DCT-DHT Sample Exam Problems with Solutions

1. Let $f(x,y)$ denote a digital image of size 256×256. In order to compress this image, we take its Discrete Cosine Transform $C(u,v), u,v = 0, \dots, 255$ and keep only the Discrete Cosine Transform coefficients for $u,v = 0, \dots, n$ with $0 \le n < 255$. The percentage of total energy of the original image that is preserved in that case is given by the formula $an + b + 85$ with a, b constants. Furthermore, the energy that is preserved if $n = 0$ is 85%. Find the constants a, b.

Solution

For $n = 0$ it is given that the preserved energy is 85%. This is the case where only the (0,0) frequency pair is kept. Therefore,

$$a \cdot 0 + b + 85 = 85 \Rightarrow b = 0$$

In case where the entire DCT is kept we have $n = 255$ and the preserved energy should be 100%. In that case: $a \cdot 255 + 85 = 100 \Rightarrow 255a = 15 \Rightarrow a = \frac{1}{17} \Rightarrow b = 0$

2. Let $f(x,y)$ denote a digital image of size $M \times N$ pixels that is zero outside $0 \le x \le M-1$, $0 \le y \le N-1$, where M and N are integers and powers of 2. In implementing the standard Discrete Hadamard Transform of $f(x,y)$, we relate $f(x,y)$ to a new $M \times N$ point sequence $H(u,v)$.
   (i) State the main disadvantage of the Discrete Hadamard Transform.
   (ii) In the case of $M = N = 2$ and $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ calculate the Hadamard transform coefficients of $f(x,y)$.

Solution

(i) Bookwork

(ii) A large fraction of the signal energy is packed within very few transform coefficients, the ones near the origin. By keeping the low index transform coefficients and replacing the rest with zero we can achieve image compression. 2. Basis functions consist of 1s and -1s and therefore the transform is more resistant to errors.

(iii) We know that $N = 2^n$ and therefore, in case of $N = 2$ we have $n = 1$, x, y 0 or 1 and $b_0(0) = b_0(0) = 0$ and $b_0(1) = b_0(1) = 1$. For $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ we calculate the Walsh transform coefficients as follows.

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{2-1} \sum_{y=0}^{2-1} f(x,y) \left[\prod_{i=0}^{n-1} (-1)^{(b_i(x) b_{n-1-i}(u) + b_i(y) b_{n-1-i}(v))} \right]$$

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{1} \sum_{y=0}^{1} f(x,y) \left[\prod_{i=0}^{1-1} (-1)^{(b_i(x) b_{-i}(u) + b_i(y) b_{-i}(v))} \right]$$

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{1} \sum_{y=0}^{1} f(x,y) (-1)^{(b_0(x) b_0(u) + b_0(y) b_0(v))}$$

$$= \frac{1}{2} f(0,0)(-1)^{(b_0(0) b_0(u) + b_0(0) b_0(v))} + \frac{1}{2} f(0,1)(-1)^{(b_0(0) b_0(u) + b_0(1) b_0(v))}$$

$$+ \frac{1}{2} f(1,0)(-1)^{(b_0(1) b_0(u) + b_0(0) b_0(v))} + \frac{1}{2} f(1,1)(-1)^{(b_0(1) b_0(u) + b_0(1) b_0(v))}$$

$$= \frac{1}{2} f(0,0)(-1)^{(0 \cdot b_0(u) + 0 \cdot b_0(v))} + \frac{1}{2} f(0,1)(-1)^{(0 \cdot b_0(u) + 1 \cdot b_0(v))}$$
$$+ \frac{1}{2} f(1,0)(-1)^{(1 \cdot b_0(u) + 0 \cdot b_0(v))} + \frac{1}{2} f(1,1)(-1)^{(1 \cdot b_0(u) + 1 \cdot b_0(v))}$$
$$= \frac{1}{2} f(0,0)(-1)^0 + \frac{1}{2} f(0,1)(-1)^{b_0(v)} + \frac{1}{2} f(1,0)(-1)^{b_0(u)} + \frac{1}{2} f(1,1)(-1)^{b_0(u)+b_0(v)}$$
$$= \frac{1}{2}(-1)^0 + \frac{1}{2} 2(-1)^{b_0(v)} + \frac{1}{2} 2(-1)^{b_0(u)} + \frac{1}{2} 3(-1)^{b_0(u)+b_0(v)}$$
$$= \frac{1}{2} + (-1)^{b_0(v)} + (-1)^{b_0(u)} + \frac{3}{2}(-1)^{b_0(u)+b_0(v)}$$
$$W(0,0) = \frac{1}{2} + (-1)^{b_0(0)} + (-1)^{b_0(0)} + \frac{3}{2}(-1)^{b_0(0)+b_0(0)}$$
$$= \frac{1}{2} + (-1)^0 + (-1)^0 + \frac{3}{2}(-1)^0 = \frac{1}{2} + 1 + 1 + \frac{3}{2} = 4$$
$$W(0,1) = \frac{1}{2} + (-1)^{b_0(0)} + (-1)^{b_0(1)} + \frac{3}{2}(-1)^{b_0(0)+b_0(1)}$$
$$= \frac{1}{2} + (-1)^0 + (-1)^1 + \frac{3}{2}(-1)^{0+1} = \frac{1}{2} + 1 - 1 - \frac{3}{2} = -1$$
$$W(1,0) = \frac{1}{2} + (-1)^{b_0(1)} + (-1)^{b_0(0)} + \frac{3}{2}(-1)^{b_0(1)+b_0(0)}$$
$$= \frac{1}{2} + (-1)^1 + (-1)^0 + \frac{3}{2}(-1)^{1+0} = \frac{1}{2} - 1 + 1 - \frac{3}{2} = -1$$
$$W(1,1) = \frac{1}{2} + (-1)^{b_0(1)} + (-1)^{b_0(1)} + \frac{3}{2}(-1)^{b_0(1)+b_0(1)}$$
$$= \frac{1}{2} + (-1)^1 + (-1)^1 + \frac{3}{2}(-1)^{1+1} = \frac{1}{2} - 1 - 1 + \frac{3}{2} = 0$$

Therefore, $W(u,v) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$.

3. Let $f(x,y)$ denote the following constant 4×4 digital image that is zero outside $0 \le x \le 3$, $0 \le y \le 3$, with r a constant value.

$$\begin{bmatrix} r & r & r & r \\ r & r & r & r \\ r & r & r & r \\ r & r & r & r \end{bmatrix}$$

(i) Give the standard Hadamard Transform of $f(x,y)$ without carrying out any mathematical manipulations.
(ii) Comment on the energy compaction property of the standard Hadamard Transform.

Solution

(i) The 1-D Hadamard transform kernel is defined as:

$$\begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the image is constant we can guess the form of the Hadamard transform. There is only one component in the (0,0) frequency pair and its value is the mean of the image.

For your own interest the proof follows:

$H(x,u) = \prod_{i=0}^{n-1} (-1)^{b_i(x) b_i(u)}$. For signals of size 2 samples the Hadamard matrix is 2x2 and the Hadamard kernel has 4 samples as follows:

$$H(x,u) = \prod_{i=0}^{1-1} (-1)^{b_i(x)b_i(u)} = (-1)^{b_0(x)b_0(u)}$$

$H(0,0) = (-1)^{0\cdot 0} = 1$

$H(0,1) = (-1)^{0\cdot 1} = 1$

$H(1,0) = (-1)^{1\cdot 0} = 1$

$H(1,1) = (-1)^{1\cdot 1} = -1$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Using the recursive relationship of the Hadamard matrix we get:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We apply this matrix in the given image row-by-row and column-by-column or the other way round. We obtain:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the intermediate image is:

$$\begin{bmatrix} r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the final image is:

$$\begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) The standard Hadamard does not possess the property of energy compaction since the basis functions in the transformation matrix are not sorted to have increasing sequency.

4. Let $f(x,y)$ denote the following constant 4×4 digital image that is zero outside $0 \le x \le 3$, $0 \le y \le 3$, with r a constant value.

$$\begin{bmatrix} r & r & r & r \\ r & r & r & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Give the standard Hadamard Transform of $f(x, y)$.

Hint: Use the recursive relationship of the Hadamard matrix and the separability property of the Hadamard Transform.

Solution

The 1-D Hadamard transform kernel is defined as $H(x, u) = \prod_{i=0}^{n-1}(-1)^{b_i(x)b_i(u)}$. For signals of size 2 samples the Hadamard matrix is 2x2 and the Hadamard kernel has 4 samples as follows:

$$H(x, u) = \prod_{i=0}^{1-1}(-1)^{b_i(x)b_i(u)} = (-1)^{b_0(x)b_0(u)}$$

$$H(0,0) = (-1)^{0\cdot 0} = 1$$

$$H(0,1) = (-1)^{0\cdot 1} = 1$$

$$H(1,0) = (-1)^{1\cdot 0} = 1$$

$$H(1,1) = (-1)^{1\cdot 1} = -1$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Using the recursive relationship of the Hadamard matrix we get:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We apply this matrix in the given image row-by-row and column-by-column or the other way round. We obtain:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the intermediate image is:

$$\begin{bmatrix} r & 0 & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} r \\ r \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2r \\ 0 \\ 2r \\ 0 \end{bmatrix}.$$

Therefore, the final image is:

$$\begin{bmatrix} \frac{1}{2}r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$