DFT Sample Exam Problems with Solutions

1. Consider an $M \times M$ -pixel gray level real image $f(x, y)$ which is zero outside $-M \leq x \leq M$ and $-M \leq y \leq M$. Show that:
   (i) $F(-u,-v) = F^*(u,v)$ with $F(u,v)$ the two-dimensional Discrete Fourier Transform of $f(x,y)$.
   (ii) In order for the image to have the imaginary part of its two-dimensional Discrete Fourier Transform equal to zero, the image must be symmetric around the origin.
   (iii) In order for the image to have the real part of its two-dimensional Discrete Fourier Transform equal to zero, the image must be antisymmetric around the origin.

Solution

1. (a)

(i) $$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{-j\frac{2\pi}{M}(ux+vy)} \qquad (1)$$

$$F(-u,-v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{j\frac{2\pi}{M}(ux+vy)} = F^*(u,v)$$

if $f(x,y)$ = real

(ii) $$F(u,v) = \frac{1}{(2M+1)^2} \sum_{M}^{-M} \sum_{M}^{-M} f(-x,-y)\, e^{j\frac{2\pi}{M}(ux+vy)}$$

$$= \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(-x,-y)\, e^{j\frac{2\pi}{M}(ux+vy)} \qquad (2)$$

If $f(x,y)$ is symmetric, (2) becomes

$$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{j\frac{2\pi}{M}(ux+vy)} \qquad (3)$$

From (1) we write

$$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y) \cos\left[\frac{2\pi}{M}(ux+vy)\right]$$

$$-j \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y) \sin\left[\frac{2\pi}{M}(ux+vy)\right] = A - jB \qquad (4)$$

From (3) we write

$F(u,v) = A + jB \quad (5)$

From (4), (5) we see that $B = \mathrm{Im}\{F(u,v)\} = 0$

(iii) If $f(x,y)$ is antisymmetric, (2) becomes

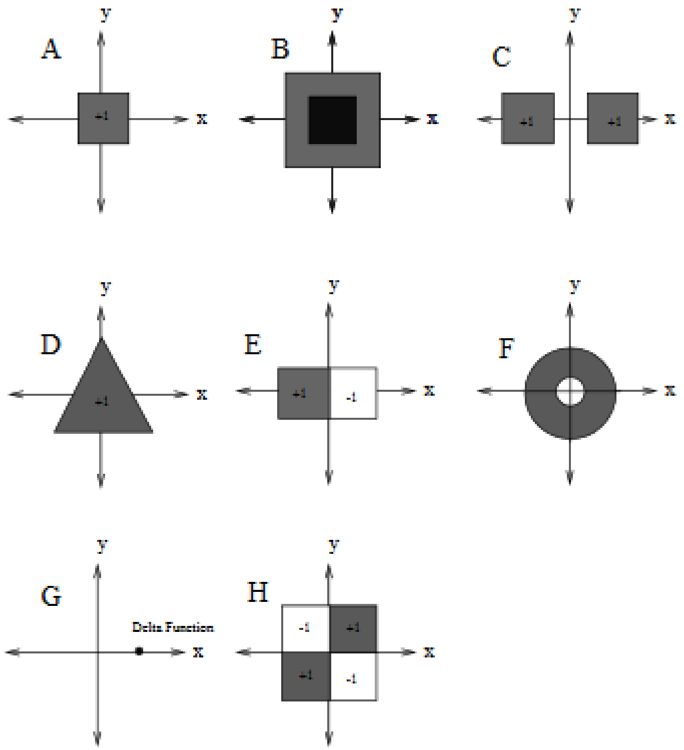
$$F(u,v) = \frac{-1}{(2M-1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{+j\frac{2\pi}{M}(ux+vy)} \qquad (6)$$

$$= -A - jB$$

From (4), (6) we see that $A = \mathrm{Re}\{F(u,v)\} = 0$

2. Consider the images shown below (A to H). Using knowledge of properties of the two-dimensional Discrete Fourier Transform symmetry and not exact calculation of it, list which image(s) will have a two-dimensional Discrete Fourier Transform $F(u,v)$ with the following properties:

(i) The imaginary part of $F(u,v)$ is zero for all u,v.
(ii) $F(0,0) = 0$
(iii) $F(u,v)$ has circular symmetry.
(iv) The real part of $F(u,v)$ is zero for all u,v.



Solution

(i) Having imaginary part 0 requires the image to be symmetric.

Therefore: A, B, C, F, H

(ii) $F(0,0) = \frac{1}{(\text{size})} \sum_x \sum_y f(x,y)$

Therefore: E, H

(iii) $F(u,v)$ has circular symmetry if $f(x,y)$ has circular symmetry, which occurs only for F.

(iv) The real part will be zero for any antisymmetric image such that $f(x,y) = -f(-x,-y)$ which is true only for E.

3. Consider an $M \times M$ -pixel gray level image $f(x, y)$ which is zero outside $0 \le x \le M-1$ and $0 \le y \le M-1$. The image intensity is given by the following relationship

$$f(x,y) = \begin{cases} c, & x = x_1, x = x_2, 0 \le y \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant value between 0 and 255 and x_1, x_2 $x_1 \ne x_2$ are constant values between 0 and $M-1$.

(i) Plot the image intensity.

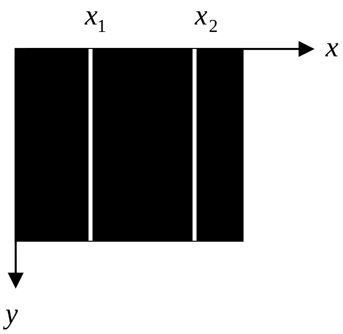
(ii) Find the $M \times M$ -point Discrete Fourier Transform (DFT) of $f(x, y)$.

(iii) Compare the original image and its Fourier Transform.

Hint: The following result holds: $\sum_{k=0}^{N-1} a^x = \frac{1-a^N}{1-a}$, $|a| \le 1$.

Solution

(i) Plot the image intensity.



(ii) For an image which contains only a single non-zero edge at $x = x_1$, the $M \times N$ -point Discrete Fourier Transform (DFT) of $f(x, y)$ is given as follows:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN}\sum_{y=0}^{N-1} f(x_1,y)e^{-j2\pi(ux_1/M+vy/N)}$$

$$= \frac{1}{MN} ce^{-j2\pi ux_1/M}\sum_{y=0}^{N-1} e^{-j2\pi vy/N} = \frac{1}{MN} ce^{-j2\pi ux_1/M}\frac{1-(e^{-j2\pi vy/N})^N}{1-e^{-j2\pi vy/N}}$$

$$= \frac{1}{MN} ce^{-j2\pi ux_1/M}\frac{1-e^{-j2\pi vy}}{1-e^{-j2\pi vy/N}}$$

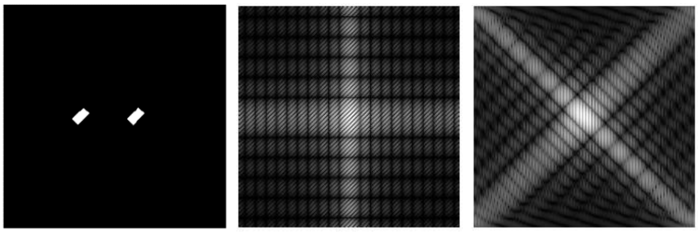
$$F(u,v) = \begin{cases} \frac{1}{M} ce^{-j2\pi ux_1/M}, & v = 0 \\ 0, & \text{otherwise} \end{cases}$$

For the image with 2 non-zero edges

$$F(u,v) = \begin{cases} \frac{1}{M} c(e^{-j2\pi ux_1/M} + e^{-j2\pi ux_2/M}), & v = 0 \\ 0, & \text{otherwise} \end{cases}$$

(iii) As seen a set of parallel straight lines in space implies a straight line perpendicular to the original one in frequency.

4. Consider the image shown in **Figure 1.1(a)** below. Two plots of magnitude of Two-Dimensional Discrete Fourier Transform (2D DFT) are shown in **Figure 1.1(b)** and **1.1(c)** below. Discuss which one is the magnitude of the 2D DFT of the image of **Figure 1.1(a)**. Justify your answer.



(a) **(b)** **(c)**

Figure 1.1

Solution

Figure (c) is the right answer since it contains edges which are perpendicular to the edges of the original image. As we know, each image in space produces a perpendicular image in the amplitude of the DFT.

5. Consider an $M \times N$-pixel image $f(x,y)$ which is zero outside $0 \le x \le M-1$ and $0 \le y \le N-1$. In transform coding, we discard the transform coefficients with small magnitudes and code only those with large magnitudes. Let $F(u,v)$ denote the $M \times N$-point Discrete Fourier Transform (DFT) of $f(x,y)$. Let $G(u,v)$ denote $F(u,v)$ modified by

$$G(u,v) = \begin{cases} F(u,v), & \text{when } |F(u,v)| \text{ is large} \\ 0, & \text{otherwise} \end{cases}$$

Let

$$\frac{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|G(u,v)|^2}{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2} = \frac{9}{10}$$

We reconstruct an image $g(x,y)$ by computing the $M \times N$-point inverse DFT of $G(u,v)$. Express $\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(f(x,y)-g(x,y))^2$ in terms of $\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)^2$.

Solution

The signal $f(x,y)-g(x,y)$ is obtained by the Inverse DFT of the signal $F(u,v)-G(u,v)$. Therefore, according to Parseval's theorem the energy of the signal $f(x,y)-g(x,y)$ is equal to the energy of the signal $F(u,v)-G(u,v)$. The signal $F(u,v)-G(u,v)$ consists of the DFT samples of $F(u,v)$ which were excluded in forming $G(u,v)$. Since, $G(u,v)$ captures 0.9 of the energy of $F(u,v)$, the signal $F(u,v)-G(u,v)$ will capture 0.1 of the energy of $F(u,v)$. Therefore,

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(f(x,y)-g(x,y))^2 = 0.1\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)^2$$