**Maths for Signals and Systems**

**Problem Sheet 2**

In the formulation , where  is a matrix of size and  is a column vector of size , then 0 is a column vector with  zero elements.

**Problem 1**

Show that a basis for the left null space of a rectangular matrix  of size  consists of the last  rows of the corresponding elimination matrix .

**Solution**

The left null space is obtained by the solutions of the system , or equivalently, , where  is a column vector and therefore,  is a row vector.

As we discussed in detail in the lecture, the reduced row echelon form (rref) of matrix  is obtained by a sequence of operations imposed on matrix . Each operation involves replacing a row of  with a linear combination of itself and another row of , with the goal of making this row look “simpler” by replacing at least one element of the row with 0. This operation is equivalent with multiplying  from the left with a matrix  of size . The exact form of matrix  is given and justified in your lecture notes. We realise a number of this type of operations until  cannot be simplified further. In that case we can write



where  is the product of all elimination matrices used in the procedure. We can find the matrix  automatically if instead of carrying elimination on  only, we carry elimination on the augmented matrix  where  is the identity matrix of size . In that case after elimination the augmented matrix  becomes:



Therefore we can easily obtain .

If the rank of matrix  is  then the last  rows of  are zero rows. Therefore, from the equation  we see that each of the last  rows of  multiplied with  from the left gives a zero row vector. This verifies the fact that the last  rows of belong to the left null space, since they satisfy the relationship . Due to the method that we use to construct , it can be shown easily that  is a full rank matrix (rank is ) and therefore its last  rows are independent. Since these rows belong to the left null space and knowing that the left null space has dimension , we can say that the last  rows of  form a basis of the left null space.

**Problem 2**

Construct a matrix with the characteristics described in (a)-(d) below or if what is required is not possible, explain why it is impossible.

1. Column space contains , , row space contains , .
2. Column space has basis , null space has basis .
3. Dimension of null space = 1 + dimension of left null space.
4. Left null space contains , row space contains .

**Solution**

1. Column space contains , , row space contains , .This question refers to a matrix  of size . This can be verified from the fact that the rows have two elements and the columns have three elements. Since the column space and the row space have the same dimension (rank), the maximum rank of this matrix is 2. The row space contains the 2 given two- dimensional vectors, which are independent. Therefore, the rank of  **is** 2. If we formulate the matrix  using the given columns we observe that the first and the third rows are independent. Furthermore, the given rows can be easily written as linear combinations of the rows of . Therefore, this  satisfied the requirements of the question.
2. The required matrix has 3 rows, since the column space has a three dimensional vector as a basis. The basis of the column space is a single vector and therefore, the dimension of the column space is 1. That means that the rank of the matrix is 1. The null space has as basis a three dimensional vector as well, and therefore, the required matrix has 3 columns. This means that the required matrix is a square matrix of size . In that case the dimensions of column (or row) space and null space should add up to three. This is not the case and therefore, it is not possible to construct the required matrix.
3. The dimension of null space is  and the dimension of the left null space is . Therefore, according to the given formula . Therefore any matrix where the number of columns is larger by 1 compared to number of rows satisfies the required conditions.
4. Since left the null space contains a two dimensional vector, the required matrix has two **rows**. Since the row space contains also a two dimensional vector, the required matrix has two **columns**. Therefore the required matrix is a square matrix of size . Suppose the matrix is . In that case  gives , and therefore,  and . This gives .

**Problem 3**

Explain why  cannot be a row of  and also in the null space.

**Solution**

The null space must consist of three dimensional vectors, therefore  must have 3 columns. Since is a row of  and a vector of the null space, the inner product of  with itself should be zero. This cannot be true for any nonzero vector since the inner product of a vector with itself is the sum of the squares of the elements of that vector.

**Problem 4**

1. Verify that the special solutions to  are perpendicular to the rows of  using the formula and matrices given below:



1. Find the dimension and a basis of the left null space for this particular example.

**Solution**

1. The size of matrix  is  and therefore, the rank of  is at most 3. Furthermore, the matrix  is



From the rref of  we observe immediately that the rank is 2 since the last row of the rref matrix is the zero vector. The pivot columns of  can be obtained immediately from . Let’s assume that the first and third column of  are the pivot columns and assume that . Therefore, in the system , the free variables are . The system  is reduced to the system . This consists of the following set of equations:



For  we obtain the special solution . For  we obtain the special solution . Both of them are perpendicular to the rows of . This is a universal result, i.e., the solutions to the system  are always perpendicular to the rows of  as seen directly from the equation .

1. As explained in detail in Problem 1, the left null space basis consists of the last  rows of the matrix  which is given by the form . In this question we can find  from the operations imposed on  and not by finding the inverse of . At that stage you should be able to do this without help. The sequence of operations on  for the goal of obtaining the rref is equivalent of multiplying  from the left with the matrix



The dimension of the left null space for this example is 1, since there is only one zero row in the rref of the original matrix. A basis of the let null space consists of the last  rows of , i.e., it can be the vector .

**Problem 5**

1. Consider matrices  and . Explain why every vector in the column space of  is also in the column space of . This will tell us an important fact: rank≤ rank.
2. Consider matrices  and . Explain why every vector in the null space of  is also in the null space of . Is this also true for every vector in the null space of  or is there an example where it’s not true?

**Solution**

1. By matrix multiplication, every column vector of  is a linear combination of column vectors of . Therefore, every vector in the column space of  is also in the column space of .
2. Suppose a vector  is in the null space of , then we get . By matrix multiplication, , therefore  is also in the null space of . This is not true for every vector in the null space of . Take ,  with , and . Then . We see that , but  is not in the null space of . This is because .

**Problem 6**

Suppose you have applied elimination to  and you have reached . From looking at , how would you be able to describe vectors that span the column space of ?

**Solution**

Since  and its reduced form  have same list of pivot colums, we can easily get the pivot columns of  from . The column space of  is spanned by those pivot column vectors.

**Problem 7**

Suppose  is a matrix of size  of rank . Let . Assume that the columns of  are rearranged so that its first columns are the pivot columns.

1. Find a general form for .
2. Find a general form for the **null space matrix** . This is defined as a matrix whose columns are the special solutions of the system , and therefore, O and O, where O is a zero matrix.

**Solution**

Assume  has  pivot columns and  free variables. Each special solution has one free variable equal to 1 and the other free variables are all zero. By setting one free variable equal to 1 and all others equal to 0, we get the values of pivot variables from equation . In general, suppose that the columns of  are rearranged so that the first  columns of  are the pivot columns. In that case  where the Os in  indicate zero matrices. The subscripts in the individual matrices reveal their corresponding sizes. Due to the special column rearrangement of  the special solution vectors contain the pivot variables in their first  elements and the free variables in their the last  elements. **As already mentioned above**, each special solution has one free variable equal to 1 and the other free variables are all zero. Therefore, the null space matrix  is given by  where  is an unknown matrix of size . Knowing that O we get:





Therefore, .

**Problem 8**

If  is an  rank one matrix , where ,  are column vectors of length  and , what vectors are in each of the 4 fundamental subspaces. What are the dimensions of the 4 subspaces?

**Solution**



The row space  is spanned by  and has dimension 1.

The column space  is spanned by  and has dimension 1.

The null space  consists of vectors  of length  such that  with dimension .

The left null space  consists of vectors  of length  such that  with dimension 