KARHUNEN-LOEVE (KLT) (continuous signals)
OR
HOTELLING TRANSFORM (discrete signals)

The term KLT is the most widely used

The concepts of ***eigenvalue*** and ***eigevector*** are necessary to understand the KL transform.

If $\underline{C}$ is a matrix of dimension $n \times n$, then a scalar λ is called an eigenvalue of $\underline{C}$ if there is a nonzero vector $\underline{e}$ in R^n such that

$$\underline{C}\,\underline{e} = \lambda \underline{e}$$

The vector $\underline{e}$ is called an eigenvector of the matrix $\underline{C}$ corresponding to the eigenvalue λ.

THE CASE OF MANY REALISATIONS OF A SIGNAL OR IMAGE

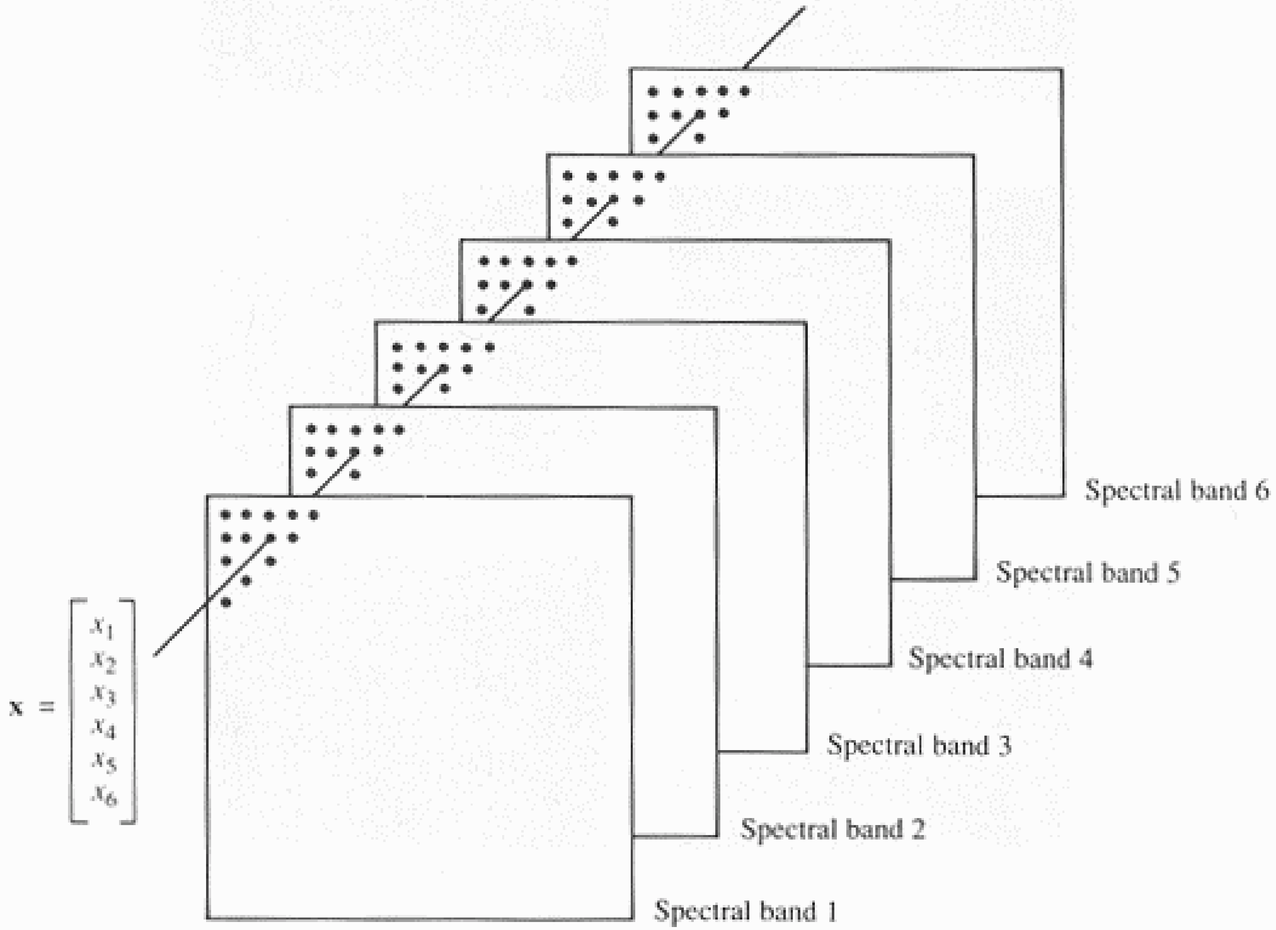
Consider a population of random vectors of the form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

x_i may represent an image

The population may refer to the number of pixels in each image.

Example: x vectors could be point values in several spectral bands (channels)



Formation of a vector from corresponding pixels in six images.

The ***mean vector*** of the population is defined as

$$\underline{m}_x = E\{\underline{x}\} \Rightarrow \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ \vdots \\ E\{x_n\} \end{bmatrix}$$

The ***covariance matrix*** of the population is defined as

$$\underline{C}_x = E\{(\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^T\}$$

For M vectors from a random population, where M is large enough

$$\underline{m}_x = \frac{1}{M}\sum_{k=1}^{M} \underline{x}_k$$

Let $\underline{A}$ be a matrix whose rows are formed from the eigenvectors of $\underline{C}_x$.

The first row of $\underline{A}$ is the eigenvector corresponding to the largest eigenvalue, and the last row the eigenvector corresponding to the smallest eigenvalue.

Suppose that $\underline{A}$ is a transformation matrix that maps the vectors $\underline{x}'s$ into vectors $\underline{y}'s$ by using the following transformation

$$\underline{y} = \underline{A}(\underline{x} - \underline{m}_x)$$

The above transform is called the ***Karhunen-Loeve*** or ***Hotelling*** transform.

$$\underline{y} = \underline{A}(\underline{x} - \underline{m}_x)$$

$$\underline{m}_y = \underline{0}$$

$$\underline{C}_y = \underline{A}\underline{C}_x\underline{A}^T$$

$$\underline{C}_y = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

The off-diagonal elements of the covariance matrix are 0, so the elements of the $\underline{y}$ vectors are uncorrelated !

To reconstruct the original vectors $\underline{x}$ from its corresponding $\underline{y}$

$$\underline{A}^{-1} = \underline{A}^{T}$$
$$\underline{x} = \underline{A}^{T}\underline{y} + \underline{m}_x$$

We form a matrix $\underline{A}_K$ from the K eigenvectors corresponding to the K largest eigenvalues, yielding a transformation matrix of order $K \times n$.

The $\underline{y}$ vectors would then be K dimensional.

The reconstruction of the original vector $\underline{\hat{x}}$ is

$$\underline{\hat{x}} = \underline{A}_K{}^{T}\underline{y} + \underline{m}_x$$

It can be proven that the mean square error between the perfect reconstruction $\underline{x}$ and the approximate reconstruction $\hat{\underline{x}}$ is given by the expression

$$e_{ms} = \left\| \underline{x} - \hat{\underline{x}} \right\|^2 = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{K} \lambda_j = \sum_{j=K+1}^{n} \lambda_j$$

By using $\underline{A}_K$ instead of $\underline{A}$ for the KL transform we achieve compression of the available data.

THE CASE OF ONE REALISATION OF A SIGNAL OR IMAGE

We assume that the 2-D signal (image) is *ergodic*.

Usually we divide the image into blocks and we apply the KLT in each block.

Mean vector inside the block:

$$m_f = \frac{1}{M^2} \sum_{k=1}^{M^2} \underline{f}(k)$$

Covariance matrix of the 2-D random field inside the block is $\underline{C}_f = \{c_{ij}\}$:

$$c_{ii} = \frac{1}{M^2} \sum_{k=1}^{M^2} \underline{f}(k)\underline{f}(k) - m_f^{\ 2}$$

$$c_{ij} = c_{i-j} = \frac{1}{M^2} \sum_{k=1}^{M^2} \underline{f}(k)\underline{f}(k+i-j) - m_f^{\ 2}$$

DRAWBACKS

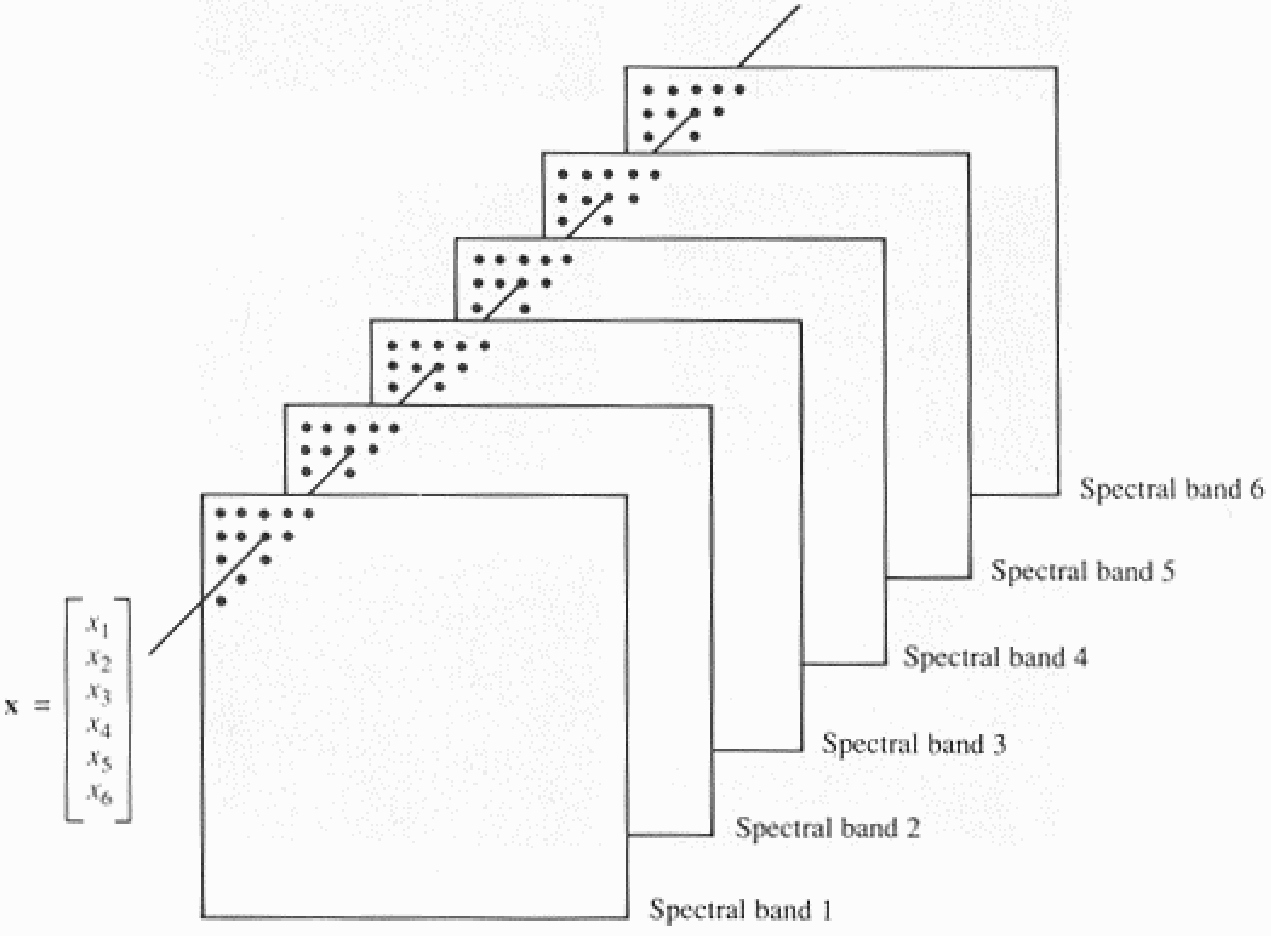
Despite its favourable theoretical properties, the KLT is not used in practice, because:

i) Its basis functions depend on the covariance matrix of the image, and hence they have to recomputed and transmitted for every image.

ii) Perfect decorrelation is not possible, since images can rarely be modelled as realisations of ergodic fields.

iii) There are no fast computational algorithms for its implementation.

Example: x vectors are point values in several spectral bands (channels)



Formation of a vector from corresponding pixels in six images.

Example: Original images

6 spectral images from an airborne Scanner.

Channel	Wavelength band (microns)
1	0.40–0.44
2	0.62–0.66
3	0.66–0.72
4	0.80–1.00
5	1.00–1.40
6	2.00–2.60

Channel 1

Channel 2

Channel 3

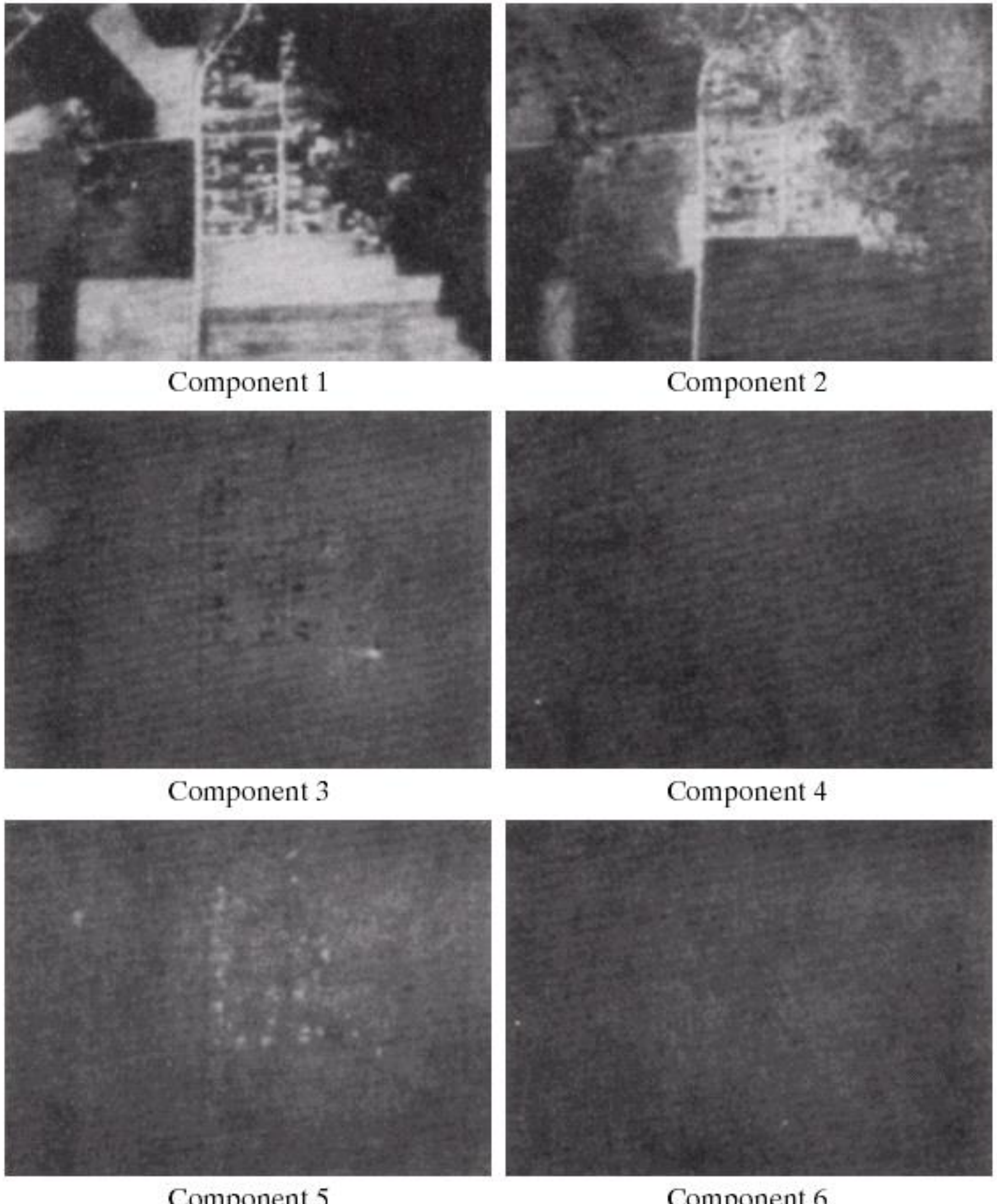
Channel 4

Channel 5

Channel 6

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

Example: Principal Components



Component	λ
1	3210
2	931.4
3	118.5
4	83.88
5	64.00
6	13.40

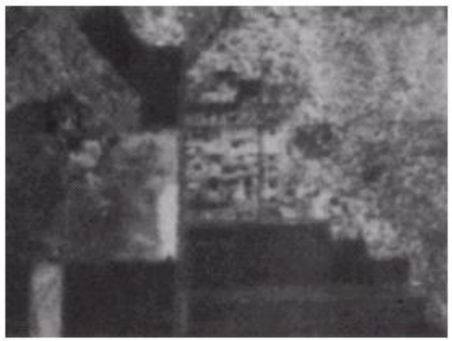
(Images from Rafael C. Gonzalez and Richard Wood, Digital Image Processing, 2nd Edition.

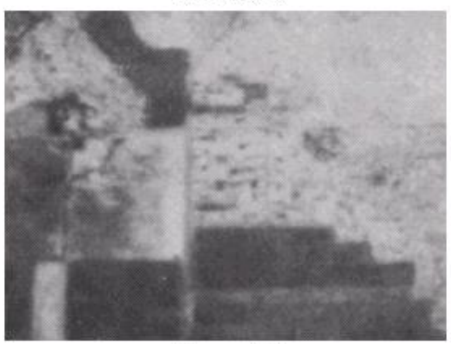
Example: Principal Components (cont.)

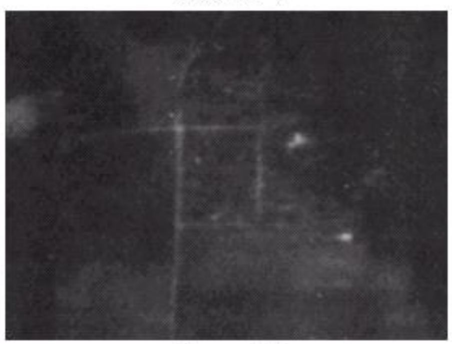

Channel 1


Channel 2

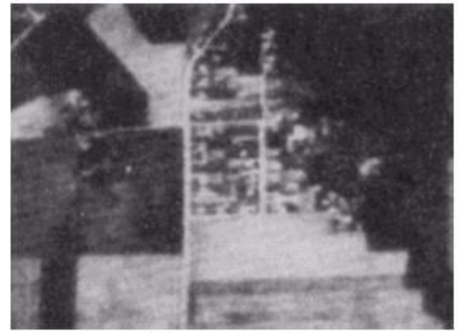

Channel 3


Channel 4

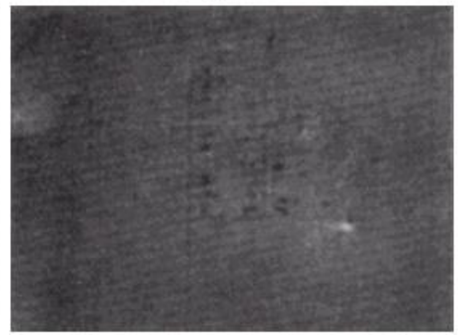

Channel 5

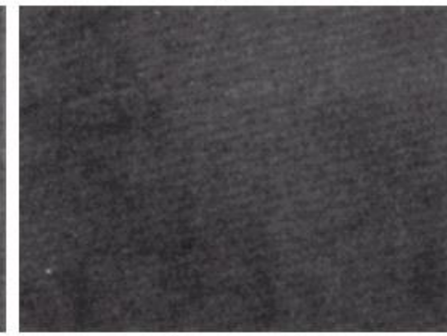

Channel 6

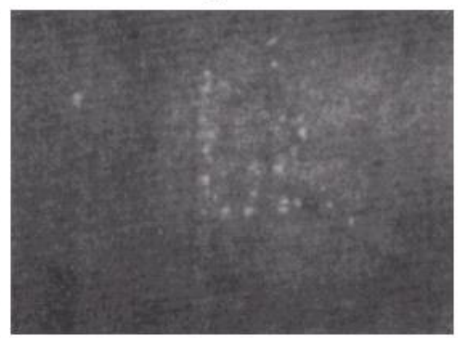
Original images (channels)


Component 1


Component 2


Component 3

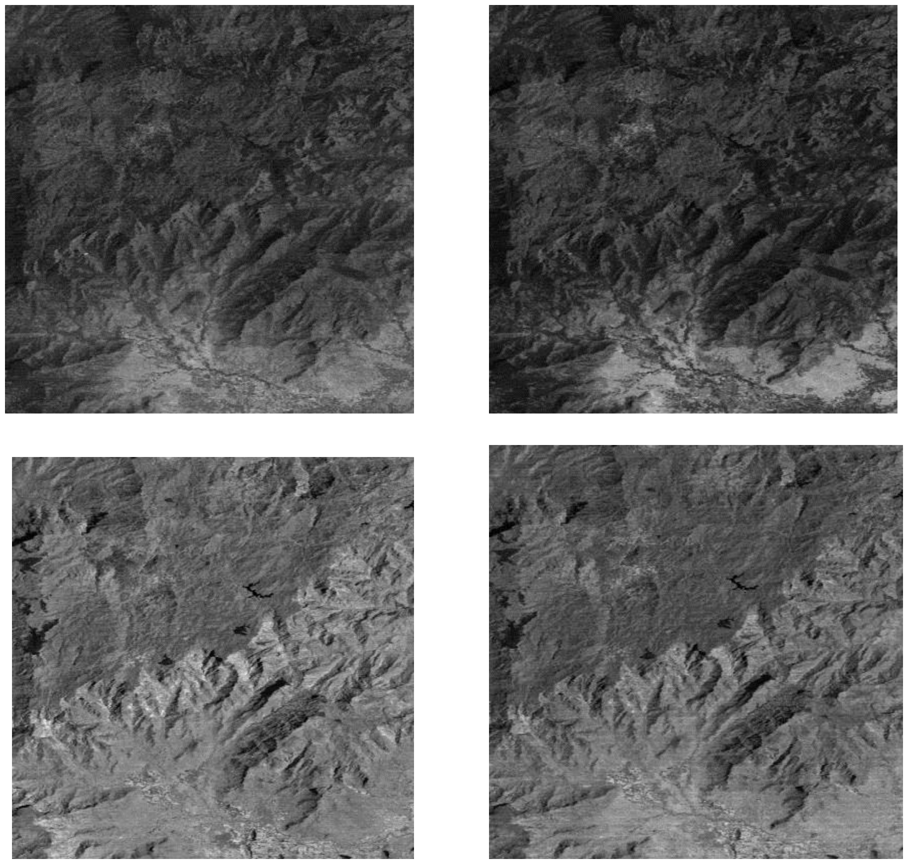

Component 4


Component 5


Component 6

Six principal components after Hotelling transform

Original Images



Karhunen-Loeve Images