# EE1 and ISE1 Communications I

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Lecture six

# Lecture Aims

- To present some properties of the Fourier transform

# Topics Covered

- **Fourier transform table**
- **Symmetry of Fourier transformation**
- **Time and Frequency shifting property**
- **Convolution,**
- **Time differentiation and time integration**
- **Please read Lathi pages “85—101”**

# Some properties of Fourier transform

| | $g(t)$ | $G(\omega)$ | |
|---|---|---|---|
| 1 | $e^{-at}u(t)$ | $\frac{1}{a+j\omega}$ | $a > 0$ |
| 2 | $e^{at}u(-t)$ | $\frac{1}{a-j\omega}$ | $a > 0$ |
| 3 | $e^{-a\|t\|}$ | $\frac{2a}{a^2+\omega^2}$ | $a > 0$ |
| 4 | $te^{-at}u(t)$ | $\frac{1}{(a+j\omega)^2}$ | $a > 0$ |
| 5 | $t^n e^{-at}u(t)$ | $\frac{n!}{(a+j\omega)^{n+1}}$ | $a > 0$ |

# Some properties of Fourier transform

| | | |
|---|---|---|
| 6 | $\delta(t)$ | $1$ |
| 7 | $1$ | $2\pi\delta(\omega)$ |
| 8 | $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |
| 9 | $\cos\ \omega_0 t$ | $\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ |
| 10 | $\sin\ \omega_0 t$ | $j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$ |
| 11 | $u(t)$ | $\pi\delta(\omega)+\frac{1}{j\omega}$ |

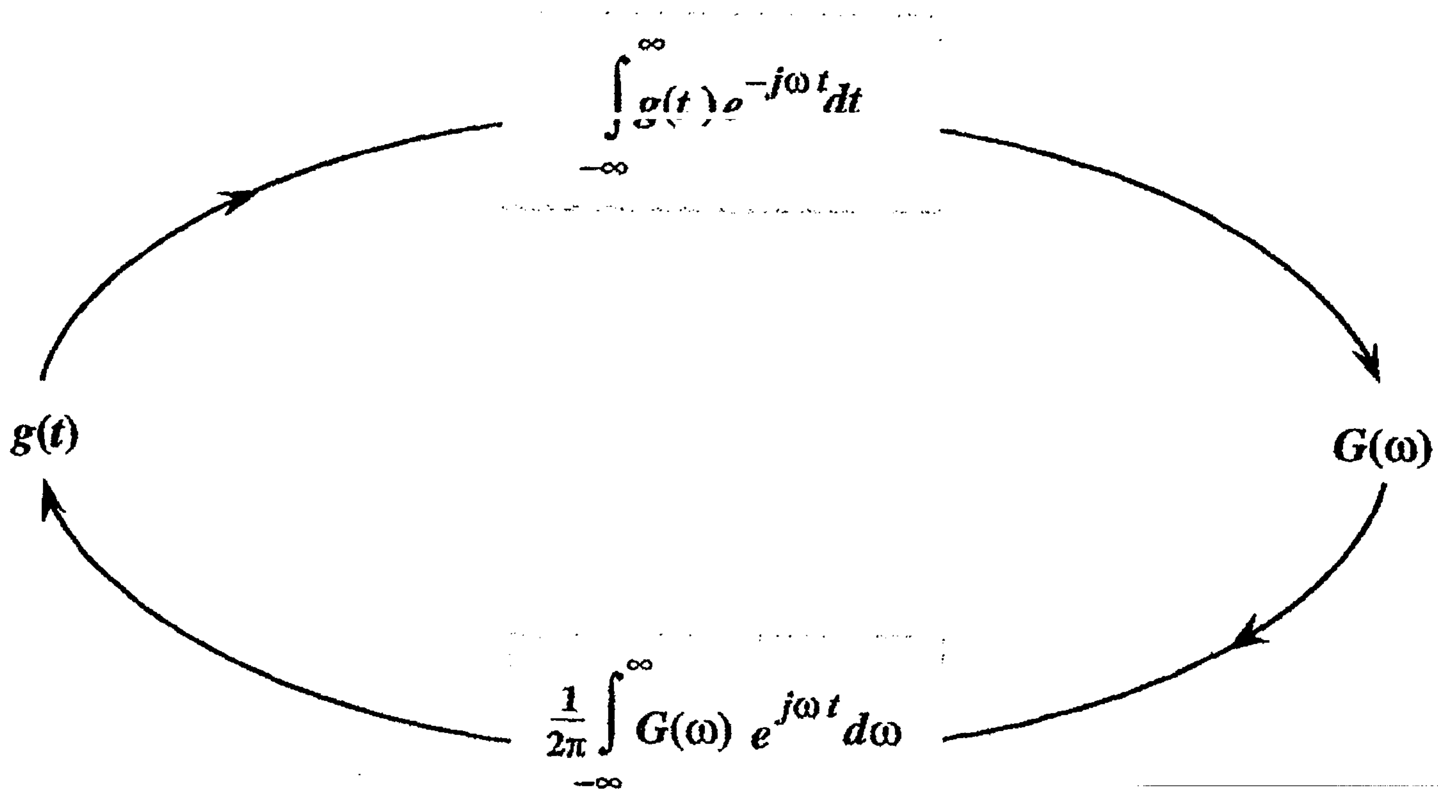
# Some properties of Fourier transform

| | | | |
|---|---|---|---|
| 12 | $\text{sgn}\, t$ | $\frac{2}{j\omega}$ | |
| 13 | $\cos \omega_0 t\, u(t)$ | $\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$ | |
| 14 | $\sin \omega_0 t\, u(t)$ | $\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$ | |
| 15 | $e^{-at} \sin \omega_0 t\, u(t)$ | $\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$ | $a > 0$ |
| 16 | $e^{-at} \cos \omega_0 t\, u(t)$ | $\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$ | $a > 0$ |
| 17 | $\text{rect}\left(\frac{t}{\tau}\right)$ | $\tau\, \text{sinc}\left(\frac{\omega\tau}{2}\right)$ | |
| 18 | $\frac{W}{\pi}\, \text{sinc}\,(Wt)$ | $\text{rect}\left(\frac{\omega}{2W}\right)$ | |

# Some properties of Fourier transform

| | | |
|---|---|---|
| 19 | $\Delta\left(\frac{t}{\tau}\right)$ | $\frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$ |
| 20 | $\frac{W}{2\pi}\,\mathrm{sinc}^2\left(\frac{Wt}{2}\right)$ | $\Delta\left(\frac{\omega}{2W}\right)$ |
| 21 | $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ | $\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$ |
| 22 | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$ |

# Fourier transform pair

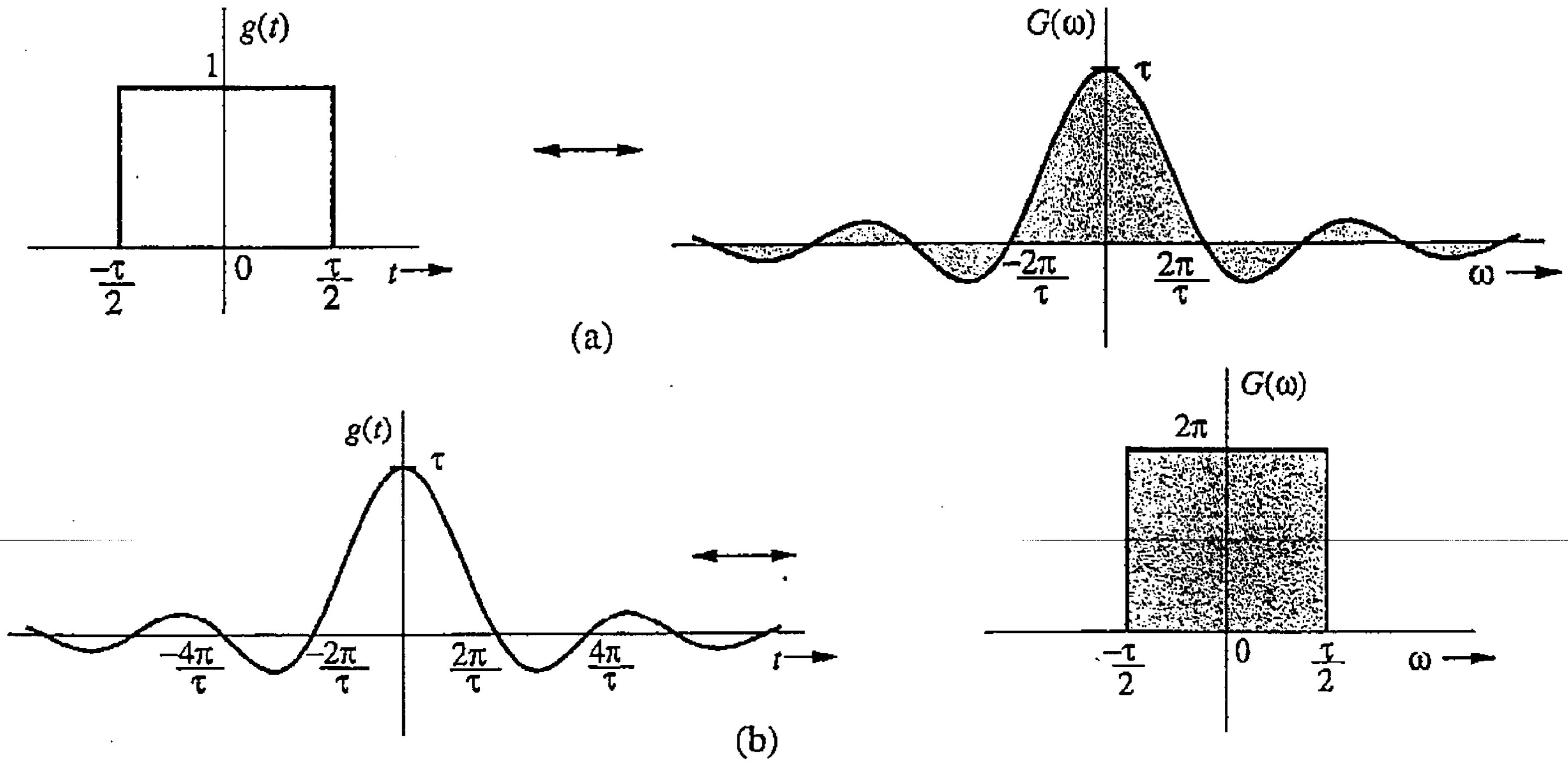


# Symmetry Property

- **Consider the Fourier transform pair**

$$g(t) \Longleftrightarrow G(\omega)$$

- **Then** $G(t) \Longleftrightarrow 2\pi g(-\omega)$
- **Example**



# Scaling Property

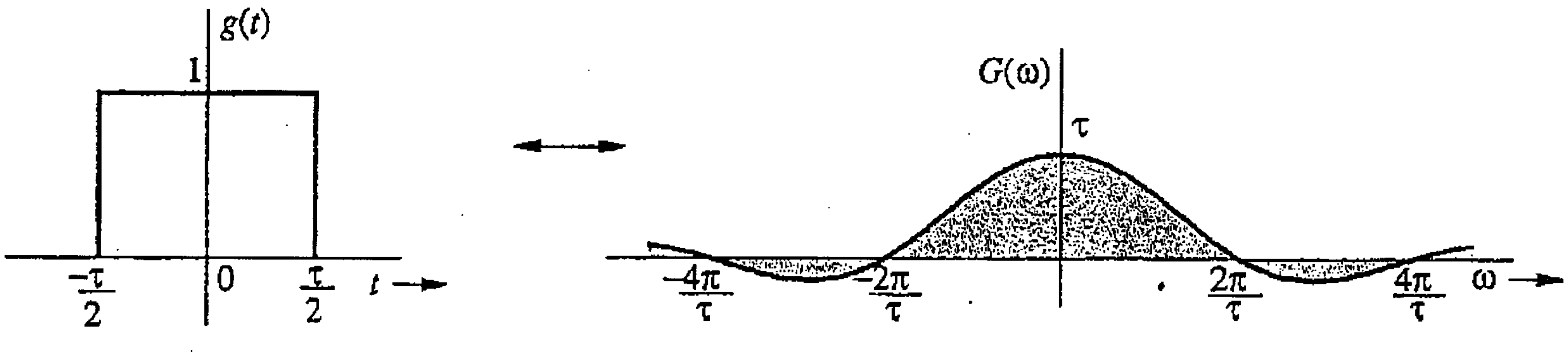
- **Consider the Fourier transform pair**

$$g(t) \Longleftrightarrow G(\omega)$$

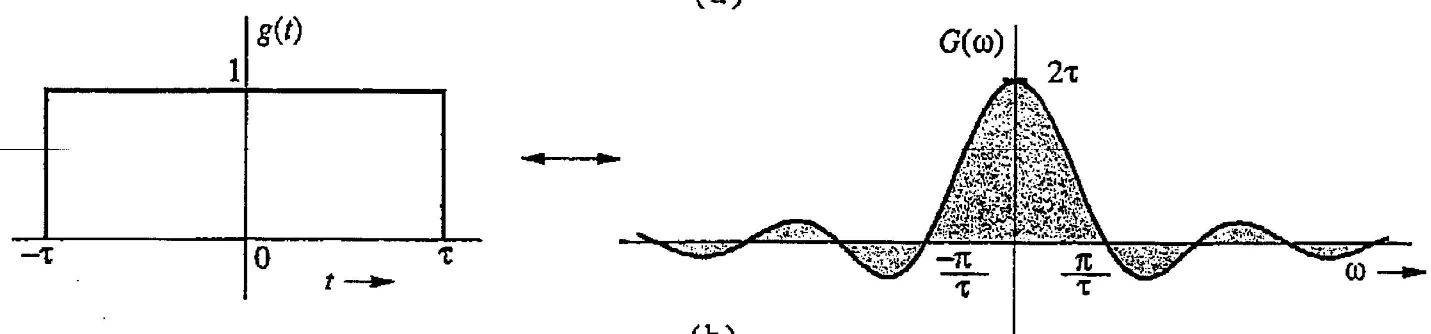
- **Then**

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$$

- **Example**




(a)




(b)

# *Time-Shifting Property*

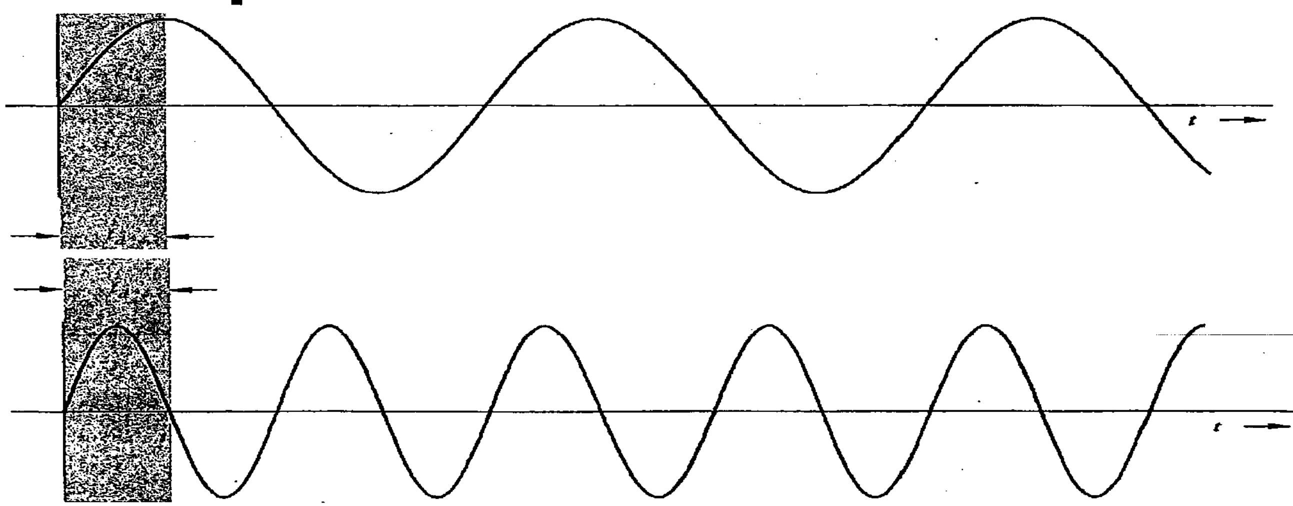
- **Consider the Fourier transform pair**

$$g(t) \Longleftrightarrow G(\omega)$$

- **Time shifting introduces phase shift**

$$g(t - t_0) \Longleftrightarrow G(\omega)e^{-j\omega t_0}$$

- **Example**



# *Frequency-Shifting Property*

- **Consider the Fourier transform pair**

$$g(t) \Longleftrightarrow G(\omega)$$

- **Exponential multiplication introduces frequency shift**

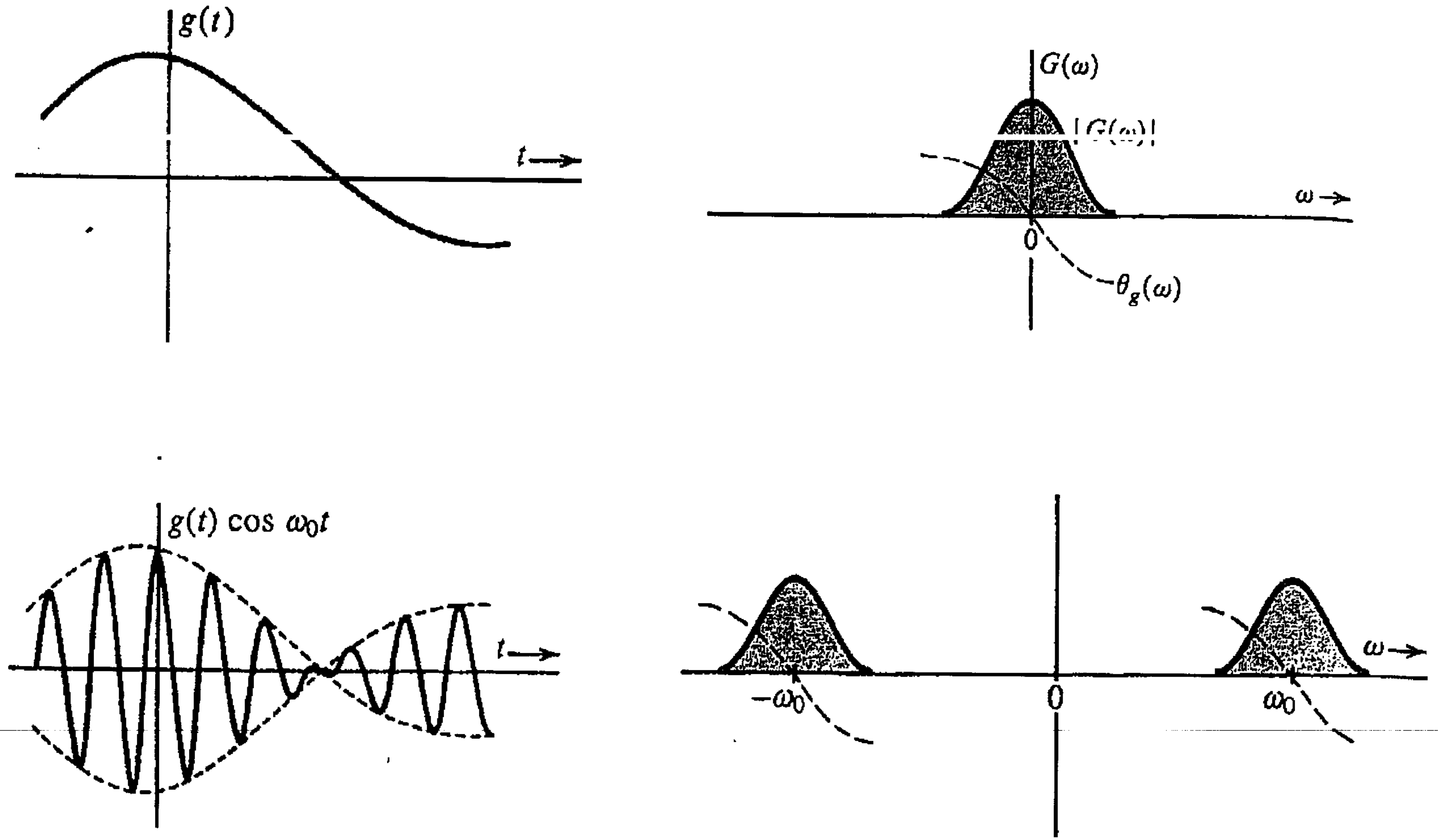
$$g(t)e^{j\omega_0 t} \Longleftrightarrow G(\omega - \omega_0) \qquad g(t)e^{-j\omega_0 t} \Longleftrightarrow G(\omega + \omega_0)$$

- **Cosine multiplication leads to**

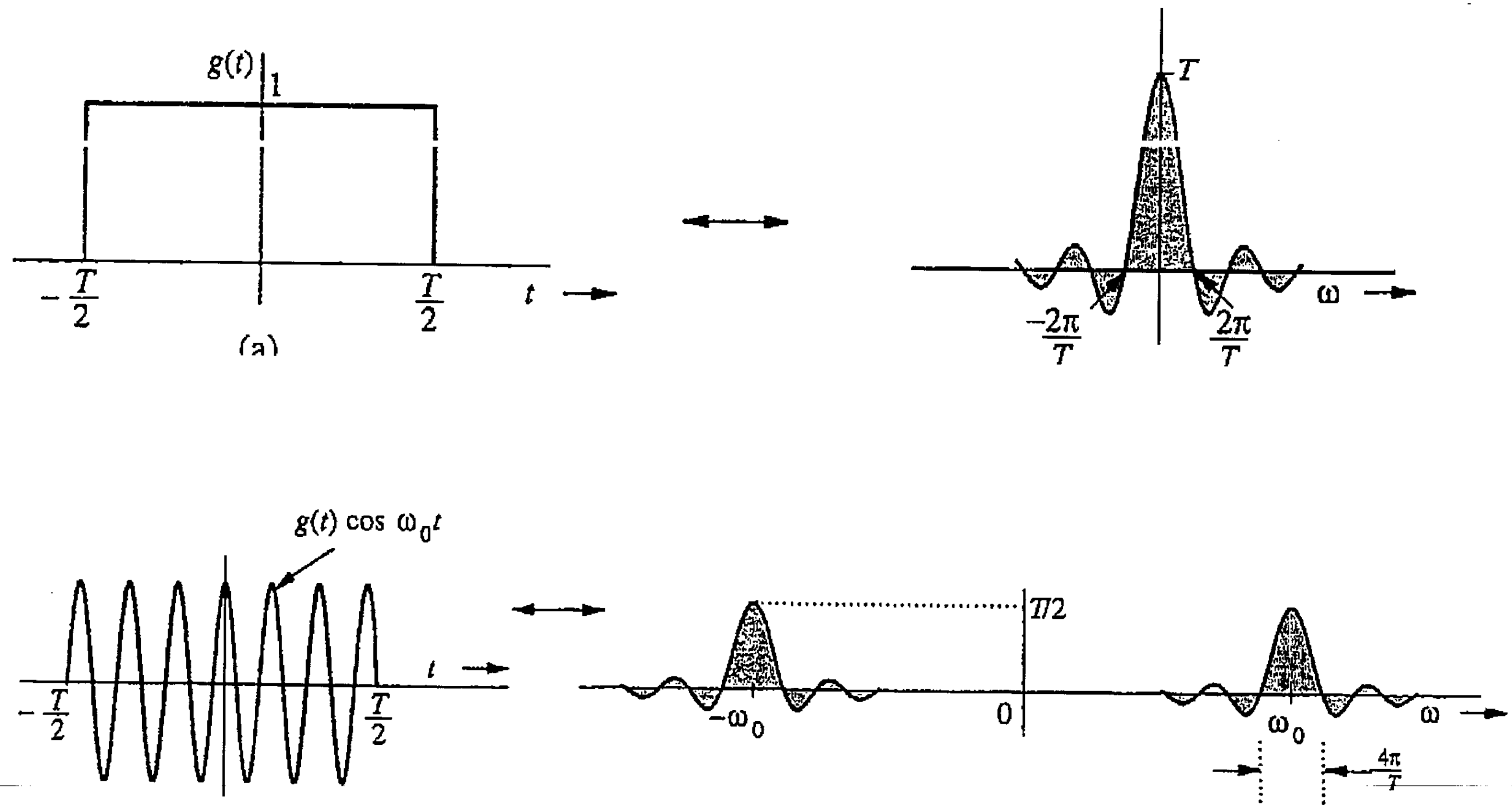
$$g(t)\cos\,\omega_0 t = \frac{1}{2}\left[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t}\right]$$

$$g(t)\cos\,\omega_0 t \Longleftrightarrow \frac{1}{2}\left[G(\omega - \omega_0) + G(\omega + \omega_0)\right]$$

# Frequency-Shifting Property



# *Frequency-Shifting Property*



# Fourier transform of periodic functions

- **Find the Fourier transform of a general periodic signal *g(t)* of period $T_o$**
- **A periodic signal *g(t)* can be expressed as an exponential Fourier series as**

$$g(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \mathcal{F}[D_n \, e^{jn\omega_0 t}]$$

$$g(t) \Longleftrightarrow 2\pi \sum_{n=-\infty}^{\infty} D_n \, \delta(\omega - n\omega_0)$$

# Fourier transform of periodic functions

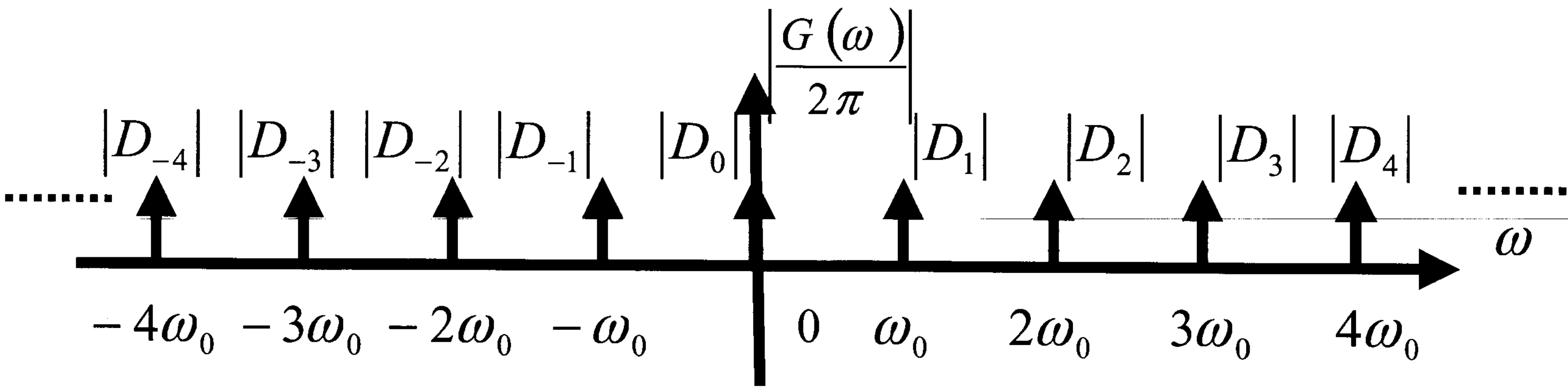
- **Consider a periodic waveform given by**

$$g(t) = \sum_{n=-\infty}^{n} w(t - nT_0) \qquad w(t) = \begin{cases} w(t) & T_0/2 \le |t| \\ 0 & \text{otherwise} \end{cases}$$

- **where**

$$g(t) = \sum_{n=-\infty}^{n} D_n \exp(j\,2\,\pi\, n\, f_0 t) \qquad w(t) \Leftrightarrow W(\omega)$$

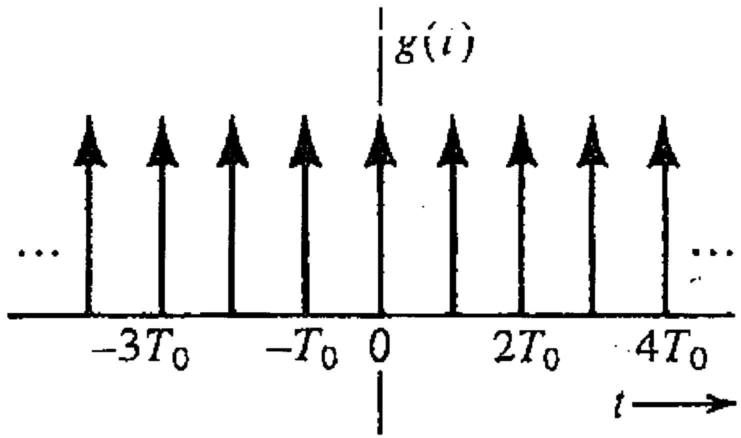
$$g(t) \Leftrightarrow G(\omega) = 2\,\pi \sum_{n=-\infty}^{n} D_n\, \delta(f - n\, f_0) \qquad D_n = \frac{W(n\omega_0)}{T_0}$$



# Fourier transform of periodic functions

- **Find the Fourier transform of a unit impulse train $\delta(t)$ of period $T_o$**

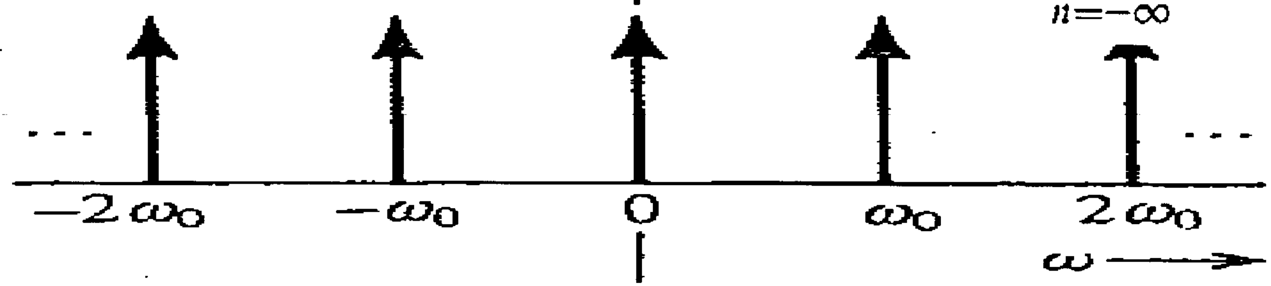
$$g(t) = \delta(t) \Leftrightarrow \quad G(\omega) = F(\delta(t))$$
$$= 1$$




$$D_n = \frac{G(n f_0)}{T_0} = \frac{1}{T_0}$$

$$\delta(t) \Leftrightarrow \frac{2\pi}{T_0} \sum_{n=-\infty}^{n} \delta(f - n f_0)$$

$$G(\omega) = \omega_0 \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$



# Convolution

The convolution of two functions $g(t)$ and $w(t)$,

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau) w(t - \tau)\, d\tau$$

- **Consider two waveforms**

$$g_1(t) \Longleftrightarrow G_1(\omega) \qquad g_2(t) \Longleftrightarrow G_2(\omega)$$

- **Convolution in time domain**

$$g_1(t) * g_2(t) \Longleftrightarrow G_1(\omega) G_2(\omega)$$

- **Convolution in the frequency domain**

$$g_1(t) g_2(t) \Longleftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$$

# Time Differentiation and Time Integration

- **Consider the Fourier transform relationship** $g(t) \Longleftrightarrow G(\omega)$
- **The following relationship exists for integration**

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

- **The following relationship exists for differentiation**

$$\frac{dg}{dt} \Longleftrightarrow j\omega G(\omega) \qquad \frac{d^n g}{dt^n} \Longleftrightarrow (j\omega)^n G(\omega)$$

# Important Fourier Transform Operations

**Fourier Transform Operations**

| Operation | $g(t)$ | $G(\omega)$ |
|---|---|---|
| Addition | $g_1(t) + g_2(t)$ | $G_1(\omega) + G_2(\omega)$ |
| Scalar multiplication | $kg(t)$ | $kG(\omega)$ |
| Symmetry | $G(t)$ | $2\pi g(-\omega)$ |
| Scaling | $g(at)$ | $\frac{1}{\lvert a\rvert} G\left(\frac{\omega}{a}\right)$ |
| Time shift | $g(t - t_0)$ | $G(\omega)e^{-j\omega t_0}$ |
| Frequency shift | $g(t)e^{j\omega_0 t}$ | $G(\omega - \omega_0)$ |
| Time convolution | $g_1(t) * g_2(t)$ | $G_1(\omega)G_2(\omega)$ |
| Frequency convolution | $g_1(t)g_2(t)$ | $\frac{1}{2\pi} G_1(\omega) * G_2(\omega)$ |
| Time differentiation | $\frac{d^n g}{dt^n}$ | $(j\omega)^n G(\omega)$ |
| Time integration | $\int_{-\infty}^{t} g(x)\, dx$ | $\frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$ |

# Conclusions

- **Examined some properties of Fourier transforms**
  - **Scaling property**
  - **Time shifting property**
  - **Frequency shifting property**
- **Examined Fourier Transform of periodic functions**
  - **General case**
  - **Unit Impulse function**
- **Examined Convolution**
- **Examined Fourier transforms for**
  - **Integration**
  - **Differentiation**