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Solving Inverse Source Problems for Linear PDEs using Sparse Sensor Measurements

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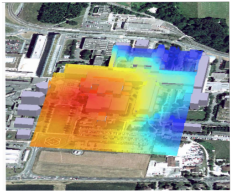
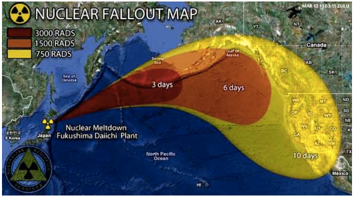
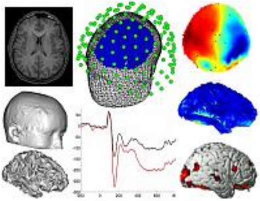
Outline

1. Introduction
   - Motivation
   - Problem Formulation
2. Source Reconstruction Framework
   - Source Reconstruction using structured least-squares methods
   - PDE-driven Inverse Problems and Sampling Theory
3. Simulation Results
4. Conclusions

Estimation of Physical Fields with Sensor Networks





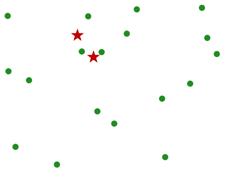


- Sensor networks are deployed to monitor and estimate various physical phenomena

Sensor Networks and Inverse Problems

Sensor networks measure:

- Leakages in/from factories,
- Temperature in server rooms,
- Nuclear fallouts (Fukushima).
- Acoustic sources localization



In the case of diffusion field: the field $u(\mathbf{x}, t)$ induced by a source distribution $f(\mathbf{x}, t)$ satisfies the following PDE:

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) - \mu \nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t). \quad (1)$$

Sensor Networks and Inverse Problems

- Different physical fields are regulated by different (linear) PDEs
- **Other PDEs:** Wave Equation, Laplace's Equation, Advection-/Convection-Diffusion Equation, Helmholtz and many more.

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$$u(\mathbf{x}, t) = (f * g)(\mathbf{x}, t)$$

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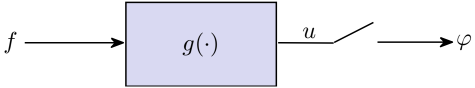
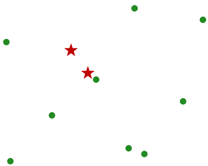
  where $g(\mathbf{x}, t)$ is the Green function of the field.
- **Goal**: Given the measurements of the field, estimate the source $f(\mathbf{x}, t)$.
  - $f(\mathbf{x}, t)$ might be **sparse**,
  - $u(\mathbf{x}, t)$ is normally not sparse

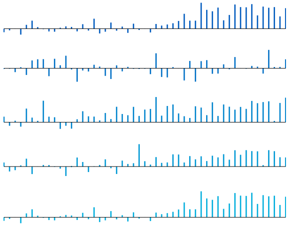
Problem Formulation: Field Measurements

Aim

Estimate $f(\mathbf{x}, t)$ from spatiotemporal samples $\{\varphi_{n,l} = u(\mathbf{x}_n, t_l)\}_{n,l}$ for $n = 1, \ldots, N$ and $l = 0, \ldots, L$, of the measured field.







Problem Formulation: Sources of Interest

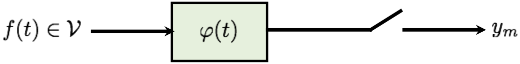
	Instantaneous	Non-Instantaneous
Point	$f(\mathbf{x},t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m)$	$f(\mathbf{x},t) = \sum_{m=1}^{M} c_m e^{\alpha_m (t-\tau_m)} \delta(\mathbf{x} - \boldsymbol{\xi}_m) H(t - \tau_m)$
Line	$f(\mathbf{x},t) = cL(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t) = cL(\mathbf{x}) e^{\alpha(t-\tau)} H(t-\tau)$
Polygonal	$f(\mathbf{x},t) = cF(\mathbf{x})\delta(t-\tau)$	$f(\mathbf{x},t) = cF(\mathbf{x}) e^{\alpha(t-\tau)} H(t-\tau)$

Where,

- $L(\mathbf{x}) \in \Omega$ describes a line with endpoints $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$.
- $F(\mathbf{x}) \in \Omega$ describes a convex polygon with vertices $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_M\}$.
- $\alpha_m, c_m, \boldsymbol{\xi}_m$ and τ_m is the release rate, intensity, location and activation time of m-th source.

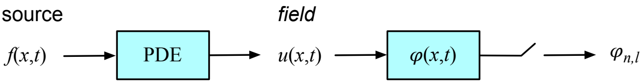
Traditional Sampling vs Sampling Physical Fields

Traditional Sampling Set-up:



-The signal f(t) lies in a subspace, is sparse (e.g., CS), is parametric (e.g., FRI)
-The acquisition device given by the set-up or by design (e.g., random matrix)

Sampling physical fields:



- No assumption on the field but on the sources,
- The acquisition device performs only temporal filtering, no spatial filtering

Source Reconstruction Framework

Recall that

$$\begin{aligned} u(\mathbf{x}, t) &= \int_{\mathbf{x}' \in \mathbb{R}^2} \int_{t' \in \mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x} - \mathbf{x}', t - t') \, \mathrm{d}t' \mathrm{d}\mathbf{x}' \\ &= \langle f(\mathbf{x}', t'), g(\mathbf{x} - \mathbf{x}', t - t') \rangle_{\mathbf{x}', t'} . \end{aligned}$$

Mathematically the spatiotemporal sample $\varphi_{n,l}$ is

$$\begin{aligned} \varphi_{n,l} &= u(\mathbf{x}_n, t_l) \\ &= \langle f(\mathbf{x}, t), g(\mathbf{x}_n - \mathbf{x}, t_l - t) \rangle_{\mathbf{x}, t} \end{aligned} \quad (2)$$

Consider a weighted-sum of the samples $\{\varphi_{n,l}\}_{n,l}$:

$$\mathcal{R}(k) = \sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}^{(k)}\varphi_{n,l} = \sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}^{(k)} \langle f(\mathbf{x},t), g(\mathbf{x}_n - \mathbf{x}, t_l - t)\rangle_{\mathbf{x},t}$$
$$= \left\langle f(\mathbf{x},t), \underbrace{\sum_{n=1}^{N}\sum_{l=0}^{L} w_{n,l}^{(k)} g(\mathbf{x}_n - \mathbf{x}, t_l - t)}_{=\Psi_k(\mathbf{x})\Gamma(t)} \right\rangle, \quad (3)$$

where $w_{n,l} \in \mathbb{C}$ are some arbitrary weights (to be determined).
We wish to find $f(\mathbf{x},t)$:

- For our source types, can we choose functions $\Psi_k(\mathbf{x})$ and $\Gamma(t)$ that makes this problem tractable? — YES!

Let these (new) *generalized measurements* be

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l}^{(k)} \varphi_{n,l} = \langle f(\mathbf{x}, t), \Psi_k(\mathbf{x})\Gamma(t) \rangle$$
$$= \int_{\Omega} \int_{t \in [0,T]} \Psi_k(\mathbf{x})\Gamma(t) f(\mathbf{x}, t) \mathrm{d}t \mathrm{d}V,$$

where $\Psi_k(\mathbf{x})$ for $k \in \mathbb{Z}^d$ and $\Gamma(t)$ a family of properly chosen *spatial* and *temporal sensing functions*, respectively.
Proper choice $\Longrightarrow$ solvability & stability of new problem.

- As an example, take the **instantaneous** source distribution

$f(\mathbf{x}, t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m)$, then:

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m \Psi_k(\boldsymbol{\xi}_m)\Gamma(\tau_m).$$

Choice of Sensing Functions: 2D Case

For $\mathbf{x} \in \mathbb{R}^2$, we may choose

- $\Gamma(t) = e^{-\mathrm{j}t/T}$, and
- $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$, for $k = 0, 1, \ldots, K$.

Then,

$$\begin{aligned}\mathcal{R}(k) &= \sum_{m=1}^{M} c_m e^{-\mathrm{j}\tau_m/T} e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})} \\ &= \sum_{m=1}^{M} c'_m v_m^k.\end{aligned}$$

Can be solved to jointly recover $c'_m = c_m e^{-\mathrm{j}\tau_m/T}$ and $v_m = e^{-(\xi_{1,m}+\mathrm{j}\xi_{2,m})}$ using Prony's method for $m = 1, \ldots, M$ providing $K \geq 2M - 1$.

Computing $\mathcal{R}(k)$ reliably from sensor Measurements?

Recall that,

$$\mathcal{R}(k) = \sum_{n=1}^{N} \sum_{l=0}^{L} w_{n,l}^{(k)} \varphi_{n,l}$$

- Thus computing $\mathcal{R}(k)$ is equivalent to finding the weights $w_{n,l}$.

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- The weights may be found using results from **sampling** theory and **approximation** theory

Approximation of Exponentials

- Let's assume a 1-D static field and uniformly spaced sensors.
- We want to find coefficients $w_n^{(k)}$ that give us a good approximation of the exponentials:

$$\sum_n w_n^{(k)} g(x-n) \simeq e^{jkx}$$

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- *Approximation error*

$$\varepsilon(x) = f(x) - e^{jkx} = \mathrm{e}^{jkx}\left[1 - w_0 \sum_{l\in\mathbb{Z}} \hat{g}(jk + j2\pi l)\mathrm{e}^{j2\pi l x}\right].$$

Generalised Strang-Fix Conditions

A function $g(x)$ can reproduce the exponential:

$$e^{j\omega_k x} = \sum_n w_n^{(k)} g(x-n)$$

if and only if

$$\hat{g}(j\omega_k) \neq 0 \text{ and } \hat{g}(j\omega_k + j2\pi l) = 0 \qquad l \in \mathbb{Z} \setminus \{0\}$$

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Note: most Green's functions approximately satisfy Strang-Fix conditions:

- if $g(\mathbf{x}, t)$ is the diffusion kernel, Strang-Fix approximately satisfied
- if $g(\mathbf{x}, t)$ is the wave kernel, Strang-Fix not satisfied in time (pre-filtering in time required)

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   - $\mathcal{R}(k)$ can be found using gossip algorithms
3. $\mathcal{R}(k) = \sum_{m=1}^{M} c'_m v_m^k$.
4. Use Prony's method to retrieve the source $f(\mathbf{x}, t)$ from $\mathcal{R}(k)$, the physical field is given by $u(\mathbf{x}, t) = g(\mathbf{x}, t) * f(\mathbf{x}, t)$

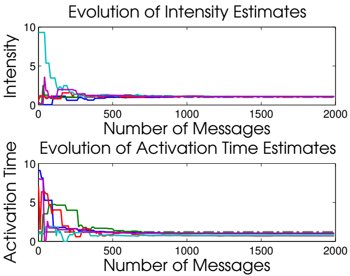
Introduction
Motivation

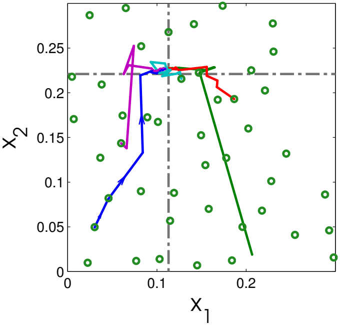
Source Reconstruction Framework
Point Sources
Computing $\mathcal{R}(k)$

Simulation Results

Conclusion

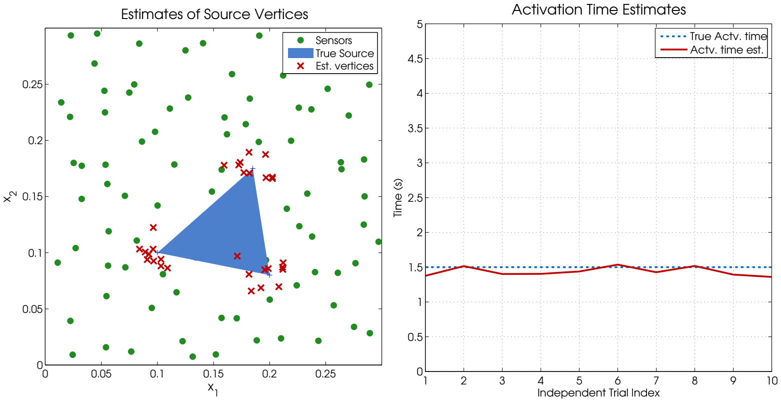
Synthetic data: Point Diffusion Source





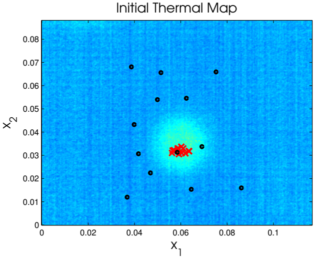
Distributed estimation for $M = 1$ source using 45 sensors, field is sampled for $T_{end} = 10s$ at $\frac{1}{\Delta t} = 1Hz$. $K = 1$.

Synthetic data: Triangular Diffusion Source

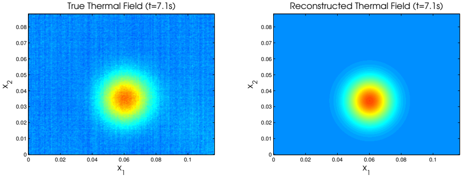


$N = 90$ arbitrarily placed sensors, field sampled at $10Hz$ for $T = 10s$ with measurement SNR$= 35$dB. $K = 6$ and $R = 5$.

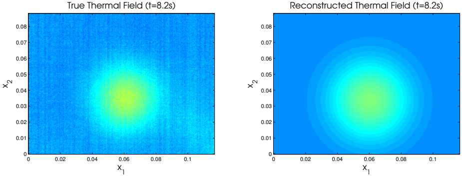
Simulation Results: Real Diffusion Data



(a) Thermal distribution (immediately after activation) and location estimates.

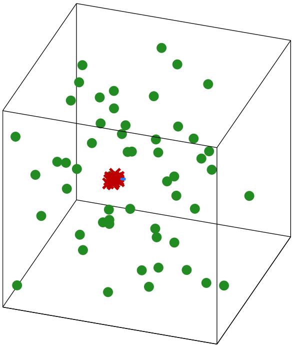


(b) Real field (left) and its reconstruction (right) at $t = 7.1s$.



(c) Real field (left) and its reconstruction (right) at $t = 8.2s$.

Simulation Results: Laplace - Synthetic data



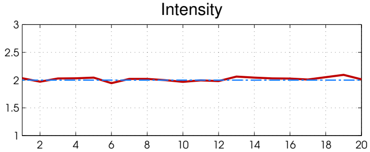


Figure: Single point source recovery in 3D using samples obtained by $N = 57$ sensors with $K_1 = K_2 = 1$ for spatial sensing function family. Results for 20 independent trials are given.

Conclusions and Future Work

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Future work

- Consider moving sensors
- Estimate simultaneously sensors' location and inducing sources

References

On Approximate Strang-Fix

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On Diffusion Fields and Sensor Networks

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Thank You.