Second Order Sequential Best Rotation Algorithm with Householder Reduction for Polynomial Matrix Eigenvalue Decomposition

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Outline

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Introduction

Motivation for PEVD

- EVD of Hermitian matrices is commonly used in
  - subspace decomposition for data compression
  - blind source separation
  - adaptive beamforming

  $\Rightarrow$ Assumption: Sources are narrowband
- Broadband signals need to model the correlation between sensor pairs across different time lags
  $\longrightarrow$ Polynomial matrices
- Development of PEVD algorithms and applications in
  - subspace decomposition using polynomial MUSIC [Alrmah et al. 2011]
  - blind source separation [Redif et al. 2017]
  - adaptive beamforming [Weiss et al. 2015]
  - source identification [Weiss et al. 2017]

Polynomial Matrices

The data vector at time index n collected from M-sensors is

$$\mathbf{x}(n) = [x_1(n), x_2(n), \ldots, x_M(n)]^T \in \mathbb{C}^M.$$

The space-time covariance matrix for N time snapshots is

$$\mathbf{A}(\tau) = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}^H(n-\tau)\} \approx \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{x}(n)\mathbf{x}^H(n-\tau) \ \in \mathbb{C}^{M\times M},$$

and its z-transform is a para-Hermitian polynomial matrix,

$$\mathbf{A}(z) = \sum_{\tau=-W}^{W}\mathbf{A}(\tau)z^{-\tau}.$$

Polynomial Eigenvalue Decomposition

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The PEVD of $\mathbf{A}(z)$ according to [McWhirter et al. 2007] is

$$\mathbf{A}(z) \approx \mathbf{U}(z)\mathbf{\Lambda}(z)\mathbf{U}^P(z),$$

where

- $\mathbf{U}^P(z) = \mathbf{U}^H(z^{-1})$,
- $\mathbf{\Lambda}(z)$ is the eigenvalue polynomial matrix and
- $\mathbf{U}(z)$ is the eigenvector polynomial matrix, such that

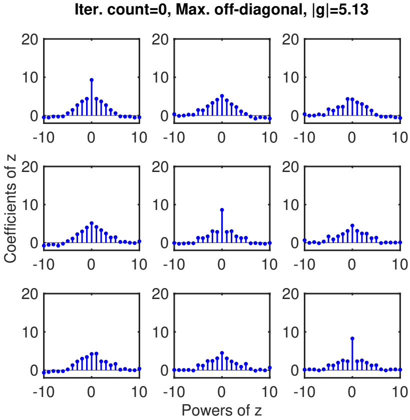
$$\mathbf{U}(z) = \mathbf{U}_L(z) \dots \mathbf{U}_2(z)\mathbf{U}_1(z),$$

constructed using L para-unitary polynomial matrices.

Comparison between EVD and PEVD

$$\begin{bmatrix} 9.30 & 5.12 & 4.23 \\ 5.12 & 8.61 & 4.50 \\ 4.23 & 4.50 & 8.27 \end{bmatrix}$$

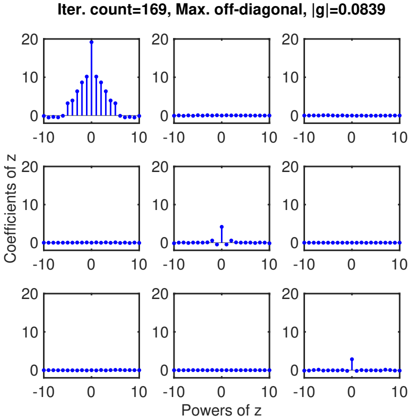
$\mathbf{A}$ taken from $\mathbf{A}(z^0)$.



$\mathbf{A}(z)$ example.

Comparison between EVD and PEVD

$$\begin{bmatrix} 18.0 & 0 & 0 \\ 0 & 4.53 & 0 \\ 0 & 0 & 3.66 \end{bmatrix}$$



$\mathbf{\Lambda}$ using EVD.

$\mathbf{\Lambda}(z)$ using SBR2 with $\delta = 0.087$.

$\delta \leq \sqrt{N_1/3} \times 10^{-2}$ where N_1 is the trace-norm of $\mathbf{A}(z^0)$ [McWhirter et al. 2007].

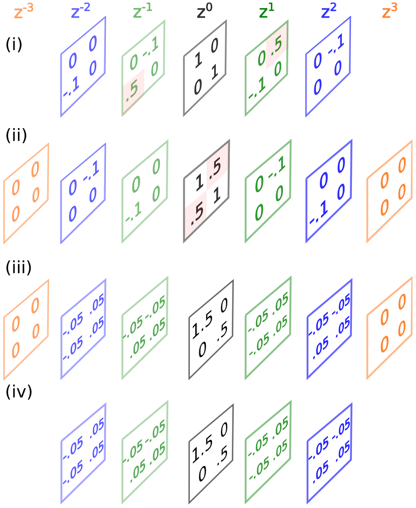
At each iteration, SBR2 will

(i) search for the largest off-diagonal, $|g|$,

(ii) delay and bring $|g|$ to the zero-lag plane,

(iii) zero $|g|$ using a Givens rotation and

(iv) trim negligible high order terms.



Family of PEVD Algorithms

SBR2 provided a framework for extensions based on (i)-(iv).

(i) search: norm-2 instead of inf-norm
  - Householder-like PEVD [Redif et al. 2011]
  - sequential matrix diagonalisation (SMD) [Redif et al. 2015]

(ii) delay: multiple-shift (MS) instead of single-shift
  - MS-SBR2 [Wang et al. 2015]
  - MS-SMD [Corr et al. 2014]

(iii) zero: one-step diagonalisation of z^0 instead of using the Givens rotation
  - SMD [Redif et al. 2015]
  - Householder-like PEVD [Redif et al. 2011]
  - approximate PEVD [Tkacenko 2011].

(iv) trim: row-shifted truncation SMD [Corr et al. 2015].

Proposed Method

Jacobi's Method for Symmetric EVD

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Consider the principal plane of a polynomial matrix, $A(z^0) \in \mathbb{C}^{M \times M}$.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,M} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,M} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,M} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{M-1,1} & a_{M-1,2} & a_{M-1,3} & \cdots & a_{M-1,M-1} & a_{M-1,M} \\ a_{M,1} & a_{M,2} & a_{M,3} & \cdots & a_{M,M-1} & a_{M,M} \end{bmatrix}$$

$\Rightarrow$ Cycling through all the off-diagonal elements using Jacobi's algorithm requires $\frac{M(M-1)}{2}$ Givens rotations.

Householder Reduction in EVD

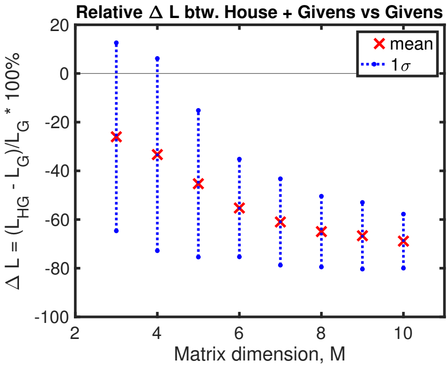
$(M-1)$ Householder reflections first reduce the principal plane to tridiagonal form [Golub et al. 1996].

$$\begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \dots & \dots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \dots & \vdots \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & a_{M-1,M-1} & a_{M-1,M} \\ 0 & \dots & \dots & \dots & a_{M,M-1} & a_{M,M} \end{bmatrix}$$

$\Rightarrow$ In this reduced form, there are fewer elements to zero.
$\Rightarrow$ Cycling through all the off-diagonal elements uses $(M-2)$ Householder reflections followed by $(M-1)$ Givens rotations.

Householder Reduction in EVD

Comparison of diagonalisation using Householder + Givens (HG) and Givens-only (G) using 1000 randomly generated symmetric matrices for every M with $\delta \leq \sqrt{N_1/3} \times 10^{-2}$.



⇒ The reduction in L achieved by Householder + Givens over Givens-only method scales with matrix dimension, M.

SBR2 with Householder Reduction

Inputs: $\mathbf{A}(z) \in \mathbb{C}^{M\times M}$, δ, maxIter, μ.

initialise: $l \leftarrow$0, g$\leftarrow 1+\delta$, $\tilde{\mathbf{\Lambda}}(z) = \mathbf{A}(z)$,$\tilde{\mathbf{U}}(z) = \mathbf{I}$.

while ($l <$maxIter **and** g$> \delta$) **do**

g $\leftarrow \max|r_{jk}(z^t)|, k > j, \forall t$. // *search*

if (g$> \delta$) **then**

$l \leftarrow l + 1$.

$\tilde{\mathbf{\Lambda}}(z) \leftarrow \mathbf{D}_j(z)\tilde{\mathbf{\Lambda}}(z)\mathbf{D}_j^P(z)$,

$\tilde{\mathbf{U}}(z) \leftarrow \mathbf{D}_j(z)\tilde{\mathbf{U}}(z)$ // *delay*

$\tilde{\mathbf{\Lambda}}(z) \leftarrow \mathbf{H}\tilde{\mathbf{\Lambda}}(z)\mathbf{H}^H$

$\tilde{\mathbf{U}}(z) \leftarrow \mathbf{H}\tilde{\mathbf{U}}(z)$ // *reflect*

$\tilde{\mathbf{\Lambda}}(z) \leftarrow \mathbf{G}(\theta,\phi)\tilde{\mathbf{\Lambda}}(z)\mathbf{G}^H(\theta,\phi)$,

$\tilde{\mathbf{U}}(z) \leftarrow \mathbf{G}(\theta,\phi)\tilde{\mathbf{U}}(z)$ // *rotate*

$\tilde{\mathbf{\Lambda}}(z) \leftarrow$trim$(\tilde{\mathbf{\Lambda}}(z),\mu)$,

$\tilde{\mathbf{U}}(z) \leftarrow$trim$(\tilde{\mathbf{U}}(z),\mu)$ // *trim*

end if

end while

return $\tilde{\mathbf{U}}(z)$, $\tilde{\mathbf{\Lambda}}(z)$.

Simulations and Results

Experiment Setup

The setup was based on the 3 sensors, 2 sources decorrelation simulation in [McWhirter et al. 2007] which used

- i.i.d. source signals of 1000 samples each and each sample was assigned ± 1 with equal probability
- each channel was modelled as a 5-th order FIR filter and each coefficent was drawn from $U[-1, 1]$
- additive white Gaussian noise with $\sigma = 1.8$
- PEVD parameters: $W = 10, \mu = 10^{-4}$, $\delta \leq \sqrt{N_1/3} \times 10^{-2}$

This was repeated 1000 times for the Monte-Carlo simulation.

Evaluation Measures

For each algorithm, we computed the

- Number of iterations, L
- Reconstruction error, $\epsilon \triangleq \sum_{\forall z} \|\tilde{\mathbf{A}}(z) - \mathbf{A}(z)\|_F$

For comparisons of both algorithms, we used

- Relative L difference, $\Delta L(\%) = \frac{L_{\text{Proposed}} - L_{\text{SBR2}}}{L_{\text{SBR2}}} \times 100\%$
- Relative ϵ difference, $\Delta\epsilon(\%) = \frac{\epsilon_{\text{Proposed}} - \epsilon_{\text{SBR2}}}{\sum_{\forall z} \|\mathbf{A}(z)\|_F} \times 100\%$

Tridiagonal Reduction in PEVD

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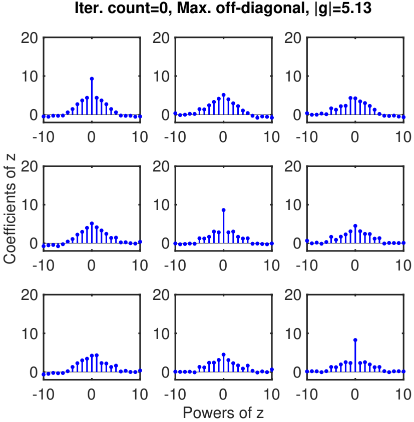
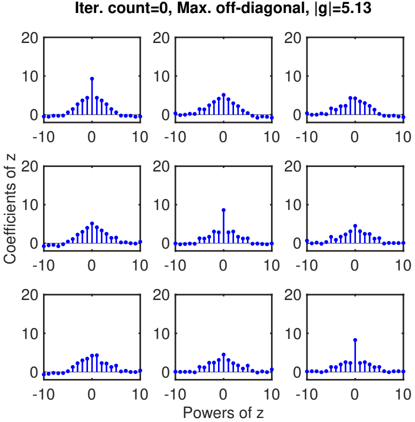
Diagonalisation target: Maximum off-diagonal $|g| \leq 0.087$



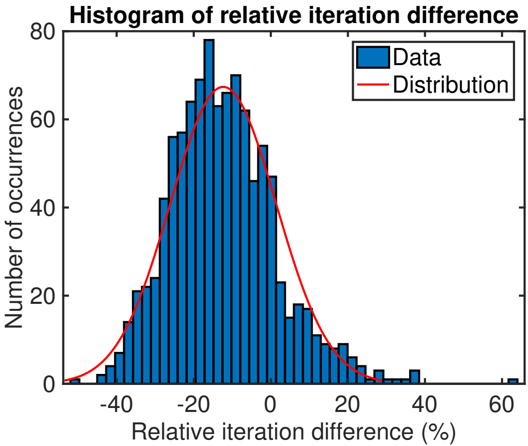
SBR2 took 169 iterations.



Our method took 101 iterations.

$\Rightarrow$ Tridiagonal reduction prior to applying the Givens rotations reduces the number of iterations for PEVD.

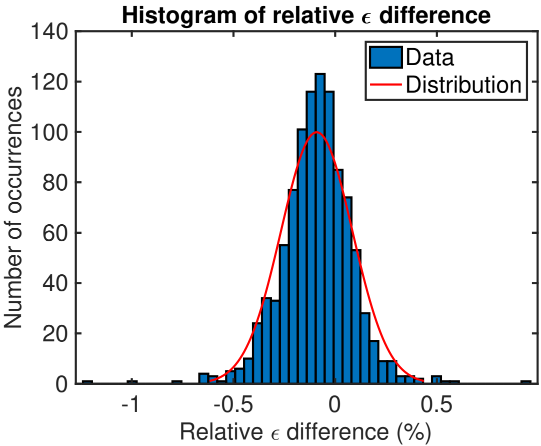
Monte-Carlo Results: Iteration Counts



⇒ Our method achieved an average of 12% reduction in L over SBR2.

⇒ Reduction in L was achieved in 82% of the trials.

Monte-Carlo Results: Reconstruction Error



$\Rightarrow$ Our method achieved an average of 0.1% reduction in ϵ.

$\Rightarrow$ Both methods were consistent to $\pm 1\%$ in ϵ.

Conclusion

Conclusion

- Proposed the use of Householder reduction before applying the Givens rotations at the zeroing step in SBR2.
- An average of 12% reduction in iteration counts is achievable.
- An average of 0.1% improvement in reconstruction error is achievable.
- Further reduction in iteration counts is expected as the matrix dimension increases.

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