Imperial College of Science Technology and Medicine

Department of Electrical and Electronic Engineering

Digital Image Processing

PART 4

IMAGE COMPRESSION

SCALAR AND VECTOR QUANTISATION

Academic responsible

Dr. Tania STATHAKI

Room 812

Ext. 46229

Email: t.stathaki@ic.ac.uk

SCALAR AND VECTOR QUANTISATION

1 SCALAR QUANTISATION

Let f represent a continuous scalar quantity which could be one of the following:

- pixel intensity
- transform coefficient
- image model parameter
- other

Suppose that only L levels are used to represent f. This process is called **amplitude quantisation**.

The process of quantisation may be classified in two main categories:

Scalar quantisation: each scalar is quantised independently.

Vector quantisation: two or more scalars are quantised jointly, i.e., the vector formed by two or more scalars is quantised.

Let $\hat{f}$ denote an f that has been quantised.

We can express $\hat{f}$ as $\hat{f} = Q(f) = r_i$, $d_{i-1} < f \le d_i$

Q: quantisation operation

r_i: the L **reconstruction levels**, $1 \le i \le L$

d_i: $L+1$ **decision boundaries**, $0 \le i \le L$

If f falls between d_{i-1} and d_i, it is mapped to the reconstruction level r_i.

$\hat{f}$ can be expressed as

$$\hat{f} = Q(f) = f + e_Q, \text{ where } e_Q = \hat{f} - f$$

e_Q: **quantisation noise**

In general, the quantity e_Q^2, with e_Q defined as above, can be viewed as a special case of a distortion measure $d(f, \hat{f})$, which is a measure of distance or dissimilarity between f and $\hat{f}$.

Other examples of $d(f, \hat{f})$

- $\left| \hat{f} - f \right|$
- $\left| \left| \hat{f} \right|^p - \left| f \right|^p \right|^{1/p}$

The reconstruction and decision levels are often determined by minimizing some error criterion based on $d(f, \hat{f})$.

Example: Minimise the **average distortion** D, defined as follows:

$$D = E[d(f,\hat{f}] = \int_{f_0=-\infty}^{\infty} d(f_0,\hat{f})p_f(f_0)df_0$$

Uniform quantisation

In uniform quantisation the reconstruction and decision levels are uniformly spaced.

$$d_i - d_{i-1} = \Delta,\ 1 \le i \le L \text{ and } r_i = \frac{d_i + d_{i-1}}{2},\ 1 \le i \le L$$

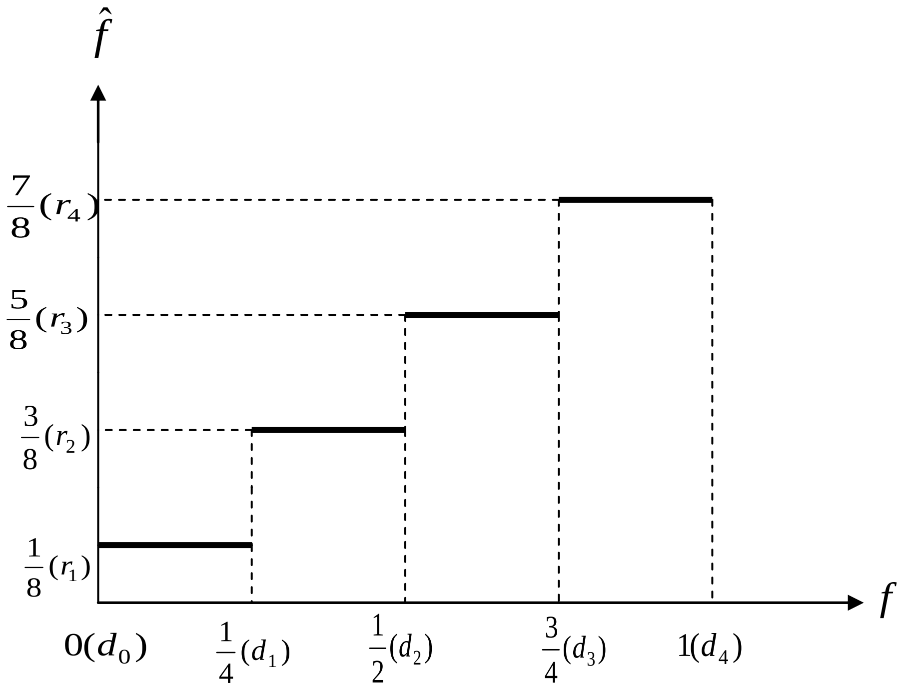

Figure 1.1: Example of uniform quantisation. The number of reconstruction levels is 4, f is assumed to be between 0 and 1, and $\hat{f}$ is the result of quantising f. The reconstruction levels and decision boundaries are denoted by r_i and d_i, respectively.

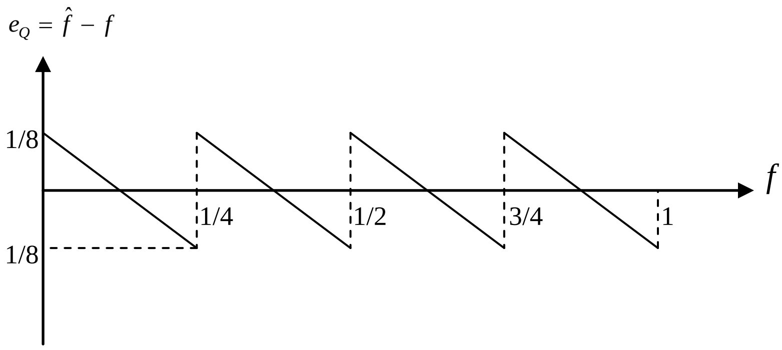

Figure 1.2: Illustration of signal dependence of quantisation noise in uniform quantisation

Uniform quantisation may not be optimal!

Suppose f is much more likely to be in one particular region that in others.

It is reasonable to assign more reconstruction levels to that region!

Quantisation in which reconstruction and decision levels do not have even spacing is called non-uniform quantisation.

Optimum determination of r_i and d_i depends on the error criterion used.

Quantisation using the MMSE criterion

Suppose f is a random variable with a pdf $p_f(f_0)$.

Using the minimum mean squared error (MMSE) criterion, we determine r_k and d_k by minimising the average distortion D given by

$$D = E[d(f,\hat{f})] = E[e_Q^2] = E[(\hat{f} - f)^2]$$
$$= \int_{f_0=-\infty}^{\infty} p_f(f_0)(\hat{f} - f_0)^2 df_0$$

Noting that $\hat{f}$ is one of the L reconstruction levels we write

$$D = \sum_{i=1}^{L} \int_{f_0=d_{i-1}}^{d_i} p_f(f_0)(r_i - f_0)^2 df_0$$

To minimize D

$$\frac{\partial D}{\partial r_k} = 0, \qquad 1 \le k \le L$$

$$\frac{\partial D}{\partial d_k} = 0, \qquad 1 \le k \le L-1$$

$$d_0 = -\infty$$

$$d_L = \infty$$

It is proven that from the above we get

$$r_k = \frac{\int_{f_0=d_{k-1}}^{d_k} f_0 p_f(f_0) df_0}{\int_{f_0=d_{k-1}}^{d_k} p_f(f_0) df_0} \qquad 1 \le k \le L$$

$$d_k = \frac{r_k + r_{k+1}}{2} \qquad 1 \le k \le L-1$$

$$d_0 = -\infty$$

$$d_L = \infty$$

Note that:

- The reconstruction level r_k is the centroid of $p_f(f_0)$ over the interval $d_{k-1} \le f_0 \le d_k$.
- The decision level d_k except d_0 and d_L is the middle point between two reconstruction levels r_k and r_{k+1}.
- The above set of equations is a necessary set of equations for the optimal solution.
- For a certain class of pdf's including uniform, Gaussian, Laplacian, is also sufficient.

A quantiser based on the MMSE criterion is often referred to as **Lloyd-Max quantiser**.

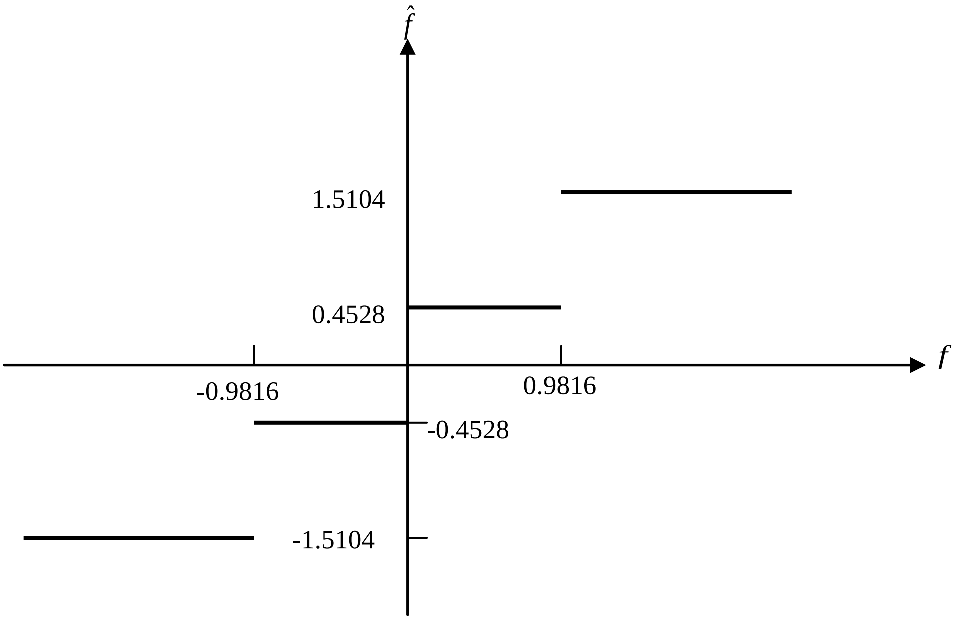

Figure 1.3: Example of a Lloyd-Max quantiser. The number of reconstruction levels is 4, and the probability for f is Gaussian with mean 0 and variance 1.

2 VECTOR QUANTISATION

Let $\mathbf{f} = [f_1, f_2, \ldots, f_k]^T$ denote an k-dimensional vector that consists of k real-valued, continuous-amplitude scalars f_i.

$\mathbf{f}$ is mapped to another k-dimensional vector $\mathbf{y} = [y_1, y_2, \ldots, y_k]^T$

$\mathbf{y}$ is chosen from N possible reconstruction or quantisation levels

$$\hat{\mathbf{f}} = \mathrm{VQ}(\mathbf{f}) = \mathbf{y}_i, \ \mathbf{f} \in C_i$$

VQ is the vector quantisation operation

C_i is called the ith cell.

distortion measure: $d(\hat{\mathbf{f}}, \mathbf{f}) = \mathbf{e}_Q^T \mathbf{e}_Q$

quantisation noise: $\mathbf{e}_Q = \hat{\mathbf{f}} - \mathbf{f} = \mathrm{VQ}(\mathbf{f}) - \mathbf{f}$

PROBLEM: determine $\mathbf{y}_i$ and boundaries of cells C_i

SOLUTION: **minimise some error criterion such as the average distortion measure** D given by

$$D = E[d(\mathbf{f}, \hat{\mathbf{f}})]$$

$$D = E[\mathbf{e}_Q^T \mathbf{e}_Q] = E[(\hat{\mathbf{f}}\text{-}\mathbf{f})^T (\hat{\mathbf{f}}\text{-}\mathbf{f})] = \int\int_{-\infty}^{\infty} \ldots \int (\hat{\mathbf{f}}\text{-}\mathbf{f}_0)^T (\hat{\mathbf{f}}\text{-}\mathbf{f}_0) p_{\mathbf{f}}(\mathbf{f}_0) d\mathbf{f}_0$$

$$= \sum_{i=1}^{N} \int\int_{\mathbf{f}_0 \in C_i} \int \ldots \int (\mathbf{r}_i\text{-}\mathbf{f}_0)^T (\mathbf{r}_i\text{-}\mathbf{f}_0) p_{\mathbf{f}}(\mathbf{f}_0) d\mathbf{f}_0$$

MAJOR ADVANTAGE OF VQ: performance improvement over scalar quantisation of a vector source.

That means

- VQ can lower the average distortion D with the number of reconstruction values held constant.
- VQ can reduce the required number of reconstruction values when D is held constant.

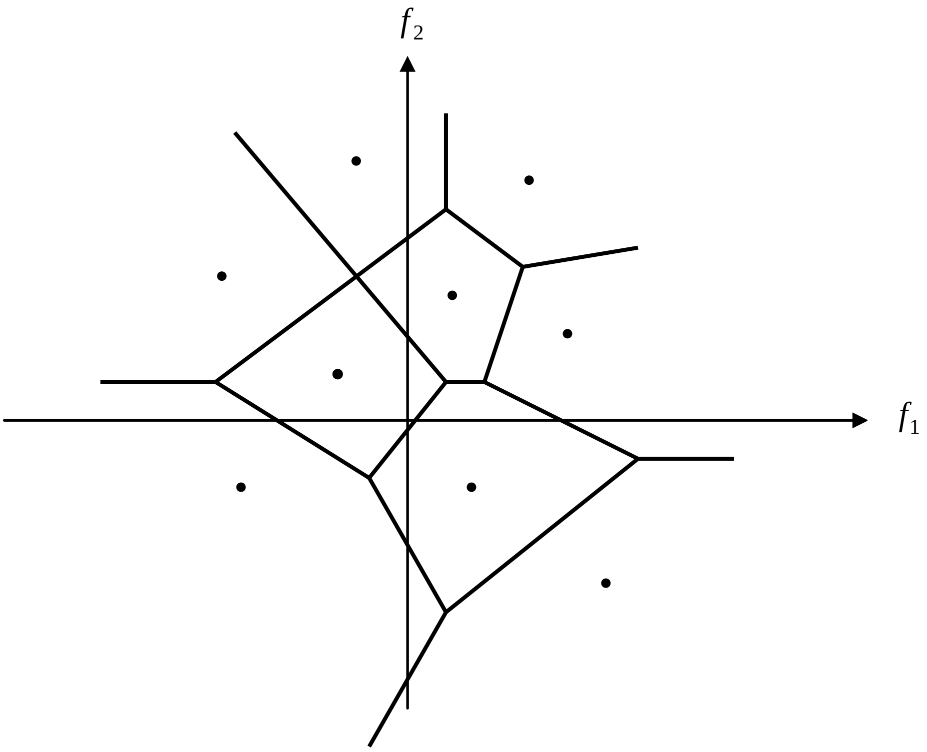

Figure 2.1: Example of vector quantisation. The number of scalars in the vector is 2, and the number of reconstruction levels is 9.