# EE1 and ISE1 Communications I

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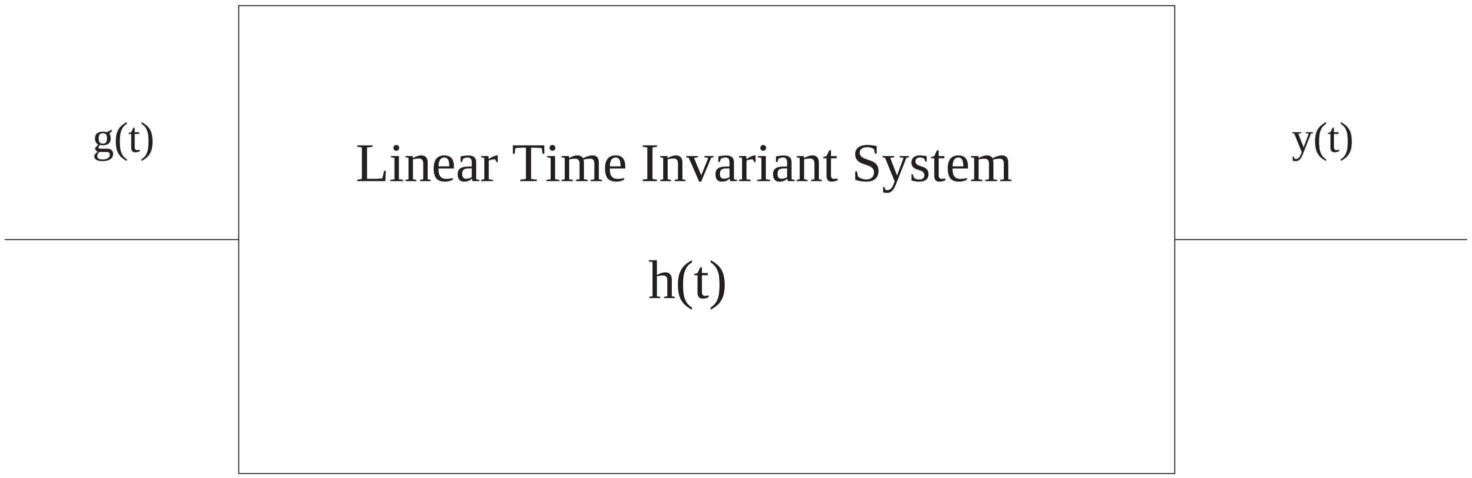
Lecture seven

# Lecture Aims

- To introduce linear systems,
- To introduce convolution,
- To examine signal transmission through a linear system,
- To give examples of real and ideal filters.

# Linear Systems



# Linear Systems (continued)

- A system is a *black box* that converts an input signal $g(t)$ in an output signal $y(t)$.
- Assume the output of a signal $g_1(t)$ is $y_1(t)$ and the output of $g_2(t)$ is $y_2(t)$. The system is linear if the output of $g_1(t) + g_2(t)$ is $y_1(t) + y_2(t)$.
- A system is time invariant if its properties do not change with the time. That is, if the response to $g(t)$ is $y(t)$, then the response to $g(t - t_0)$ is going to be $y(t - t_0)$

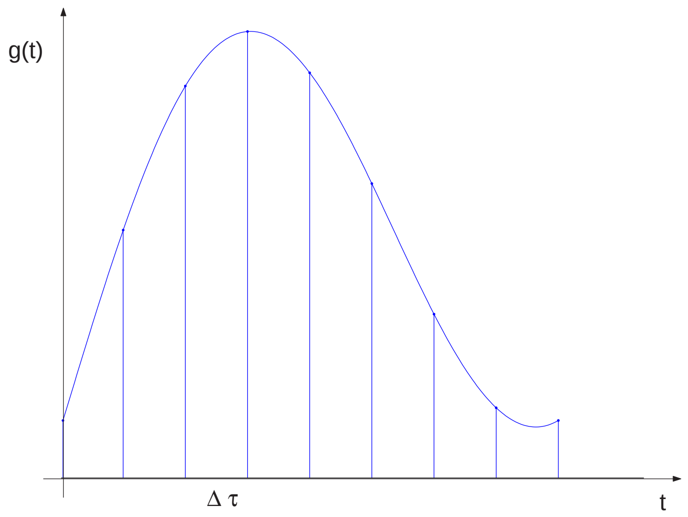
# Unit impulse response of a LTI system

Consider a linear time invariant (LTI) system. Assume the input signal is a Dirac function $\delta(t)$. Call the observed output $h(t)$.

- $h(t)$ is called the **unit impulse response** function.
- With $h(t)$, we can relate the input signal to its output signal through the convolution formula:

$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(\tau) g(t - \tau) d\tau.$$

# *Intuitive* explanation of the convolution formula




- $g(t)$ can be approximated as $g(t) \simeq \sum_n g(n\Delta\tau)\Delta\tau\delta(t - n\Delta\tau)$.
- In the limit as $\Delta\tau \longrightarrow 0$ this approximation approaches the true function $g(t)$.
- The response $\hat{y}(t)$ of the LTI system to the input $\sum_n g(n\Delta\tau)\Delta\tau\delta(t - n\Delta\tau)$ is going to be $\sum_n g(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$.
- Thus, $y(t) = \lim_{\Delta\tau\to 0} \sum_n g(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$.

# Convolution in the frequency domain

The convolution of two functions $g(t)$ and $h(t)$, denoted by $g(t) * h(t)$, is defined by the integral

$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(x)g(t-x)dx.$$

If $g(t) \Longleftrightarrow G(\omega)$ and $h(t) \Longleftrightarrow H(\omega)$ then the convolution reduces to a product in the Fourier domain

$$y(t) = h(t) * g(t) \Longleftrightarrow Y(\omega) = H(\omega)G(\omega).$$

$H(\omega)$ is called the **system transfer function** or the **system frequency response** or the **spectral response**.

Notice that, for symmetry, a product in the time domain corresponds to a convolution in frequency domain. That is

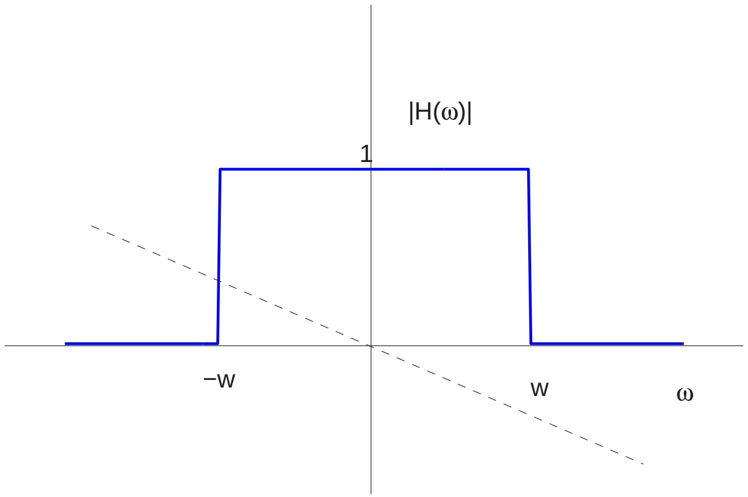
$$g_1(t)g_2(t) \Longleftrightarrow \frac{1}{2\pi}G_1(\omega) * G_2(\omega).$$

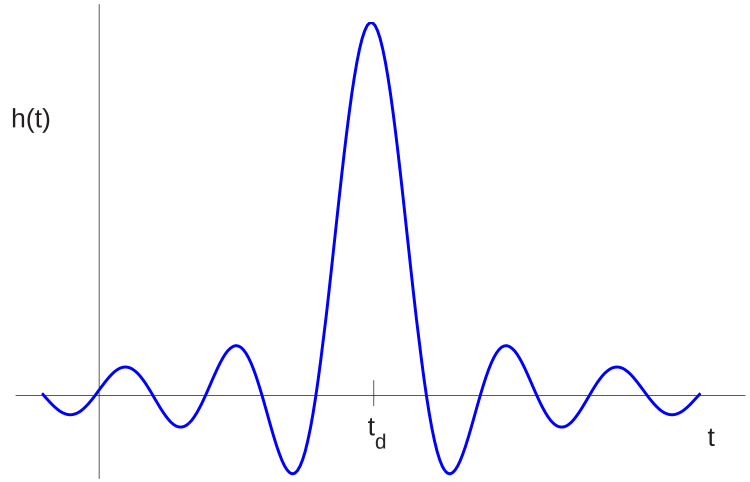
## Bandwidth of the product of two signals

If $g_1(t)$ and $g_2(t)$ have bandwidths $B_1$ and $B_2$ Hz, respectively.

The bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz.

# Ideal Low-Pass Filter




Ideal low-pass filter response

$$H(\omega) = rect\left(\frac{\omega}{2w}\right) e^{-j\omega t_d}$$

Ideal low-pass filter impulse response

$$h(t) = \frac{w}{\pi} sinc[(t - t_d)]$$

# Ideal High-Pass and Band-pass filters

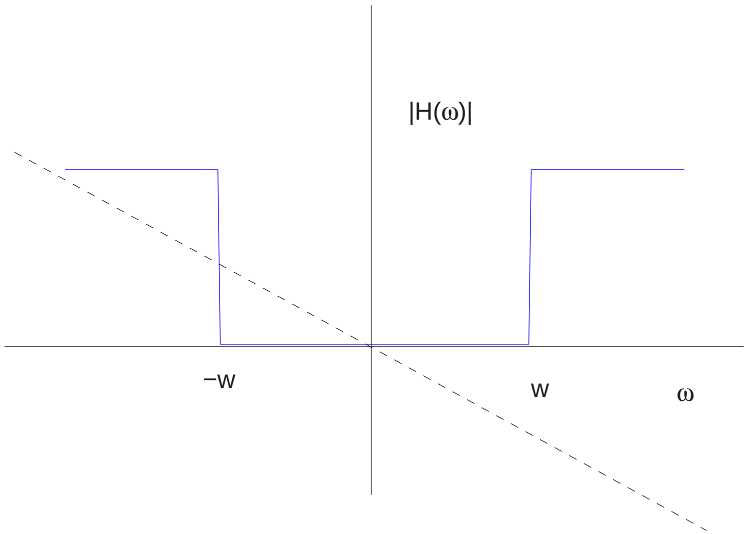



Figure 1: Ideal high-pass filter

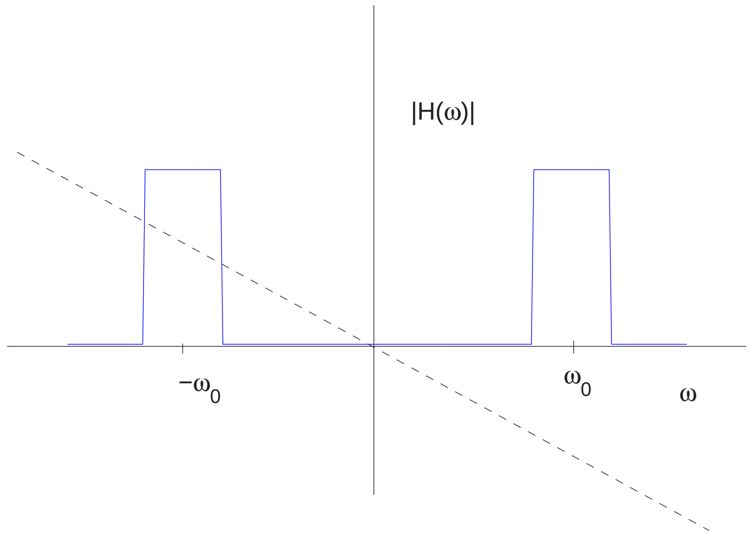
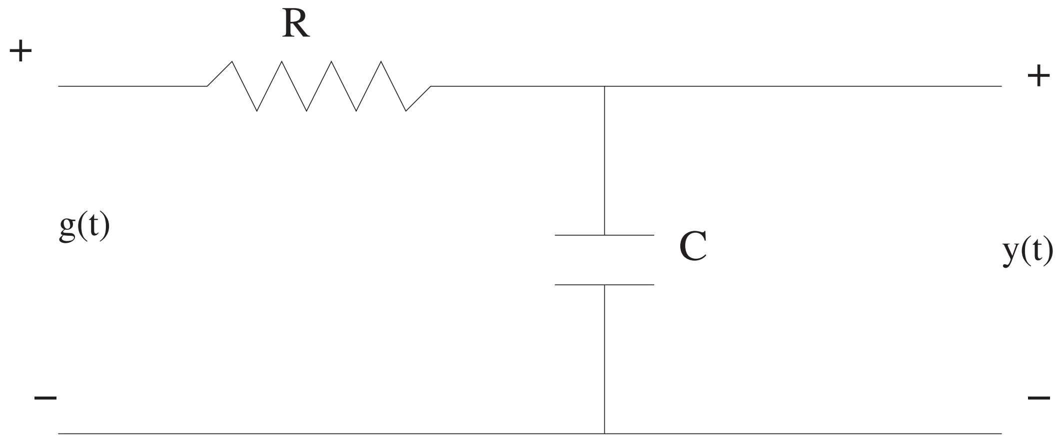



Figure 2: Ideal band-pass filter

# Practical filters

- The filters in the previous examples are ideal filters.
- They are not realizable since their unit impulse responses are everlasting (Think of the sinc function).
- Physically realizable filter impulse response $h(t) = 0$ for $t < 0$.
- Therefore, we can only obtain approximated version of the ideal low-pass, high-pass and band-pass filters.

# Example of a linear system: RC circuit



# Example: RC circuit (continued)

$$H(\omega) = \frac{1/j\omega C}{R + (1/j\omega C)} = \frac{1}{1 + j\omega RC} = \frac{a}{a + j\omega}$$

$$a = \frac{1}{RC}$$

and

$$|H(\omega)| = \frac{a}{\sqrt{a^2 + \omega^2}} \Rightarrow |H(0)| = 1, \lim_{\omega \to \infty} |H(\omega)| = 0.$$

$$\theta_h(\omega) = -tan^{-1}\frac{\omega}{a}$$

Therefore, this circuit behaves as a low-pass filter.

# Summary

- Linear time invariant systems
- Unit impulse response function
- Convolution formula: $y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(\tau) g(t - \tau) d\tau$
- Low-pass, high-pass and band-pass filters