1. (i) For a very small $k_2(x,y)$ the factor $e^{-k_2(x,y)(m^2+n^2)}$ tends to 1 and therefore, we can write $h_{(x,y)}(m,n)=k_1(x,y)w(m,n)$.

(ii) For a very large $k_2(x,y)$ we can write

$$e^{-k_2(x,y)(m^2+n^2)}=\begin{cases}1 & m=n=0\\ 0 & \text{otherwise}\end{cases}$$

Therefore, we can write

$$h_{(x,y)}(m,n)=k_1(x,y)\delta(m,n)$$

(iii) In image regions of high detail, such as edge regions, we don't want to do any filtering and therefore we want large k_2. In image regions of low detail, such as uniform background regions, we want to de-noise and therefore we want small k_2. Therefore, we can choose $k_2(x,y)=\sigma_g^2(x,y)$

(iv) We wish the sum of the filter coefficients to be 1 so that we don't distort the mean intensity of the image, and therefore,

$$k_1(x,y)=\frac{1}{\sum_{m=-2}^{2}\sum_{n=-2}^{2}e^{-\sigma_g^2(x,y)(m^2+n^2)}}$$

(v) A possible scenario is to filter both high detail and bright areas. Bright areas can be identified from the local mean m_g (A large m_g indicates a bright area). Therefore, we can normalize σ_g^2 and m_g so that they occupy the same range of values (for example from 0 to 1) to obtain $\tilde{\sigma}_g^2$ and $\tilde{m}_g$ and then we can consider $k_2=\tilde{\sigma}_g^2+\tilde{m}_g$.

2. (i) Bookwork.

(ii) Inverse Filtering cannot be applied if $H(u,v)=0$.

$$H(u,v)=\frac{1}{NN}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}h(x,y)e^{-j2\pi(ux/N+vy/N)}=\frac{1}{NN}(e^{-j2\pi u/N}+e^{j2\pi u/N}+1)$$

$$=\frac{1}{NN}[2\cos(2\pi u/N)+1]$$

The values of (u,v) for which $H(u,v)$ cannot be estimated are found by:

$$2\cos(2\pi u/N)+1=0\Rightarrow\cos(2\pi u/N)=-1/2\Rightarrow 2\pi u/N=2k\pi\pm2\pi/3$$

$$\Rightarrow u=\frac{N(2k\pi\pm2\pi/3)}{2\pi}=N(k\pm1/3),k=0,1,2,\dots$$

Therefore, the valid values are $u=N/3$ and $u=2N/3$.

3. (i) Bookwork

(ii) $H^*(u,v)=\sqrt{2\pi}\sigma(u^2+v^2)e^{j2\pi^2\sigma^2(u^2+v^2)}$

$$|H(u,v)|^2=H(u,v)H^*(u,v)=2\pi\sigma^2(u^2+v^2)^2$$

The CLS filter in frequency domain is $\dfrac{\sqrt{2\pi}\sigma(u^2+v^2)e^{j2\pi^2\sigma^2(u^2+v^2)}}{2\pi\sigma^2(u^2+v^2)^2+\alpha|C(u,v)|^2}$ where $C(u,v)$ is the Laplacian filter in frequency domain.

4. (i) The noise is white with variance σ_n^2 and therefore the autocorrelation function of the noise is

$$R_{nn}(k,l) = \underset{(x,y)}{E}\{n(x,y)n(x+k,y+l)\} = \sigma_n^2\delta(k,l)$$

and the power spectrum of the noise is σ_n^2.

$$S_{ff}(u,v) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_{ff}(k,l)e^{-j(uk+vl)} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\rho^{|k|+|l|}e^{-j(uk+vl)} = \sum_{k=-\infty}^{\infty}\rho^{|k|}e^{-juk}\sum_{l=-\infty}^{\infty}\rho^{|l|}e^{-jvl}$$

We find that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{-j\omega n} = \sum_{n=-\infty}^{-1} r^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n} = \sum_{n=1}^{\infty} r^{n}e^{j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n} = \sum_{n=0}^{\infty} r^{n}e^{j\omega n} + \sum_{n=0}^{\infty} r^{n}e^{-j\omega n} - 1$$

$$= \sum_{n=0}^{\infty}(re^{j\omega})^n + \sum_{n=0}^{\infty}(re^{-j\omega})^n - 1 = \frac{1}{1-re^{j\omega}} + \frac{1}{1-re^{-j\omega}} - 1 = \frac{2-2r\cos\omega}{1+r^2-2r\cos\omega} - 1 = \frac{1-r^2}{1+r^2-2r\cos\omega}$$

Therefore,

$$S_{ff}(u,v) = \frac{(1-\rho^2)^2}{(1+\rho^2-2\rho\cos u)(1+\rho^2-2\rho\cos v)}$$

and

$$W(u,v) = \frac{(1-\rho^2)^2}{(1-\rho^2)^2 + (1+\rho^2-2\rho\cos u)(1+\rho^2-2\rho\cos v)\sigma_n^2}$$

(ii) Develop a method of estimating ρ_1 and ρ_2 from $g(x,y)$.

$$R_{gg}(k,l) = R_{ff}(k,l) + R_{nn}(k,l) = \rho^{|k|+|l|} + \sigma_n^2\delta(k,l)$$

If we choose two pairs of values for the parameters k,l as $(k,l)=(1,0)$ and $(k,l)=(0,0)$ we have the following relationships:

For $(k,l)=(1,0)$: $R_{gg}(1,0)=\rho$ where $R_{gg}(1,0)$ is estimated from the available data.

For $(k,l)=(0,0)$: $R_{gg}(0,0)=1+\sigma_n^2$ where $R_{gg}(0,0)$ is estimated from the available data.

5. (i) For a very small $k_2(x,y)$ the factor $e^{-k_2(x,y)(m^2+n^2)}$ tends to 1 and therefore, we can write $h_{(x,y)}(m,n) = k_1(x,y)w(m,n)$.

(ii) For a very large $k_2(x,y)$ we can write

$$e^{-k_2(x,y)(m^2+n^2)} = \begin{cases} 1 & m=n=0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we can write $h_{(x,y)}(m,n) = k_1(x,y)\delta(m,n)$.

(iii) In image regions of high detail, such as edge regions, we don't want to do any filtering and therefore we want large k_2. In image regions of low detail, such as uniform background regions, we want to de-noise and therefore we want small k_2. Therefore, we can choose $k_2(x,y) = \sigma_g^2(x,y)$.

(iv) We wish the sum of the filter coefficients to be 1 so that we don't distort the mean intensity of the image, and therefore,

$$k_1(x,y) = \frac{1}{\sum_{m=-2}^{2}\sum_{n=-2}^{2} e^{-\sigma_g^2(x,y)(m^2+n^2)}}$$

(v) A possible scenario is to filter both high detail and bright areas. Bright areas can be identified from the local mean m_g (A large m_g indicates a bright area). Therefore, we can normalize

σ_g^2 and m_g so that they occupy the same range of values (for example from 0 to 1) to obtain $\tilde{\sigma}_g^2$ and $\tilde{m}_g$ and then we can consider $k_2 = \tilde{\sigma}_g^2 + \tilde{m}_g$.

6. (i) $h(x,y) = \begin{cases} 1 & x=0, y=0 \\ 2 & x=0, y=1 \\ 1 & x=0, y=2 \end{cases}$

$$H(u,v) = \frac{1}{N^2}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1} h(x,y)e^{-j\frac{2\pi}{N}(ux+vy)} = \frac{1}{N^2}(e^{-j\frac{2\pi}{N}v0} + 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v})$$

$$= \frac{1}{N^2}(1+2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v}) = \frac{1}{M^2}e^{-j\frac{2\pi}{N}v}(e^{j\frac{2\pi}{N}v} + 2 + e^{-j\frac{2\pi}{N}v})$$

$$= \frac{2}{N^2}e^{-j\frac{2\pi}{N}v}[\cos(\frac{2\pi v}{N}) + 1]$$

The expressions for Inverse Filtering are book work.

(ii) In order to find the frequency points for which $H(u,v) = 0$ we set

$$\cos(\frac{2\pi v}{N}) = -1 \Rightarrow \frac{2\pi v}{N} = k\pi, k \text{ odd}.$$

Therefore, for $k = 1 \Rightarrow v = \frac{N}{2}$

For $k = 3 \Rightarrow v = \frac{3N}{2}$ this value is invalid since $v \in [0, N-1]$.

(iii)

$$C(u,v) = \frac{1}{N^2}\sum_{x=0}^{M-1}\sum_{y=0}^{M-1} h(x,y)e^{-j\frac{2\pi}{N}(ux+vy)} = \frac{1}{N^2}(e^{-j\frac{2\pi}{N}v0} - 2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v})$$

$$= \frac{1}{N^2}(1-2e^{-j\frac{2\pi}{N}v} + e^{-j\frac{2\pi}{N}2v}) = \frac{1}{M^2}e^{-j\frac{2\pi}{N}v}(e^{j\frac{2\pi}{N}v} - 2 + e^{-j\frac{2\pi}{N}v})$$

$$= \frac{2}{N^2}e^{-j\frac{2\pi}{N}v}[\cos(\frac{2\pi v}{N}) - 1]$$

The expression for the CLS estimator is book work.

(iv) $\cos(\frac{2\pi v}{N}) = 1 \Rightarrow \frac{2\pi v}{N} = 2k\pi, k$ any integer.

Therefore, for $k = 0 \Rightarrow v = 0$

For $k = 1 \Rightarrow v = N$ this value is invalid since $v \in [0, N-1]$.

By comparing the frequency points for which $H(u,v)$ and $C(u,v)$ we see that there are not any frequency points for which both functions are zero and therefore the CLS estimator can be obtained for all frequencies.