1.

(i) The first image $f_1(x,y)$ has a solid horizontal edge. Its mean is $\frac{r_1+s_1}{2}$. The zero-mean version of it is $f_1(x,y) = \begin{cases} \frac{r_1-s_1}{2} & 1 \le x \le M, 1 \le y \le \frac{M}{2} \\ \frac{s_1-r_1}{2} & 1 \le x \le M, \frac{M}{2} < y \le M \end{cases}$. The second image $f_2(x,y)$ has a solid vertical edge. Its mean is $\frac{r_2+s_2}{2}$. The zero-mean version of it is $f_2(x,y) = \begin{cases} \frac{r_2-s_2}{2} & 1 \le x \le M, 1 \le y \le \frac{M}{2} \\ \frac{s_2-r_2}{2} & 1 \le x \le M, \frac{M}{2} < y \le M \end{cases}$. The variance of $f_1(x,y)$ is $\frac{(r_1-s_1)^2}{4}$. The variance of $f_2(x,y)$ is $\frac{(r_2-s_2)^2}{4}$. The covariance between the two images is zero (this is the mean of the product of the two images). This is because $f_1(x,y)$ is of the form $\begin{bmatrix} a \\ \cdots \\ -a \end{bmatrix}$ and $f_2(x,y)$ is of the form $[b \quad \vdots \quad -b]$ therefore $f_1(x,y)f_2(x,y) = \begin{bmatrix} ab & \vdots & -ab \\ \cdots & \vdots & \cdots \\ -ab & \vdots & ab \end{bmatrix}$. So the mean of $f_1(x,y)f_2(x,y)$ is zero. In that case the covariance matrix of the population is $C = \begin{bmatrix} \frac{(r_1-s_1)^2}{4} & 0 \\ 0 & \frac{(r_2-s_2)^2}{4} \end{bmatrix}$. The eigenvalues of the covariance matrix are $\frac{(r_1-s_1)^2}{4}$ and $\frac{(r_2-s_2)^2}{4}$. The images $g_1(x,y)$ and $g_2(x,y)$ are simply the zero mean versions of the original images.

(ii) There is no point of using the KL transform since it is obvious visually that the images are uncorrelated.

2.

(i) $\boldsymbol{f_1(x,y) = \begin{cases} r_1 & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ r_2 & \frac{M}{2} < x \le M, 1 \le y \le M \end{cases}}$

Mean value of $f_1(x,y)$ is $m_1 = \frac{r_1}{2} + \frac{r_2}{2}$. Zero-mean version of $f_1(x,y)$ is

$$f_1(x,y) - m_1 = \begin{cases} \frac{r_1}{2} - \frac{r_2}{2} & 1 \le x \le \frac{M}{2}, 1 \le y \le M \\ \frac{r_2}{2} - \frac{r_1}{2} & \frac{M}{2} < x \le M, 1 \le y \le M \end{cases}$$

Mean value of $f_2(x,y)$ is r_3. Zero-mean version of $f_2(x,y)$ is $f_2(x,y) - m_2 = 0$.
Mean value of $f_3(x,y)$ is r_4. Zero-mean version of $f_3(x,y)$ is $f_3(x,y) - m_3 = 0$.
Variance of $f_1(x,y)$ is $\frac{1}{2}\frac{1}{4}(r_1 - r_2)^2 + \frac{1}{2}\frac{1}{4}(r_1 - r_2)^2 = \frac{1}{4}(r_1 - r_2)^2$.
Variance of $f_2(x,y) - m_2$ is 0.
Variance of $f_3(x,y) - m_3$ is 0.
Covariance between $f_1(x,y) - m_1$ and the other two images is 0. Therefore, the covariance matrix is $\begin{bmatrix} \frac{1}{4}(r_1 - r_2)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with eigenvalues $\frac{1}{4}(r_1 - r_2)^2$ and 0. Therefore, by using the Karhunen Loeve transform we produce three new images, with two of them being 0 and the other being $f_1(x,y) - m_1$.

(ii) The above result is expected since two of the given images are constant and therefore they don't carry any information. This means that there is only one principal component in the given set.

3.

(i) Values of a, b, c must be real and positive.

The eigenvalues of the covariance matrix $\underline{C_f}$ are obtained by solving the equation $det\left|\underline{C_f} - \lambda I\right| = 0 \Rightarrow (a-\lambda)^3 - 2b^4(a-\lambda) = 0 \Rightarrow \lambda = a, \lambda = a \pm \sqrt{2b^2}$.

The eigenvalues are sorted as follows: $a + \sqrt{2b^2} \geq a \geq a - \sqrt{2b^2}$.

If we keep one image the error will be $2a - \sqrt{2b^2}$ and if we keep two images the error will be $a - \sqrt{2b^2}$.

(ii) No, since there isn't any redundancy among the images.

4.

The eigenvalues of the covariance matrix $\underline{C_f} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$ are found by the following relationship:

$$\det\begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} = (c-\lambda)[(a-\lambda)^2 - b^2]$$

$c - \lambda)[(a-\lambda) - b](c-\lambda)[(a-\lambda) + b] = 0 \Rightarrow \lambda_1 = c, \lambda_2 = a - b, \lambda = a + b$

(i) If $c < a - b$ then because $c \geq 0$ since it represents variance of an image the eigenvalues will be sorted according to magnitude as $a + b \geq a - b > c$ and therefore by using only one principal component the error of reconstruction will be $a - b + c$.

(ii) If $c > a + b$ the eigenvalues will be sorted according to magnitude as $c > a + b \geq a - b$ and therefore by using only two principal components the error of reconstruction will be $a - b$.

5.

(i) Values of a, b, c must be real and positive.

The eigenvalues of the covariance matrix $\underline{C}_{\underline{f}}$ are obtained by solving the equation $\det\left|\underline{C}_{\underline{f}} - \lambda I\right| = 0 \Rightarrow (a-\lambda)^3 - b^2(a-\lambda) - c^2(a-\lambda) = 0 \Rightarrow \lambda = a, \quad \lambda = a \pm \sqrt{b^2 + c^2}$. The eigenvalues are sorted as follows: $a + \sqrt{b^2 + c^2} \geq a \geq a - \sqrt{b^2 + c^2}$.

(ii) If we keep one image the error will be $2a - \sqrt{b^2 + c^2}$ and if we keep two images the error will be $a - \sqrt{b^2 + c^2}$.