tau \le t$ (2)

- Condition (1) forms Interval 1 shown in blue line below with fixed bounds 0 and 2π.
- Condition (2) forms Interval 2 shown in red line below with moving bounds $-2\pi + t$ and t.
- The green line below represents the variable of integration τ.

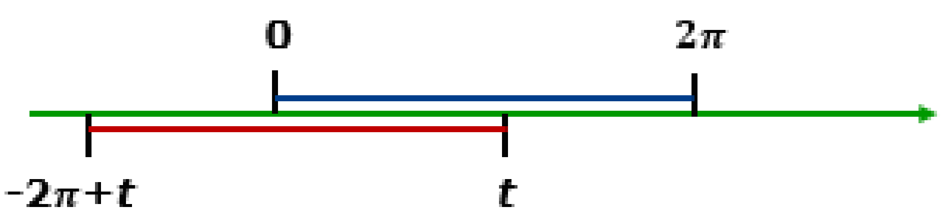


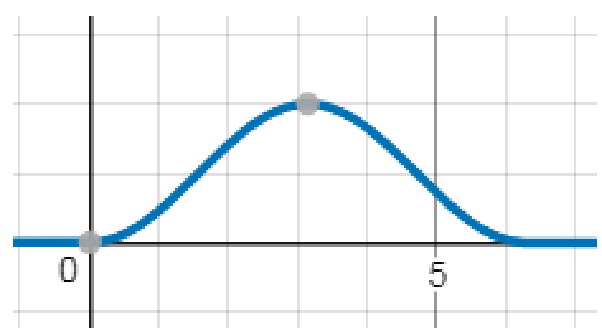
Problem 7 cont.

Scenario I:

- **Interval 2** overlaps with Interval 1 from the left.
- This means that the upper bound of **Interval 2** lies within **Interval 1** and the lower bound of **Interval 2** is outside **Interval 1**.
- The above implies that $t \geq 0$ and $-2\pi + t \leq 0 \Rightarrow t \leq 2\pi$ and therefore, by combining the above conditions we obtain $0 \leq t \leq 2\pi$.
- The overlapping area is from 0 to t.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario I.

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau = \int_0^t \sin(\tau) d\tau = -\cos(t) + 1$$





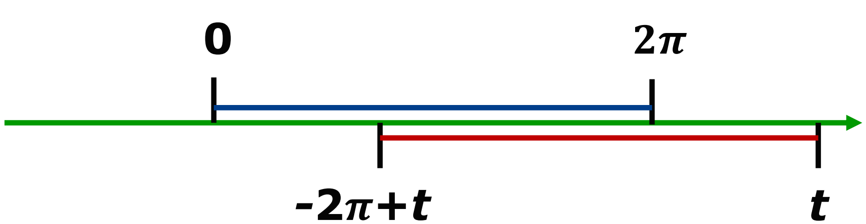
Problem 7 cont.

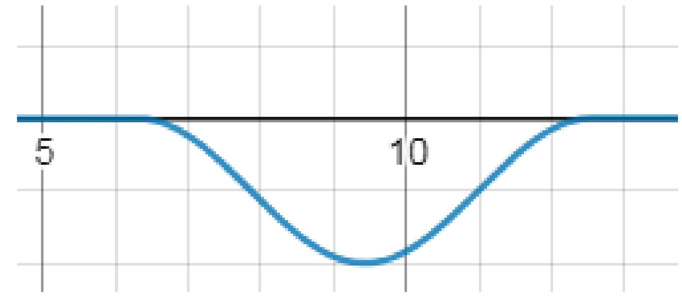
Scenario II:

- **Interval 2** overlaps with **Interval 1** from the right.
- This means that the lower bound of **Interval 2** lies within **Interval 1** and the upper bound of **Interval 2** is outside **Interval 1**.
  - The above implies that $-2\pi + t \leq 2\pi \Rightarrow t \leq 4\pi$ and $t \geq 2\pi$ and therefore, by combining the above conditions we obtain $2\pi \leq t \leq 4\pi$.
- The overlapping area is from $-2\pi + t$ to 2π.
- The overlapping of the two intervals specifies the two bounds of the convolution integral for Scenario II.

$$f_1(t) * f_2(t) = \int_{-2\pi+t}^{2\pi} f_1(\tau) f_2(t-\tau) d\tau = \int_{-2\pi+t}^{2\pi} \sin(\tau) d\tau$$

$$= -\cos(\tau)|_{-2\pi+t}^{2\pi} = -\cos(2\pi) + \cos(-2\pi + t) = \cos(t) - 1$$





Problem 7 cont.

By combining Scenarios I and II above, we obtain the result of convolution for the entire range of time as depicted in the figure below.