EE1 and ISE1 Communications I

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Lecture five

Lecture Aims

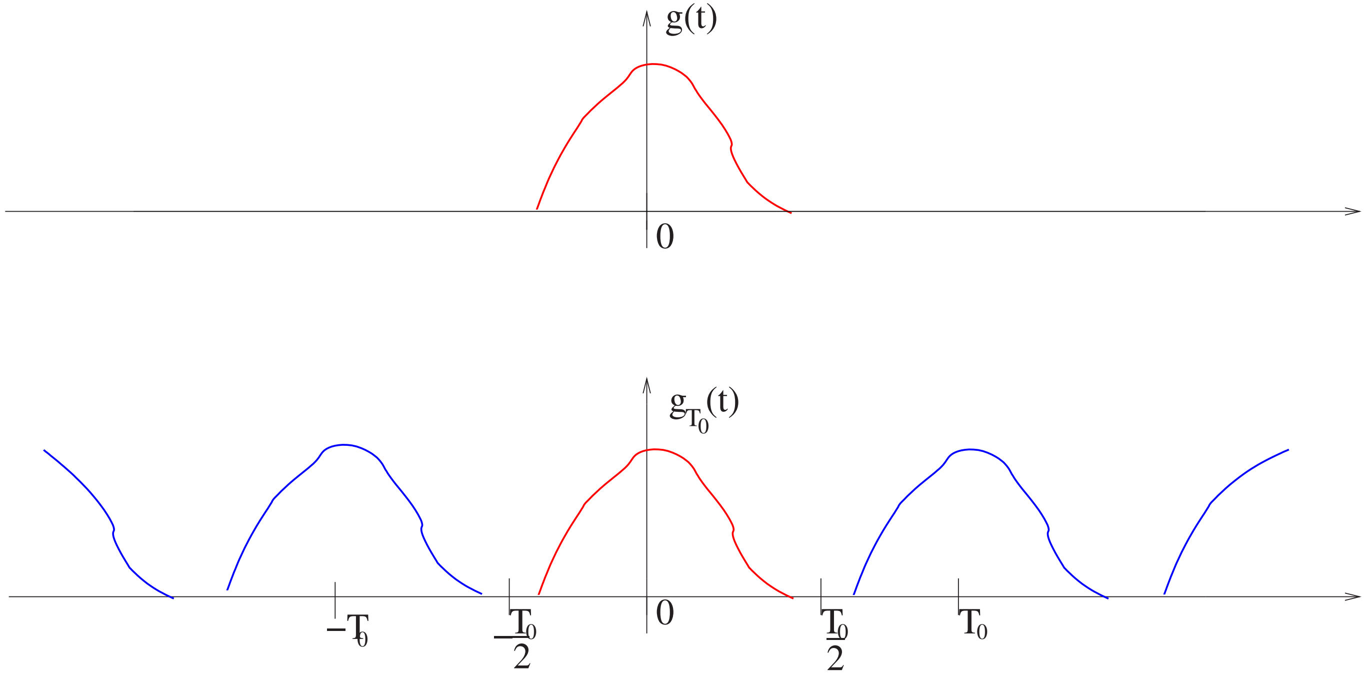
- To introduce Fourier integral, Fourier transformation
- To present transforms of some useful functions
- To discuss some properties of the Fourier transform

Introduction

- We electrical engineers think of signals in terms of their spectral content.
- We have studied the spectral representation of periodic signals.
- We now extend this spectral representation to the case of aperiodic signals.

Aperiodic signal representation

We have an aperiodic signal $g(t)$ and we consider a periodic version $g_{T_0}(t)$ of such signal obtained by repeating $g(t)$ every T_0 seconds.



The periodic signal $g_{T_0}(t)$

The periodic signal $g_{T_0}(t)$ can be expressed in terms of $g(t)$ as follows:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0).$$

Notice that, if we let $T_0 \to \infty$, we have that

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t).$$

The Fourier representation of $g_{T_0}(t)$

The signal $g_{T_0}(t)$ is periodic, so it can be represented in terms of its Fourier series. The basic intuition here is that the Fourier series of $g_{T_0}(t)$ will also represent $g(t)$ in the limit for $T_0 \to \infty$.
The exponential Fourier series of $g_{T_0}(t)$ is

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

where

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jn\omega_0 t} dt$$

and

$$\omega_0 = \frac{2\pi}{T_0}.$$

The Fourier representation of $g_{T_0}(t)$

Integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. So we can write

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}dt.$$

If we define a function

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

then we can write the Fourier coefficients D_n as follows:

$$D_n = \frac{1}{T_0}G(n\omega_0).$$

Computing the $\lim_{T_0 \to \infty} g_{T_0}(t)$

Thus $g_{T_0}(t)$ can be expressed as:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} e^{jn\omega_0 t}.$$

Assuming $\frac{1}{T_0} = \frac{\Delta\omega}{2\pi}$, we get

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{j(n\Delta\omega)t}.$$

In the limit for $T_0 \to \infty$, $\Delta\omega \to 0$ and $g_{T_0}(t) \to g(t)$.
We thus get:

$$\begin{aligned} g(t) &= \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{\Delta\omega \to 0} \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{j(n\Delta\omega)t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega. \end{aligned}$$

Fourier Transform and Inverse Fourier Transform

What we have just learned is that, from the spectral representation $G(\omega)$ of $g(t)$, that is, from

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt,$$

we can obtain $g(t)$ back by computing

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega.$$

Fourier transform of $g(t)$:

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt.$$

Inverse Fourier transform:

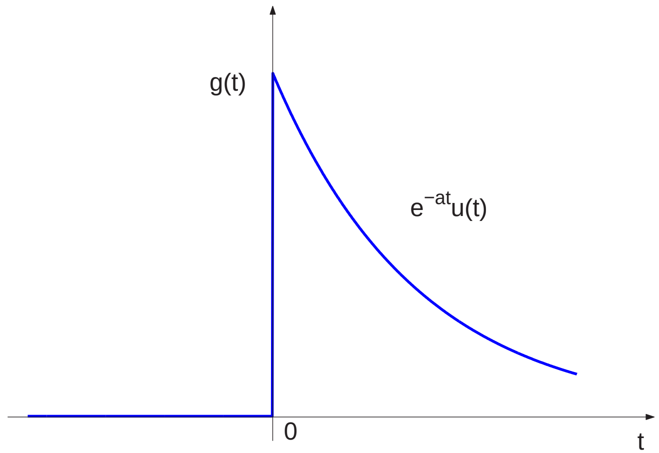
$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega.$$

Fourier transform relationship:

$$g(t) \Longleftrightarrow G(\omega).$$

Example

Find the Fourier transform of $g(t) = e^{-at}u(t)$.



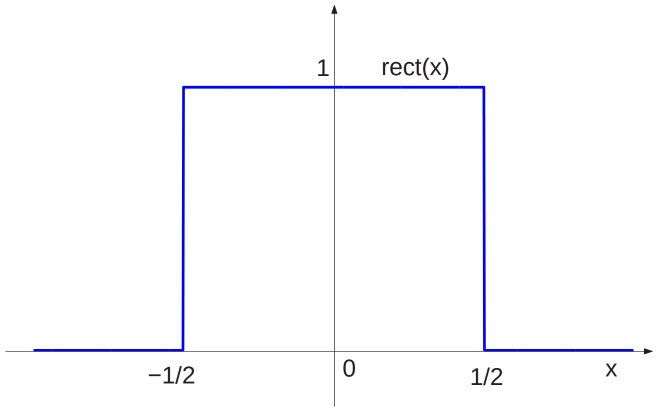
$$G(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}.$$

Since $|e^{-j\omega t}| = 1$, we have that $\lim_{t\to\infty} e^{-at}e^{-j\omega t} = 0$. Therefore:

$$G(\omega) = \frac{1}{a+j\omega}, \quad |G(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Some useful functions

The Unit Gate Function:

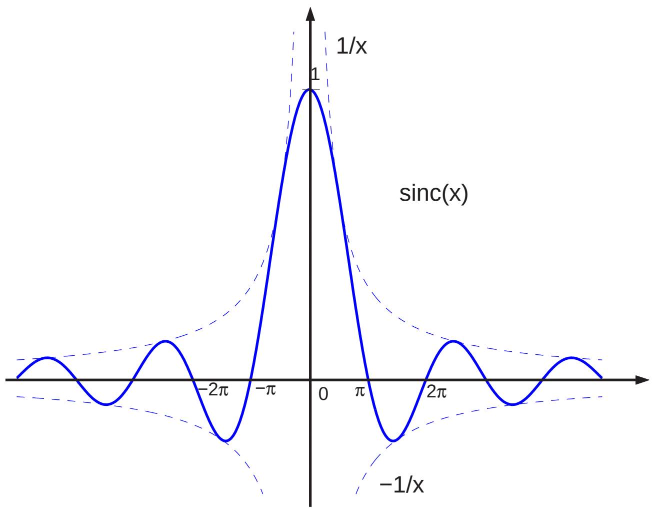


The unit gate function rect(x) is defined as:

$$\text{rect}(x) = \begin{cases} 0 & |x| > 1/2 \\ 1/2 & |x| = 1/2 \\ 1 & |x| < 1/2 \end{cases}$$

Some useful functions

The function $\sin(x)/x$ 'sine over argument' function is denoted by sinc(x):



- sinc(x) is an even function of x.
- sinc$(x) = 0$ when $\sin(x) = 0$ and $x \neq 0$.
- Using L'Hopital's rule, we find that sinc$(0) = 1$
- sinc(x) is the product of an oscillating signal $\sin(x)$ and a monotonically decreasing function $1/x$.

Example

Find the Fourier transform of $g(t) = \text{rect}(t/\tau)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \text{rect}\left(\tfrac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \\ &= -\tfrac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \tfrac{2\sin(\omega\tau/2)}{\omega} \\ &= \tau \tfrac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau \text{sinc}(\omega\tau/2). \end{aligned}$$

Therefore

$$\text{rect}(t/\tau) \Longleftrightarrow \tau\text{sinc}(\omega\tau/2)$$

Example

Find the Fourier transform of the unit impulse $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1.$$

Therefore

$$\delta(t) \Longleftrightarrow 1$$

Find the inverse Fourier transform of $\delta(\omega)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}.$$

Therefore

$$1 \Longleftrightarrow 2\pi\delta(\omega)$$

Example

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}.$$

Therefore

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0)$$

Example

Find the Fourier transform of the everlasting sinusoid $\cos(\omega_0 t)$.
Since

$$\cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

and using the fact that $e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0)$ and $e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0)$, we discover that

$$\cos(\omega_0 t) \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

Summary

Fourier transform of $g(t)$:

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt.$$

Inverse Fourier transform:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega.$$

Fourier transform relationship:

$$g(t) \Longleftrightarrow G(\omega).$$

Important Fourier transforms:

$$\text{rect}(t/\tau) \Longleftrightarrow \tau\text{sinc}(\omega\tau/2)$$

$$\delta(t) \Longleftrightarrow 1$$

$$\cos(\omega_0 t) \Longleftrightarrow \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)].$$