KLT Sample Exam Problems

1. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2$ represents an image of size $M \times M$, where M is even. The population arises from their formation across the entire collection of pixels.
The two images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le M, 1 \le y \le \frac{M}{2} \\ s_1 & 1 \le x \le M, \frac{M}{2} < y \le M \end{cases}, f_2(x,y) = \begin{cases} r_2 & 1 \le y \le M, 1 \le x \le \frac{M}{2} \\ s_2 & 1 \le y \le M, \frac{M}{2} < x \le M \end{cases}$$

Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

(i) Find the images $g_1(x,y)$ and $g_2(x,y)$ using the Karhunen-Loeve (KL) transform.
(ii) Comment on whether you could obtain the result of c)-(i) above using intuition.

2. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y)$, $i = 1,2,3$ represents an image of size $M \times M$ where M is even. The population arises from the formation of the vectors $\underline{f}$ across the entire collection of pixels (x,y).
The three images are defined as follows:

$$f_1(x,y) = \begin{cases} r_1 & 1 \le x \le \dfrac{M}{2}, 1 \le y \le M \\ r_2 & \dfrac{M}{2} < x \le M, 1 \le y \le M \end{cases}$$

$$f_2(x,y) = r_3,\ 1 \le x \le M,\ 1 \le y \le M$$

$$f_3(x,y) = r_4,\ 1 \le x \le M,\ 1 \le y \le M$$

The parameters r_1, r_2, r_3, r_4 are constants.
Consider now a population of random vectors of the form

$$\underline{g}(x,y) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve (KL) transforms of the vectors $\underline{f}$.

(i) Find the images $g_1(x,y)$, $g_2(x,y)$ and $g_3(x,y)$ using the Karhunen-Loeve (KL) transform.
(ii) Comment on whether you could obtain the result of c)-(i) above using intuition rather than by explicit calculation.

3. Consider the population of random vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ \vdots \\ f_n(x,y) \end{bmatrix}.$$

Each component $f_i(x,y)$ represents an image. The population arises from their formation across the entire collection of pixels. Suppose that $n > 2$, i.e. you have at least three images.
Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ \vdots \\ g_n(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.
The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & 0 & b^2 \\ 0 & a & b^2 \\ b^2 & b^2 & a \end{bmatrix}$$

Suppose that a credible job could be done of reconstructing approximations to the three original images by using one or two principal component images. What would be the mean square error incurred in doing so in each case?

4. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2,3$ represents an image. The population arises from the formation of the vectors across the entire collection of pixels.
Consider now a population of vectors $\underline{g}$ of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ g_3(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.
The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

with $a, b, c > 0$.

(i) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one principal component image. What would be the mean square error incurred in doing so, if it is known that $c < a - b$.

(ii) Suppose that a credible job could be done of reconstructing approximations to the three original images by using two principal component images. What would be the mean square error incurred in doing so, if it is known that $c > a + b$.

5. Consider the population of vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{bmatrix}.$$

Each component $f_i(x,y), i = 1,2,3$ represents a real image. The population arises from the formation of the vectors across the entire collection of pixels. Consider now a population of vectors $\underline{g}$ which are the Karhunen-Loeve transforms of the vectors $\underline{f}$.

The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & b & c \\ b & a & 0 \\ c & 0 & a \end{bmatrix}$$

(i) Find the covariance matrix of the population $\underline{g}$. Provide conditions for a, b, c so that both covariance matrices are valid.

(ii) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one or two principal component images. What would be the mean square error incurred in each case?