**Maths for Signals and Systems**

**Problem Sheet 4**

**Problems**

1. Consider a matrix  of size . Using the properties of determinants, find the determinants of three matrices  which are obtained from  through the following operations:
(i)  is obtained by multiplying each element of  with .

(ii)  is obtained when rows 1,2,3 of  are subtracted from rows 2,3,1.

(iii)  is obtained when rows 1,2,3 of  are added to rows 2,3,1.

**Solution**

1. The matrix  is given as follows:

. This can be written as



Therefore .

1. . The rows of add up to 0 and therefore, they are dependent. The determinant of  is 0.
2. 



1. Find the determinant of the following matrix and investigate whether the matrix is singular for certain values of the parameter .



**Solution**

We subtract row 3 from row 1. In that case the matrix becomes:



Then we subtract row 3 from row 2. The matrix now becomes:



We obtain the determinant using the cofactors of the first row. Note that when we replace a row with a linear combination of rows the determinant doesn’t change. In that case we have .

The determinant is zero in the following two cases:

* . In that case the original matrix is the “all ones” matrix which is obviously singular.
* . In that case the original matrix becomes . In that case the rows or columns add up to 0, and therefore, the matrix is singular.
1. (i) If the entries in every row of a matrix  add to zero, prove that  by commenting on the solutions of the system .

(ii) If the entries of every row of a matrix add up to 1, show that . Does this mean that ?

**Solution**

(i) If the entries in every row add up to zero the “all ones” column vector is in the null space. Therefore, the null space is a non-zero subspace and therefore, the original matrix is singular.

(ii) If the entries of every row of the matrix add up to 1, the entries in every row of add up to zero. Based on the comments of (i) we immediately see that . That doesn’t mean that , since in general .

1. A Hessenberg matrix is a square triangular matrix with one extra non-zero diagonal. The ,  and Hessenberg matrices are shown below:

, , 

Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci’s rule . The same rule continues for all sizes, i.e., . Which Fibonacci number is ?

**Solution**

The cofactor  for  is the determinant . We also need the cofactor . This is . If we consider matrix , this can be written as . Therefore,





We have , , . By considering the Fibonacci sequence we immediately see that .

1. (i) If  is the 10 by 10 “all-ones” matrix, how does the big formula for determinant give ?

(ii) If we multiply all  permutation matrices of size  is the resulting matrix’s determinant +1 or -1?

1. If we multiply each element  of a matrix with the fraction , how is the determinant of the matrix affected?

**Solution**

1. In the big formula (please look at your notes), in case of a square matrix with even number of rows and columns, half of the products will be +1 and half of them will be -1. Therefore, the determinant will be zero. This is anyway expected, since the 10 by 10 “all-ones” matrix is singular.
2. In the case of 2 by 2 matrices we have two distinct permutation matrices. These are  and  with determinants +1 and -1 respectively. Therefore, the determinant of the product is -1.

In case of 3 by 3 matrices we have the following permutations:

 with determinants .

There is an odd number of -1s since in that case there are  permutations with an odd number of row exchanges. In that case the determinant of the product is .

For matrices of size greater than 3 we have that the number of permutation matrices with odd number of row exchanges is . This is always even and therefore we have an even number of determinants equal to  and therefore, the total determinant is 1.

1. In the “big formula” for the estimation of the determinant we have sums of products of the form  where all rows and all columns participate once. Therefore, if we multiply each element  of a matrix with the fraction , the term  will be multiplied by all row numbers and divided by all column numbers and therefore will remain unchanged. Therefore, in that case the determinant remains unchanged.