Imperial College London

Parametric Sparse Sampling and its Applications in Neuroscience and Sensor Networks

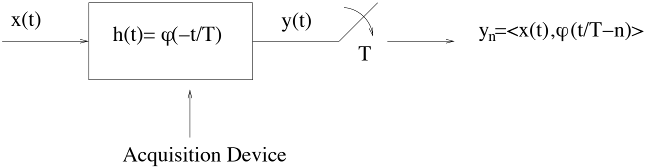
Pier Luigi Dragotti

April 24, 2014[1]

[1]This research is supported by European Research Council ERC, project 277800 (RecoSamp)

Problem Statement

You are given a class of functions. You have a sampling device. Given the measurements $y_n = \langle x(t), \varphi(t/T - n) \rangle$, you want to reconstruct $x(t)$.

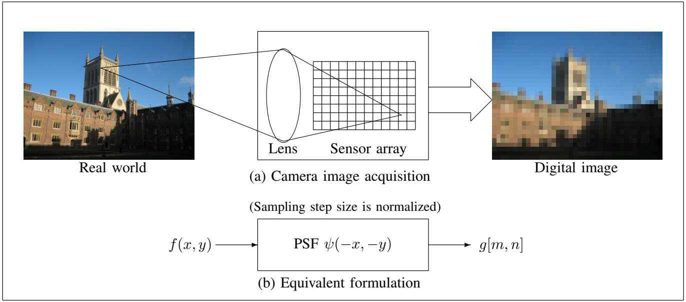


Natural questions:

- When is there a one-to-one mapping between $x(t)$ and y_n?
- What signals can be sampled and what kernels $\varphi(t)$ can be used?
- What reconstruction algorithm?

Imperial College London

The Information Acquisition Process



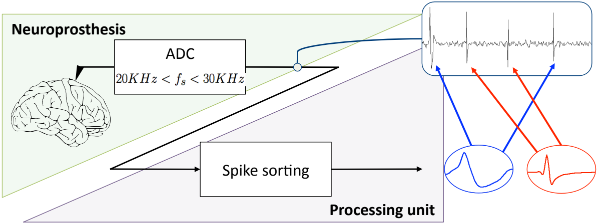
(a) Camera image acquisition

(b) Equivalent formulation

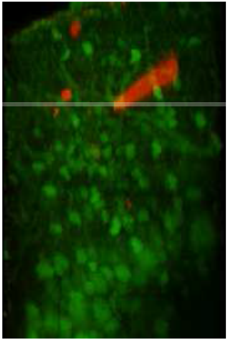
- The lens blurs the image.
- The image is sampled ('pixelized') by the CCD array.
- You want to develop techniques that give you the sharpest and highest possible resolution images given the available acquisition device

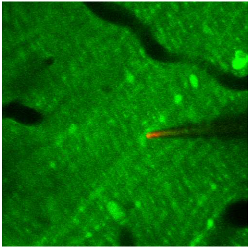
Motivation: Sampling Everywhere

Applications in Neuroscience



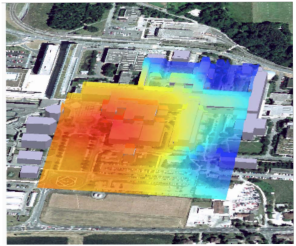
Neural Activity Detection

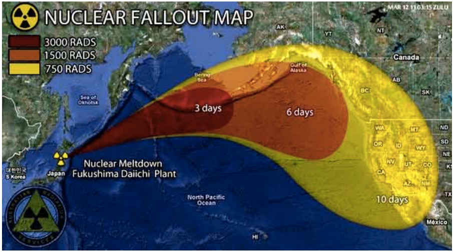




Imperial College London

Motivation: Sensor Networks



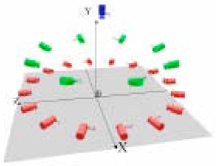


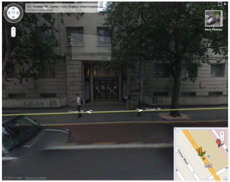
- The source (phenomenon) is distributed in space and time.
- The phenomenon is sampled in space and time.
- How many sensors? How can we localise the diffusion source?

Imperial College London

Motivation: Free Viewpoint Video

Multiple cameras are used to record a scene or an event. Users can freely choose an arbitrary viewpoint for 3D viewing.





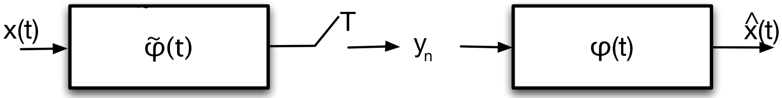
- This is a multi-dimensional sampling and interpolation problem.

Outline

- Classical Sampling Formulation and Signals with FRI
- Sampling Kernels and Approximate Strang-Fix Conditions
- From Samples to Signals
- Robust and Universal Sparse Sampling
- Applications in
  - Image Super-Resolution
  - Neuroscience
  - Sensor Networks
- Conclusions and Outlook

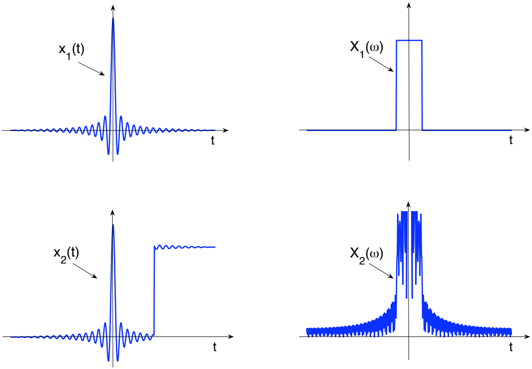
Classical Sampling Formulation

- Sampling of $x(t)$ is equivalent to projecting $x(t)$ into the shift-invariant subspace $V = \text{span}\{\varphi(t/T - n)\}_{n \in \mathbb{Z}}$.
- If $x(t) \in V$, perfect reconstruction is possible.
- Reconstruction process is linear: $\hat{x}(t) = \sum_n y_n \varphi(t/T - n)$.
- For bandlimited signals $\varphi(t) = \text{sinc}(t)$.



Nyquist Sampling Rate vs Rate of Information

Here, $x_1(t)$ and $x_2(t)$ have the same rate of innovation. However, one discontinuity and no sampling theorems ;-)



Signals with Finite Rate of Innovation

- The signal $x(t) = \sum_n y_n \varphi(t/T - n)$ is exactly specified by one parameter y_n every T seconds, $x(t)$ has a finite number $\rho = 1/T$ of degrees of freedom per unit of time.
- In the classical formulation, innovation is uniform. How about signals where the rate of innovation is finite but non-uniform? E.g.
  - Piecewise sinusoidal signals (Frequency Hopping modulation)
  - Pulse position modulation (UWB)
  - Edges in images

Signals with Finite Rate of Innovation

Consider a signal of the form:

$$x(t) = \sum_{k \in \mathbb{Z}} \gamma_k \varphi(t - t_k). \quad (1)$$

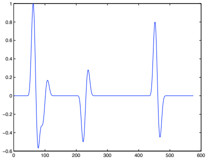
The rate of innovation of $x(t)$ is then defined as

$$\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_x \left(-\frac{\tau}{2}, \frac{\tau}{2} \right), \quad (2)$$

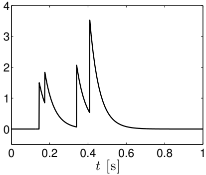
where $C_x(-\tau/2, \tau/2)$ is a function counting the number of free parameters in the interval τ.

Definition [VetterliMB:02] A signal with a **finite rate of innovation** is a signal whose parametric representation is given in (1) and with a finite ρ as defined in (2).

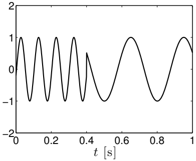
Examples of Signals with Finite Rate of Innovation



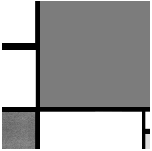
Filtered Streams of Diracs



Decaying Exponentials

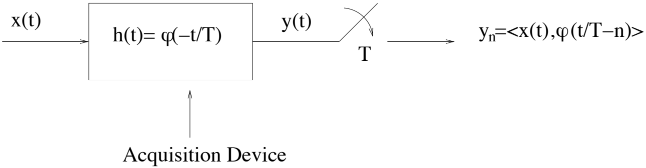


Piecewise Sinusoidal Signals



Mondrian paintings ;-)

Sampling Kernels



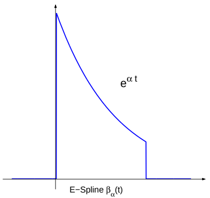
- Given by nature
  - Diffusion equation, Green function. Ex: sensor networks.
- Given by the set-up
  - Designed by somebody else. Ex: Hubble telescope, digital cameras.
- Given by design
  - Pick the best kernel. Ex: engineered systems.

Sampling Kernels

Any kernel $\varphi(t)$ that can reproduce exponentials:

$$\sum_n c_{m,n}\varphi(t-n) = e^{\alpha_m t}, \qquad \alpha_m = \alpha_0 + m\lambda \text{ and } m = 0, 1, ..., L.$$

This includes any composite kernel of the form $\gamma(t) * \beta_{\vec{\alpha}}(t)$ where $\beta_{\vec{\alpha}}(t) = \beta_{\alpha_0}(t) * \beta_{\alpha_1}(t) * ... * \beta_{\alpha_L}(t)$ and $\beta_{\alpha_i}(t)$ is an Exponential Spline of first order [UnserB:05].

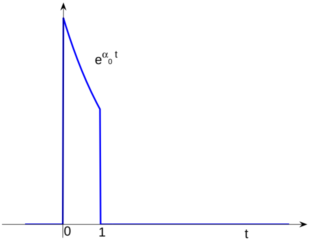


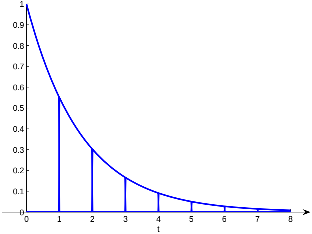
$$\beta_\alpha(t) \Leftrightarrow \hat{\beta}(\omega) = \frac{1 - e^{\alpha - j\omega}}{j\omega - \alpha}$$

Notice:

- α can be complex.
- E-Spline is of compact support.
- E-Spline reduces to the classical polynomial spline when $\alpha = 0$.

Exponential Reproducing Kernels





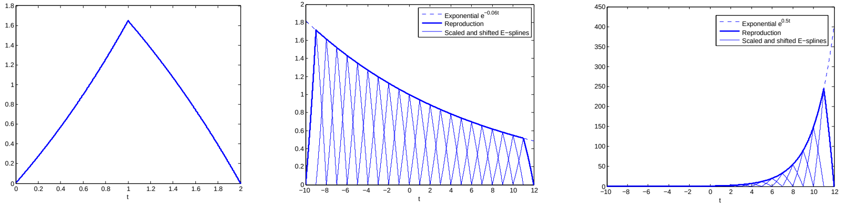
The E-spline of first order $\beta_{\alpha_0}(t)$ reproduces the exponential $e^{\alpha_0 t}$:

$$\sum_n c_{0,n}\beta_{\alpha_0}(t-n) = e^{\alpha_0 t}.$$

In this case $c_{0,n} = e^{\alpha_0 n}$. In general, $c_{m,n} = c_{m,0}e^{\alpha_m n}$.

Imperial College London

Exponential Reproducing Kernels



Here the E-spline is of second order and reproduces the exponential $e^{\alpha_0 t}$, $e^{\alpha_1 t}$: with $\alpha_0 = -0.06$ and $\alpha_1 = 0.5$.

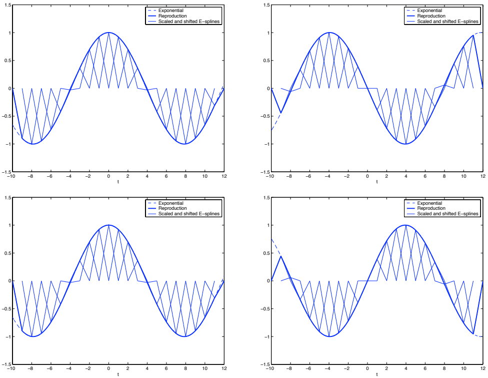
Exponential Reproducing Kernels

- The exponent α of the E-splines can be complex. This means $\beta_\alpha(t)$ can be a complex function.
- However if pairs of exponents are chosen to be complex conjugate then the spline stays real.
- Example:

$$\beta_{\alpha_0+j\omega_0}(t) * \beta_{\alpha_0-j\omega_0}(t) = \begin{cases} \frac{\sin\omega_0 t}{\omega_0} e^{\alpha_0 t} & 0 \leq t < 1 \\ -\frac{\sin\omega_0(t-2)}{\omega_0} e^{\alpha_0 t} & 1 \leq t < 2 \\ 0 & \text{Otherwise} \end{cases}$$

When $\alpha_0 = 0$ (i.e., purely imaginary exponents), the spline is called trigonometric spline.

Exponential Reproducing Kernels



Here $\vec{\alpha} = (-j\omega_0, j\omega_0)$ and $\omega_0 = 0.2$. $\sum_n c_{n,m}\beta_{\vec{\alpha}}(t-n) = e^{jm\omega_0} \qquad m = -1, 1$.

Notice: $\beta_{\vec{\alpha}}(t)$ is a real function, but the coefficients $c_{m,n}$ are complex.

Generalised Strang-Fix Conditions

A function $\varphi(t)$ can reproduce the exponential:

$$e^{j\omega_m t} = \sum_n c_{m,n}\varphi(t-n)$$

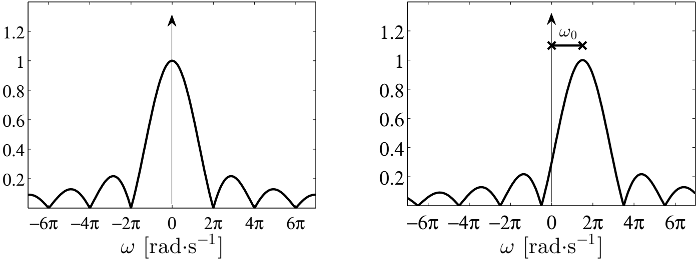
if and only if

$$\hat{\varphi}(j\omega_m) \neq 0 \text{ and } \hat{\varphi}(j\omega_m + j2\pi l) = 0 \qquad l \in \mathbb{Z} \setminus \{0\}$$

where $\hat{\varphi}(\cdot)$ is the Fourier transform of $\varphi(t)$.

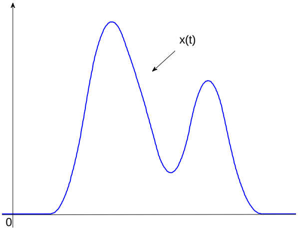
Also note that $c_{m,n} = c_{m,0}e^{j\omega_m n}$ with $c_{m,0} = \hat{\varphi}(j\omega_m)^{-1}$.

Generalised Strang-Fix Conditions



(a) $|\hat{\beta}_\alpha(\omega)|$ with $\alpha = 0$ **(b)** $|\hat{\beta}_\alpha(\omega)|$ with $\alpha = i\omega_0$

From Samples to Signals



- Consider any $x(t)$ with $t \in [0, N)$ and sampling period $T = 1$.
- The sampling kernel $\varphi(t)$ satisfies

$$\sum_n c_{m,n}\varphi(t-n) = e^{j\omega_m t} \qquad m = 1, ..., L,$$

- We want to retrieve $x(t)$, from the samples $y_n = \langle x(t), \varphi(t-n)\rangle$, $n = 0, 1, ..., N-1$.

From Samples to Signals

We have that

$$
\begin{aligned}
s_m &= \sum_{n=0}^{N-1} c_{m,n} y_n \\
&= \langle x(t), \sum_{n=0}^{N-1} c_{m,n}\varphi(t-n)\rangle \\
&= \int_{-\infty}^{\infty} x(t) e^{j\omega_m t} dt, \qquad m = 1, ..., L.
\end{aligned}
$$

- Note that s_m is the Fourier transform of $x(t)$ evaluated at $j\omega_m$.

From Samples to Signals

- Consider FRI signals which are completely specified by a finite number of free parameters
- This is an 'analogue' sparsity model
- For classes of parametrically sparse signals there is a one-to-one mapping between samples and signal:

$$x(t) \Leftrightarrow \hat{x}(j\omega_m) \qquad m = 1, 2, ..., L$$

- The number d of degrees of freedom of the signal must satisfy $d \leq L$

Sampling Streams of Diracs

- Assume $x(t)$ is a stream of K Diracs on the interval of size N: $x(t)=\sum_{k=0}^{K-1} x_k\delta(t-t_k),\ t_k\in[0,N)$.
- We restrict $j\omega_m = j\omega_0 + jm\lambda \quad m=1,...,L$ and $L\geq 2K$.
- We have N samples: $y_n=\langle x(t),\varphi(t-n)\rangle,\ n=0,1,...N-1$:
- We obtain

$$
\begin{aligned}
s_m &= \sum_{n=0}^{N-1} c_{m,n}y_n \\
&= \int_{-\infty}^{\infty} x(t)e^{j\omega_m t}dt, \\
&= \sum_{k=0}^{K-1} x_k e^{j\omega_m t_k} \\
&= \sum_{k=0}^{K-1} \hat{x}_k e^{j\lambda m t_k} = \sum_{k=0}^{K-1} \hat{x}_k u_k^m, \qquad m=1,...,L.
\end{aligned}
$$

Prony's Method

- The quantity
$$s_m = \sum_{k=0}^{K-1} \hat{x}_k u_k^m, \qquad m = 1, ..., L$$
is a sum of exponentials.
- Retrieving the locations u_k and the amplitudes $\hat{x}_k$ from $\{s_m\}_{m=1}^{L}$ is a classical problem in spectral estimation and was first solved by Gaspard de Prony in 1795.
- Given the pairs $\{u_k, \hat{x}_k\}$, then $t_k = (\ln u_k)/\lambda$ and $x_k = \hat{x}_k / e^{\alpha_0 t_k}$.

Overview of Prony's Method

Assume: $y_n = \sum_{k=0}^{K-1} \alpha_k u_k^m$ and consider the polynomial:

$$P(x) = \prod_{k=1}^{K}(x - u_k) = x^K + h_1 x^{K-1} + h_2 x^{K-2} + \ldots + h_{K-1}x + h_K.$$

It is easy to verify that

$$y_{n+K} + h_1 y_{n+K-1} + h_2 y_{n+K-2} + \ldots + h_K y_n = \sum_{1 \leq k \leq K} \alpha_k u_k^n P(u_k) = 0.$$

In matrix-vector form for indices n such that $\ell \leq n < \ell + K$, we get

$$\begin{bmatrix} y_{\ell+K} & y_{\ell+K-1} & \cdots & y_\ell \\ y_{\ell+K+1} & y_{\ell+K} & \cdots & y_{\ell+1} \\ \vdots & \ddots & \ddots & \vdots \\ y_{\ell+2K-2} & \ddots & \ddots & \vdots \\ y_{\ell+2K-1} & y_{\ell+2K-2} & \cdots & y_{\ell+K-1} \end{bmatrix} \begin{bmatrix} 1 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix} = \mathbf{T}_{K,\ell}\mathbf{h} = \mathbf{0}$$

Overview of Prony's Method

The vector of polynomial coefficients $\mathbf{h} = [1, h_1, ..., h_K]^T$ is in the null space of $\mathbf{T}_{K,\ell}$. Moreover, $\mathbf{T}_{K,\ell}$ has size $K \times (K+1)$ and has full row rank when the u_k's are distinct. Therefore $\mathbf{h}$ is unique.

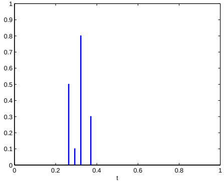
□

Prony's method summary:

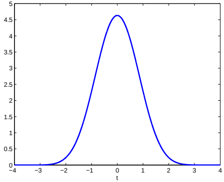
1. Given the input y_n, build the Toeplitz matrix $\mathbf{T}_{K,\ell}$ and solve for $\mathbf{h}$. This can be achieved by taking the SVD of $\mathbf{T}_{K,\ell}$.
2. Find the roots of $P(x) = 1 + \sum_{n=1}^{K} h_k x^{K-k}$. These roots are exactly the exponentials $\{u_k\}_{k=0}^{K-1}$.
3. Given the $\{u_k\}_{k=0}^{K-1}$, find the corresponding amplitudes $\{\alpha_k\}_{k=0}^{K-1}$ by solving K linear equations.

Imperial College London

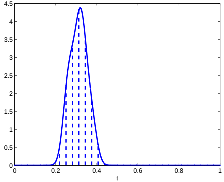
Sampling Streams of Diracs: Numerical Example



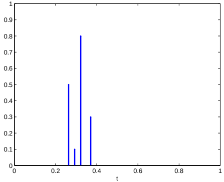
(a) Original Signal



(b) Sampling Kernel ($\beta_7(t)$)

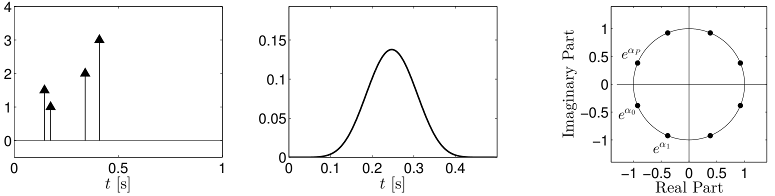


(c) Samples

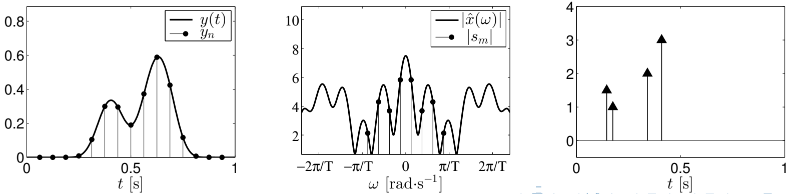


(d) Reconstructed Signal

Sampling Streams of Diracs: Numerical Example



(a) Input signal, $x(t)$ **(b)** Sampling kernel, $h(t)$ **(c)** e^{α_m} reproduced by $h(t)$



Note on the proof

Linear vs Non-linear

- Problem is **Non-linear** in t_k, but **linear** in x_k given t_k
- The key to the solution is the separability of the non-linear from the linear problem using the annihilating filter.

The proof is based on a constructive algorithm:

1. Given the N samples y_n, compute the moments s_m using the exponential reproduction formula. In matrix vector form $S = \mathbf{C}Y$.
2. Solve a $K \times K$ Toeplitz system to find $H(z)$
3. Find the roots of $H(z)$
4. Solve a $K \times K$ Vandermonde system to find the a_k

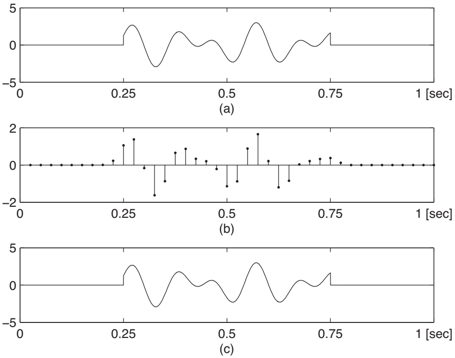
Complexity

1. $O(KN)$
2. $O(K^2)$
3. $O(K^3)$
4. $O(K^2)$

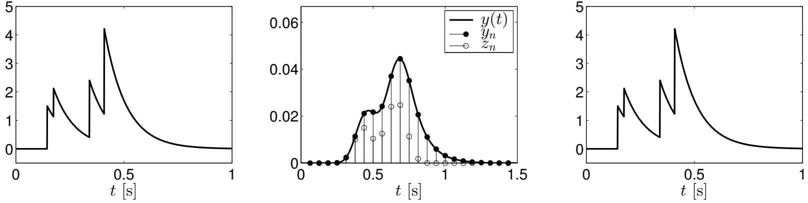
Thus, the algorithm complexity is polynomial with the signal innovation.

Sparse Sampling: Extensions

Using variations of Prony's method other signals can be sampled such as for example piecewise sinusoidal signals [BerentDragotti:10].

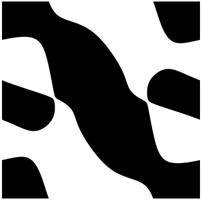
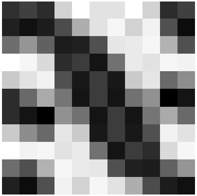
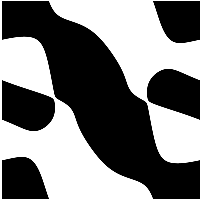


Stream of Decaying Exponentials



(a) Input signal, $x(t)$ **(b)** Filtered and sampled signal **(c)** Reconstructed signal

Sampling 2-D domains

The curve is implicitly defined through the equation [PanBluDragotti:11,14]:

$$f(x, y) = \sum_{k=1}^{K} \sum_{i=1}^{I} b_{k,i} e^{-j2\pi xk/M} e^{-j2\pi yi/N} = 0.$$

The coefficients $b_{k,i}$ are the only free parameters in the model.

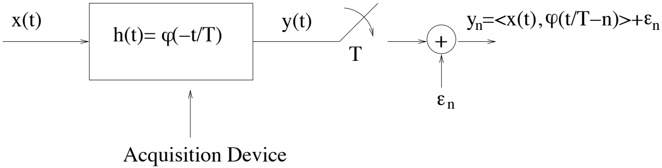
Sampling 2-D domains

samples

interpolation

inter+ curve constraint

Robust and Universal Sparse Sampling



- The acquisition device is arbitrary
- The measurements are noisy
- The noise is additive and i.i.d. Gaussian
- Many robust versions of Prony's method exist (e.g., Cadzow, matrix pencil)

Approximate Strang-Fix

- How restrictive are the Strang-Fix conditions?
- Assume $\varphi(t)$ cannot reproduce exponentials, we want to find the coefficients $c_n = c_0 e^{j\omega_m n}$ such that:

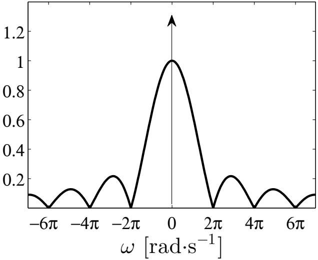
$$\sum_{n\in\mathbb{Z}} c_n \varphi(t-n) \approxeq \mathrm{e}^{j\omega_m t}.$$

- Approximation error

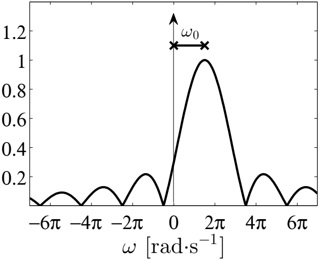
$$\varepsilon(t) = f(t) - e^{j\omega_m t} = \mathrm{e}^{j\omega_m t}\left[1 - c_0 \sum_{l\in\mathbb{Z}} \hat{\varphi}(j\omega_m + j2\pi l)\mathrm{e}^{j2\pi l t}\right].$$

- We only need $\hat{\varphi}(j\omega_m + j2\pi l) \approxeq 0 \quad l \in \mathbb{Z}\setminus\{0\}$, which is satisfied when $\varphi(t)$ has an essential bandwidth of size 2π.

Generalised Strang-Fix Conditions

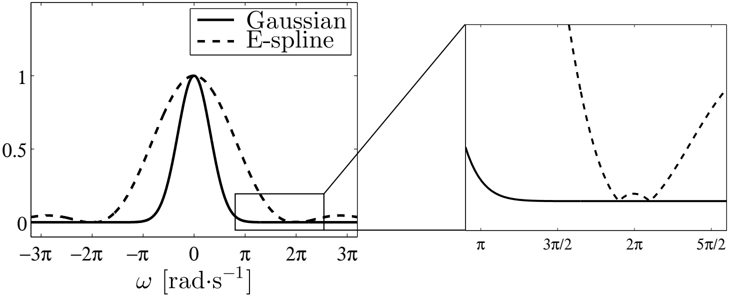


(a) $|\hat{\beta}_\alpha(\omega)|$ with $\alpha = 0$



(b) $|\hat{\beta}_\alpha(\omega)|$ with $\alpha = i\omega_0$

Approximate Strang-Fix



Approximate Strang-Fix

- Assume $\varphi(t)$ cannot reproduce exponentials, we want to find the coefficients $c_n = c_0 e^{j\omega_m n}$ such that:

$$\sum_{n\in\mathbb{Z}} c_n \varphi(t-n) \approx \mathrm{e}^{j\omega_m t}.$$

- Approximation error

$$\varepsilon(t) = f(t) - e^{j\omega_m t} = \mathrm{e}^{j\omega_m t}\left[1 - c_0 \sum_{l\in\mathbb{Z}} \hat{\varphi}(j\omega_m + j2\pi l)\mathrm{e}^{j2\pi l t}\right].$$

- *Constant Least-squares approximation*

$$c_0 = \hat{\varphi}(j\omega_m)^{-1} \Rightarrow c_n = \hat{\varphi}(j\omega_m)^{-1} e^{j\omega_m n}$$

- **Advantage**: only need to know the Fourier transform of $\varphi(t)$ at $j\omega_m$.

Approximate vs Exact Strang-Fix

Exact

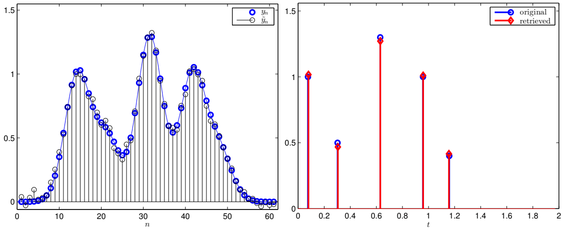
- Any device with unit input response of the form $\gamma(t) * \beta_{\vec{\alpha}}(t)$ where $\beta_{\vec{\alpha}}(t)$ is an E-spline of order L
- The order L and the exponents $\alpha_0, \alpha_1, .., \alpha_L$ are decided a-priori and cannot be changed.

Approximate

- Any acquisition device $h(t)$ can be used within this framework
- The essential bandwidth of $h(t) = \varphi(-t/T)$ must be at most $2\pi/T$
- We do not need to know $h(t)$ exactly. We only need to know $\hat{h}(j\omega_m)$ $m = 0, 1, .., L$
- The number L of exponentials reproduced is arbitrary

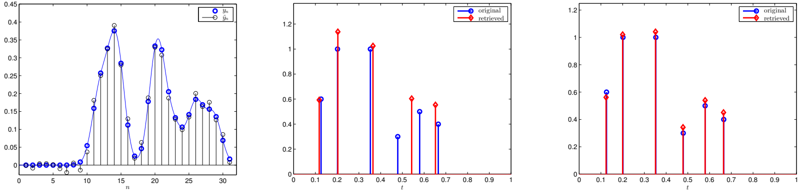
Approximate FRI recovery: Numerical Example

Gaussian Kernel



Approximate FRI with the Gaussian kernel. $K = 5$, $N = 61$, SNR=25dB. Recovery using the approximate method with $\alpha_m = j\frac{\pi}{3.5(P+1)}(2m - P)$, $m = 0, \ldots, P$ where $P + 1 = 21$.

Approximate Strang-Fix: when 'Mr Approximate' is better than 'Mr Exact'

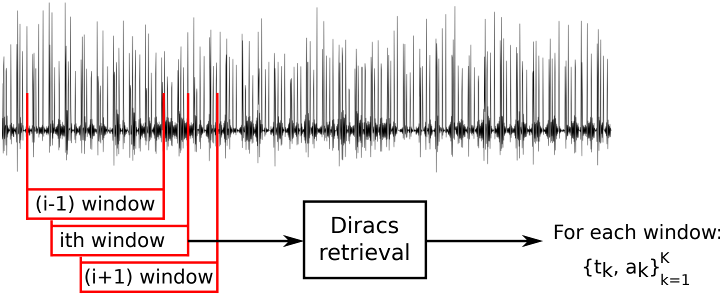


(a) y_n and $\tilde{y}_n$ (b) Default FRI retrieval (c) Approx. FRI retrieval

Estimation of $K = 6$ Dirac with the B-Spline kernel of order $L = 16$, $N = 31$. (b) Default polynomial recovery. (c) Approximate recovery with $\alpha_m = j\frac{\pi}{1.5(P+1)}(2m - P)$, $m = 0, \ldots, P$ where $P + 1 = 21$, SNR=25dB.

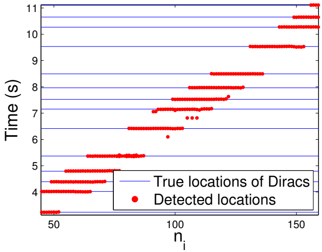
Imperial College London

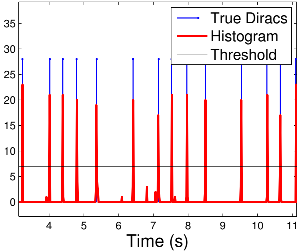
Retrieving 1000 Diracs with Strang-Fix Kernels



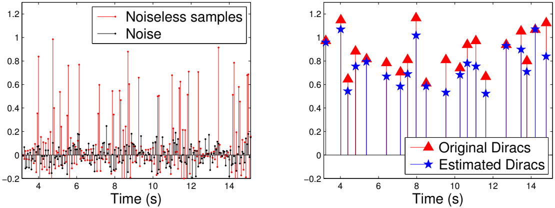
- Retrieve Diracs using a sliding window
- Locations of true Diracs are consistent across windows [Onativia-Uriguen-Dragotti-13]

Retrieving 1000 Diracs with Strang-Fix kernels





Retrieving 1000 Diracs with Strang-Fix Kernels



(a) y_n samples **(b)** Reconstructed stream

- $K = 1000$ Diracs in an interval of 630 seconds, $N = 10^5$ samples, $T = 0.06$ and $SNR = 10$dB
- 9997 Diracs retrieved with an error $\epsilon < T/2$
- Average accuracy $\Delta t = 0.005$, execution time 105 seconds.

Application: Image Super-Resolution [BaboulazD:09]

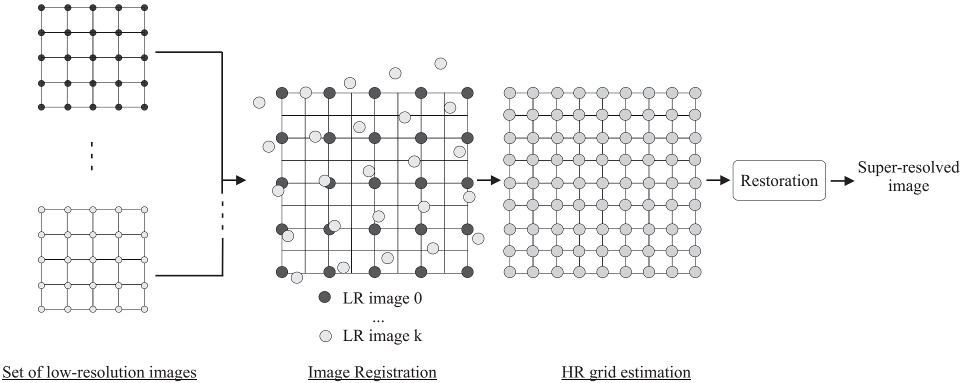
Super-Resolution is a multichannel sampling problem with unknown shifts. Use moments to retrieve the shifts or the geometric transformation between images.

(a)Original (512 × 512) (b) Low-res. (64 × 64) (c) Super-res (PSNR=24.2dB)

- Forty low-resolution and shifted versions of the original.
- The disparity between images has a finite rate of innovation and can be retrieved.
- Accurate registration is achieved by retrieving the continuous moments of the

Application: Image Super-Resolution

Image super-resolution basic building blocks



Application: Image Super-Resolution

- For each blurred image $I(x,y)$:
  - A pixel $P_{m,n}$ in the blurred image is given by

$$P_{m,n} = \langle I(x,y), \varphi(x/T - n, y/T - m) \rangle,$$

    where $\varphi(t)$ represents the point spread function of the lens.
  - We assume $\varphi(t)$ is a spline that can reproduce polynomials:

$$\sum_n \sum_m c_{m,n}^{(l,j)} \varphi(x-n, y-m) = x^l y^j \qquad l = 0,1,...,N; j = 0,1,...,N.$$

  - We retrieve the exact moments of $I(x,y)$ from $P_{m,n}$:

$$\tau_{l,j} = \sum_n \sum_m c_{m,n}^{(l,j)} P_{m,n} = \int \int I(x,y) x^l y^j dxdy.$$

- Given the moments from two or more images, we estimate the geometrical transformation and register them. Notice that moments of up to order three along the x and y coordinates allows the estimation of an affine transformation.

Application: Image Super-Resolution

Acquisition with Nikon D70

(a)Original (2014 $\times$ 3040)

(b) ROI (128 $\times$ 128)

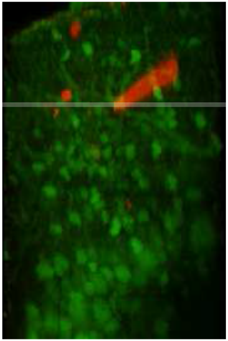
(b) Super-res (1024 $\times$ 1024)

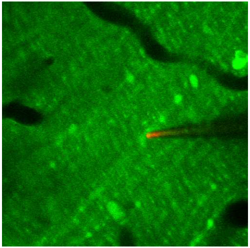
Application: Image Super-Resolution

(a)Original (48 $\times$ 48)

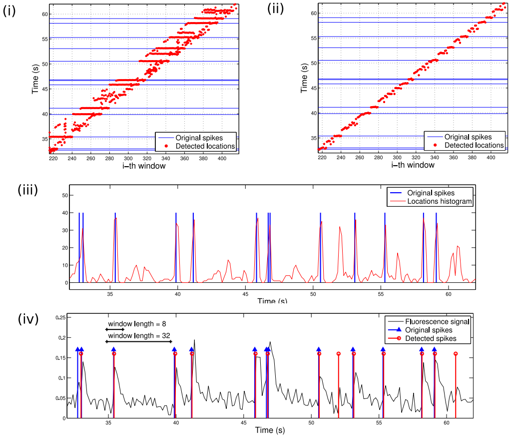
(b) Super-res (480 $\times$ 480)

Neural Activity Detection [OnativiaSD:13]

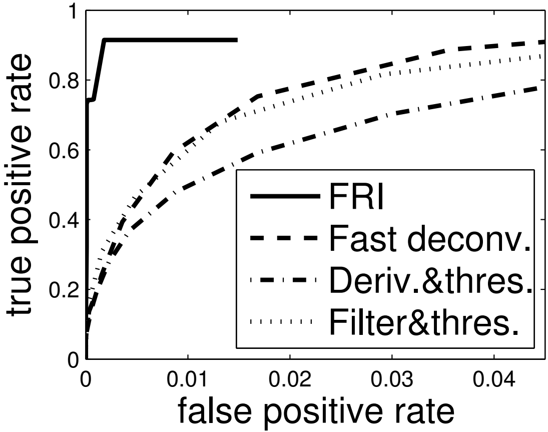




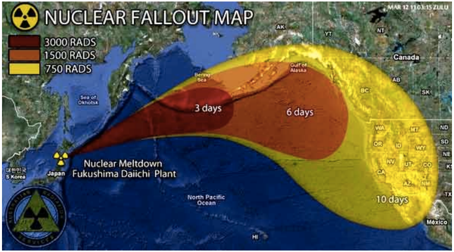
Calcium Transient Detection



Calcium Transient Detection



Localisation of Diffusion Sources using Sensor Networks [Murray-BruceD:14]

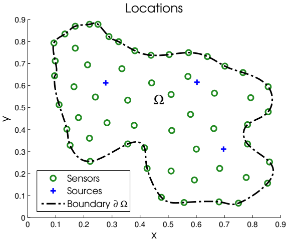


- The diffusion equation models the dispersion of chemical plumes, smoke from forest fires, radioactive materials
- The phenomenon is sampled in space and time using a sensor network.
- Sources often localised in space. Can we retrieve their location and the time of activation?

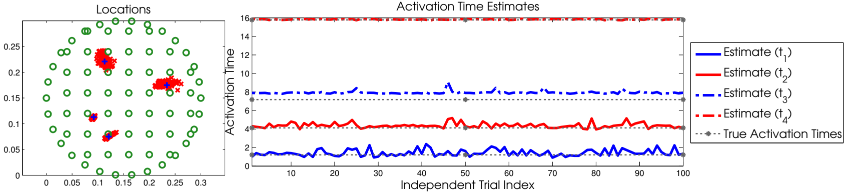
Localisation of Diffusion Sources using Sensor Networks

Good news:

- When sources are localised in space and time, the field inversion is equivalent to an FRI problem
- Proper linear combinations of sensors measurements in time and space leads to a Prony-type problem



Localisation of Diffusion Sources: Numerical Results



(b) 100 independent trials using noisy sensor measurement samples (SNR=15dB).

Conclusions

Sampling signals using sparsity models:

- New framework that allows the sampling and reconstruction of infinite-dimensional continuous-time signals at a rate smaller than Nyquist rate.
- It is a non-linear problem
- Different possible algorithms with various degrees of efficiency and robustness
- Approximate Strang-Fix method: universal and robust to noise

Outlook:

- Promising applications in neuroscience
- Applications to the inversion of physical fields from sensors' measurements

Still many open questions from theory to practice!

References

On sampling

- J. Uriguen, T. Blu, and P.L. Dragotti ‘FRI Sampling with Arbitrary Kernels’, IEEE Trans. on Signal Processing, November 2013
- T. Blu, P.L. Dragotti, M. Vetterli, P. Marziliano and L. Coulot ‘Sparse Sampling of Signal Innovations: Theory, Algorithms and Performance Bounds,’ IEEE Signal Processing Magazine, vol. 25(2), pp. 31-40, March 2008
- P.L. Dragotti, M. Vetterli and T. Blu, ‘Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon meets Strang-Fix’, IEEE Trans. on Signal Processing, vol.55 (5), pp.1741-1757, May 2007.
- J.Berent and P.L. Dragotti, and T. Blu,‘Sampling Piecewise Sinusoidal Signals with Finite Rate of Innovation Methods,’ IEEE Transactions on Signal Processing, Vol. 58(2),pp. 613-625, February 2010.
- J. Uriguen, P.L. Dragotti and T. Blu,‘On the Exponential Reproducing Kernels for Sampling Signals with Finite Rate of Innovation’ in Proc. of Sampling Theory and Application Conference, Singapore, May 2011.
- H. Pan, T. Blu, and P.L. Dragotti, ‘Sampling Curves with Finite Rate of Innovation’ IEEE Trans. on Signal Processing, January 2014.

References (cont'd)

On Image Super-Resolution

- L. Baboulaz and P.L. Dragotti, 'Exact Feature Extraction using Finite Rate of Innovation Principles with an Application to Image Super-Resolution', IEEE Trans. on Image Processing, vol.18(2), pp. 281-298, February 2009.

On Calcium Transient Detection

- Jon Onativia, Simon R. Schultz, and Pier Luigi Dragotti, A Finite Rate of Innovation algorithm for fast and accurate spike detection from two-photon calcium imaging, Journal of Neural Engineering, August 2013 .

On Diffusion Fields and Sensor Networks

- John Murray-Bruce and Pier Luigi Dragotti, Spatio-Temporal Sampling and Reconstruction of Diffusion Fields induced by Point Sources, to be presented at ICASSP, Florence (It), May 2014 .