EE1 and ISE1 Communications I

Pier Luigi Dragotti

Lecture three

Lecture Aims

- To introduce some useful signals,
- To present analogies between vectors and signals,
  - Signal comparison: correlation,
  - Energy of the sum of orthogonal signals,
  - Signal representation by orthogonal signal set.

Useful Signals: Unit impulse function

The unit impulse function or Dirac function is defined as

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Multiplication of a function by an impulse.

$$g(t)\delta(t-T) = g(T)\delta(t-T)$$

$$\int_{-\infty}^{\infty} g(t)\delta(t-T)dt = g(T).$$

Useful Signals: Unit step function

Another useful signal is the unit step function $u(t)$, defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Observe that

$$\int_{-\infty}^{t} \delta(\alpha) d\alpha = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Therefore

$$\frac{du}{dt} = \delta(t).$$

If you don't understand this proof, use your intuition! The derivative of a 'jump' is a Dirac.

Useful Signals: Sinusoids

Consider the sinusoid

$$x(t) = C\cos(2\pi f_0 t + \theta)$$

f_0 (measured in Hertz) is the frequency of the sinusoid and $T_0 = 1/f_0$ is the period.

Sometimes we use ω_0 (radiant per second) to express $2\pi f_0$.

Important identities

$$e^{\pm jx} = \cos x \pm j\sin x, \quad \cos x = \frac{1}{2}[e^{jx} + e^{-jx}], \quad \sin x = \frac{1}{2j}[e^{jx} - e^{-jx}],$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$
$$a\cos x + b\sin x = C\cos(x+\theta)$$

with $C = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \frac{-b}{a}$

Signals and Vectors

- Signals and vectors are closely related. For example,
  - A vector has components,
  - A signal has also its components.
- Begin with some basic vector concepts,
- Apply those concepts to signals.

Inner product in vector spaces

x is a certain vector.
It is specified by its magnitude or length $|x|$ and direction.
Consider a second vector $\mathbf{y}$
We define the inner or scalar product of two vectors as

$$\langle \mathbf{y}, \mathbf{x} \rangle = |x||y| \cos \theta.$$

Therefore, $|x|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$.

When $\langle \mathbf{y}, \mathbf{x} \rangle = 0$, we say that $\mathbf{y}$ and $\mathbf{x}$ are orthogonal (geometrically, $\theta = \pi/2$).

Signals as vectors

The same notion of inner product can be applied for signals.
What is the useful part of this analogy?
We can use some geometrical interpretation of vectors to understand signals!
Consider two (energy) signals $y(t)$ and $x(t)$.
The inner product is defined by

$$\langle y(t), x(t)\rangle = \int_{-\infty}^{\infty} y(t)x(t)dt$$

For complex signals

$$\langle y(t), x(t)\rangle = \int_{-\infty}^{\infty} y(t)x^*(t)dt.$$

Two signals are orthogonal if $\langle y(t), x(t)\rangle = 0$.

Energy of orthogonal signals

If vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, and if $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$$|z|^2 = |x|^2 + |y|^2 \quad \text{(Pythagorean Theorem)}.$$

If signals $x(t)$ and $y(t)$ are orthogonal and if $z(t) = x(t) + y(t)$ then

$$E_z = E_x + E_y.$$

Proof:

$$\begin{aligned}
E_z &= \int_{-\infty}^{\infty} (x(t) + y(t))^2 dt \\
&= \int_{-\infty}^{\infty} x^2(t) dt + \int_{-\infty}^{\infty} y^2(t) dt + 2 \int_{-\infty}^{\infty} x(t) y(t) dt \\
&= E_x + E_y + 2 \int_{-\infty}^{\infty} x(t) y(t) dt \\
&= E_x + E_y
\end{aligned}$$

since $\int_{-\infty}^{\infty} x(t) y(t) dt = 0$.

Power of orthogonal signals

The same concepts of orthogonality and inner product extend to power signals. For example, $g(t) = x(t) + y(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)$ and $\omega_1 \neq \omega_2$.

$$P_x = \frac{C_1^2}{2}, \qquad\qquad P_y = \frac{C_2^2}{2}.$$

The signal $x(t)$ and $y(t)$ are orthogonal: $\langle x(t), y(t) \rangle = 0$. Therefore,

$$P_g = P_x + P_y = \frac{C_1^2}{2} + \frac{C_2^2}{2}.$$

Signal comparison: Correlation

If vectors $\mathbf{x}$ and $\mathbf{y}$ are given, we have the correlation measure as

$$c_n = \cos\theta = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{|y||x|}$$

Clearly, $-1 \leq c_n \leq 1$.
In the case of energy signals:

$$c_n = \frac{1}{\sqrt{E_y E_x}} \int_{-\infty}^{\infty} y(t)x(t)dt$$

again $-1 \leq c_n \leq 1$.

Best friends, worst enemies and complete strangers

- $c_n = 1$. **Best friends**. This happens when $g(t) = Kx(t)$ and K is positive. The signals are aligned, maximum similarity.

- $c_n = -1$. **Worst Enemies**. This happens when $g(t) = Kx(t)$ and K is negative. The signals are again aligned, but in opposite directions. The signals *understand* each others, but they do not like each others.

- $c_n = 0$. **Complete Strangers** The two signals are orthogonal. We may view orthogonal signals as unrelated signals.

Correlation

Why bother poor undergraduate students with correlation?
Correlation is widely used in engineering.
For instance

- To design receivers in many communication systems
- To identify signals in radar systems
- For classifications.

Correlation examples

Find the correlation coefficients between:

- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$.
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_1 t)$ and $\omega_0 \neq \omega_1$.
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_0 t)$.
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$ and $\omega_0 \neq \omega_1$.
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_0 t)$.
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = -A_1 \sin(\omega_0 t)$.

Correlation examples

Find the correlation coefficients between:

- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_1 t)$ and $\omega_0 \neq \omega_1$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_0 t)$ $\quad c_{x,y} = 1.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$ and $\omega_0 \neq \omega_1$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_0 t)$ $\quad c_{x,y} = 1.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = -A_1 \sin(\omega_0 t)$ $\quad c_{x,y} = -1.$

Signal representation by orthogonal signal sets

- Examine a way of representing a signal as a sum of orthogonal signals.
- We know that a vector can be represented as the sum of orthogonal vectors.
- The results for signals are parallel to those for vectors.
- Review the case of vectors and extend to signals.

Orthogonal vector space

Consider a three-dimensional Cartesian vector space described by three mutually orthogonal vectors, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

$$\langle \mathbf{x_m}, \mathbf{x_n} \rangle = \begin{cases} 0 & m \neq n \\ \\ |\mathbf{x}_m|^2 & m = n \end{cases}$$

Any three-dimensional vector can be expressed as a linear combination of those three vectors: $\mathbf{g} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$.
Where $c_i = \frac{\langle \mathbf{g}, \mathbf{x_i} \rangle}{|x_i|^2}$.

In this case, we say that this set of vector is *complete*.
Such vectors are known as a **basis** vector.

Orthogonal signal space

Same notions of completeness extend to signals.
A set of mutually orthogonal signals $x_1(t), x_2(t), ..., x_N(t)$ is complete if it can represent any signal belonging to a certain space. For example:

$$g(t) \simeq c_1 x_1(t) + c_2 x_2(t) + ... + c_N x_N(t)$$

If the approximation error is zero for any $g(t)$ then the set of signals $x_1(t), x_2(t), ..., x_N(t)$ is complete. In general, the set is complete when $N \to \infty$. Infinite dimensional space (this will be more clear in the next lecture).

Summary

- Analogies between vectors and signals
- Inner product and correlation
- Energy and Power of orthogonal signals
- Signal representation by means of orthogonal signal