Signal Processing 149 (2018) 236–245

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

A perspective on CLMS as a deficient length augmented CLMS: Dealing with second order noncircularity

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ARTICLE INFO

Article history:
Received 30 November 2017
Revised 12 March 2018
Accepted 15 March 2018
Available online 16 March 2018

Keywords:
Complex LMS (CLMS)
Widely linear estimation (WLE)
Second order noncircularity (improperness)
Convergence analysis
Approximate uncorrelating transform (AUT)

ABSTRACT

Mean and mean square performance analyses of the strictly linear complex least mean square (CLMS) algorithm are addressed for widely linear estimation (WLE) of second order noncircular (improper) Gaussian inputs, for which both the covariance and pseudo-covariance matrices contain nonzero off-diagonal elements. A detailed performance analysis of standard CLMS in this 'suboptimal' context is not trivial but is important for practical applications. To this end, we here consider the strictly linear CLMS as a 'deficient length' version of the widely linear augmented CLMS (ACLMS) algorithm, which is second order optimal for improper inputs. Rigorous performance analysis is provided, which also statistically quantifies the suboptimality of CLMS in both the transient and steady state stages. The recently introduced approximate uncorrelating transform (AUT) is employed to derive closed-form expressions for the mean square stability and the steady-state performance of CLMS. In addition, since WLE with second order noncircular inputs is a general linear estimation problem in the complex domain $\mathbb{C}$, these results provide a generalised framework from which current statistical descriptions of CLMS for strictly linear estimation (SLE) can be deduced as special cases. Simulations in system identification settings validate the findings.

1. Introduction

The complex least mean square (CLMS) algorithm is the most often used adaptive signal processing algorithm in the complex domain $\mathbb{C}$. Based on a stochastic version of gradient descent applied to a simple mean square error (MSE) cost function, for an $N \times 1$ weight vector $\mathbf{h}(k) = [h_1(k), h_2(k), \ldots, h_N(k)]^T$ at a time instant k, the CLMS weight update is given by Widrow et al. [1]

$$e(k) = d_{\mathrm{SL}}(k) - \mathbf{h}^H(k)\mathbf{x}(k) \tag{1}$$

$$\mathbf{h}(k+1) = \mathbf{h}(k) + \mu e^*(k)\mathbf{x}(k) \tag{2}$$

where $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_N(k)]^T \in \mathbb{C}^{N\times 1}$ is the input vector, $e(k)$ the output error, and μ the step-size. Within the standard CLMS, the desired signal, $d_{\mathrm{SL}}(k)$, is generated by a strictly linear estimation (SLE) model, given by

$$d_{\mathrm{SL}}(k) = \mathbf{h}^{\mathrm{o}H}\mathbf{x}(k) + q(k) \tag{3}$$

where $\mathbf{h}^{\mathrm{o}} = [h_1^{\mathrm{o}}, h_2^{\mathrm{o}}, \ldots, h_N^{\mathrm{o}}]^T$ is the optimal system impulse response vector to be estimated and $q(k) \in \mathbb{C}$ is zero-mean independent identically distributed Gaussian noise, $q(k) \sim \mathcal{N}(0, \sigma_q^2)$.

The analysis of the CLMS algorithm for this strictly linear estimation (SLE) problem was introduced by Horowitz and Senne in their seminal paper [2]. A related analysis in the context of an adaptive line enhancer appears in [3]. In both papers, the pair of the desired and input signals $\{d_{\mathrm{SL}}(k), \mathbf{x}(k)\}$ is considered to be jointly Gaussian, and the zero-mean input $\mathbf{x}(k)$ is implicitly assumed to be second order circular (proper) with a vanishing pseudo-covariance matrix, $\mathbf{P} = E[\mathbf{x}(k)\mathbf{x}^T(k)] = \mathbf{0}$ [4,5]. This assumption on the signal circularity has been implicitly or explicitly inherited in performance analyses of CLMS and its variants in various applications [6–12], since in this way, complex-valued analyses have a similar statistical form to their real-valued counterparts, which reduces the complexity of the performance analysis of CLMS. On the other hand, recent advances in the so-called *augmented complex statistics* allow for the analysis of a general case of second order noncircular random signals with non-vanishing pseudo-covariance matrix $\mathbf{P}$. Such signals may arise from different powers of their real and imaginary parts or a degree of correlation between the real and imaginary parts, and in order to make use of all the available second order information, we also need to

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https://doi.org/10.1016/j.sigpro.2018.03.009
0165-1684/

consider the pseudo-covariance matrix, $\mathbf{P}$, alongside with the standard covariance matrix $\mathbf{R} = E[\mathbf{x}(k)\mathbf{x}^H(k)]$ [13–18].

A concept closely related to augmented complex statistics is the so-called widely linear estimation (WLE), which considers the desired signal, $d_{\mathrm{WL}}(k)$, generated by the following widely linear model [14,15,19]

$$d_{\mathrm{WL}}(k) = \mathbf{h}^{oH}\mathbf{x}(k) + \mathbf{g}^{oH}\mathbf{x}^*(k) + q(k) \tag{4}$$

where $\mathbf{g}^o = [g_1^o, g_2^o, \ldots, g_N^o]^T$ is the so-called conjugate optimal system impulse response vector, associated with the input conjugate $\mathbf{x}^*(k)$. The WLE can be considered as a generalised estimation framework in the complex domain, and has provided modelling advantages over SLE in numerous applications in signal processing, communications, power systems, biomedical engineering and renewable energy [20–32]. The augmented complex statistics have opened the possibility to design LMS-type adaptive algorithms based on the widely linear model, leading to the so-called augmented CLMS (ACLMS), given by Mandic et al. [23], Javidi et al. [25], Xia et al. [26], Khalili et al. [28], Xia and Mandic [29], Korpi et al. [30], Shi et al. [31]

$$e(k) = d_{\mathrm{WL}}(k) - \mathbf{h}^H(k)\mathbf{x}(k) - \mathbf{g}^H(k)\mathbf{x}^*(k) \tag{5}$$

$$\mathbf{h}(k+1) = \mathbf{h}(k) + \mu e^*(k)\mathbf{x}(k) \tag{6}$$

$$\mathbf{g}(k+1) = \mathbf{g}(k) + \mu e^*(k)\mathbf{x}^*(k) \tag{7}$$

Compared with standard CLMS, the ACLMS updates an additional weight vector $\mathbf{g}(k)$ in order to track the optimal system impulse response vector, $\mathbf{g}^o$, associated with the desired signal $d_{\mathrm{WL}}(k)$ generated by the widely linear model in (4). The ACLMS has been proved to be second order optimal for the WLE of both second order circular and noncircular input signals [33]. Although preliminary results do exist which show that CLMS yields suboptimal estimates in the steady-state when employed for WLE [34,35], a detailed quantitative assessment of its statistical behaviour for WLE is still missing.

In this paper, we provide a comprehensive mean and mean square performance analysis of CLMS for WLE with both second order circular and noncircular Gaussian signals. For the first time, we provide a statistical framework to quantify the suboptimality of CLMS in both transient and steady state stages under the same umbrella, by considering CLMS as a special case of a *deficient* 'half-length' ACLMS, whereby the weight vector $\mathbf{g}(k)$ vanishes. In addition, since WLE is the most general framework for the processing of complex-valued data in terms of second order statistics, this work provides a rigourous generalised performance analysis of CLMS, from which current statistical descriptions for SLE can be deduced as special cases. Closed-form expressions for the steady state performance of CLMS are subsequently obtained from the mean square analysis, where the recently introduced approximate uncorrelating transform (AUT) [33,36] is employed in order to achieve a single joint singular value decomposition (SVD) of both the covariance and pseudo-covariance matrices, $\mathbf{R}$ and $\mathbf{P}$, if the data are second order noncircular.

The rest of the paper is organised as follows. The mean and mean square convergence of CLMS for WLE are respectively addressed in Sections 2 and 3. A detailed mean square stability analysis to quantify the bound on the step-size μ is given in Section 4. Section 5 provides the steady state mean square error (SSMSE) performance of CLMS for WLE. Simulations in system identification setting are given in Section 6 to support the analysis. Finally, Section 7 concludes this paper.

2. Mean convergence analysis

Consider the use of CLMS, summarised in (1) and (2), for WLE of the desired widely linear response $d_{\mathrm{WL}}(k)$ given in (4). The output error $e(k)$ in (1) then becomes

$$e(k) = (\mathbf{h}^o - \mathbf{h}(k))^H\mathbf{x}(k) + \mathbf{g}^{oH}\mathbf{x}^*(k) + q(k) \tag{8}$$

Upon introducing the $N \times 1$ weight error vector

$$\widetilde{\mathbf{h}}(k) = \mathbf{h}(k) - \mathbf{h}^o \tag{9}$$

the output error $e(k)$ in (8) can be rewritten as

$$e(k) = \mathbf{g}^{oH}\mathbf{x}^*(k) - \widetilde{\mathbf{h}}^H(k)\mathbf{x}(k) + q(k) \tag{10}$$

while for the conjugate error we have

$$e^*(k) = \mathbf{g}^{oT}\mathbf{x}(k) - \widetilde{\mathbf{h}}^T(k)\mathbf{x}^*(k) + q^*(k) \tag{11}$$

From (2), the recursion for the update of the weight error vector $\widetilde{\mathbf{h}}(k)$ becomes

$$\begin{aligned}\widetilde{\mathbf{h}}(k+1) &= \widetilde{\mathbf{h}}(k) + \mu(\mathbf{g}^{oT}\mathbf{x}(k) - \widetilde{\mathbf{h}}^T(k)\mathbf{x}^*(k) + q^*(k))\mathbf{x}(k)\\ &= (\mathbf{I} - \mu\mathbf{x}(k)\mathbf{x}^H(k))\widetilde{\mathbf{h}}(k) + \mu\big(\mathbf{g}^{oT}\mathbf{x}(k)\big)\mathbf{x}(k)\\ &\quad + \mu q^*(k)\mathbf{x}(k)\end{aligned} \tag{12}$$

where $\mathbf{I}$ is an $N \times N$ identity matrix.

The mean behaviour of $\widetilde{\mathbf{h}}(k)$ can now be determined by applying the statistical expectation operator to both sides of (12) and upon employing the standard independence assumptions, that is, the noise $q(k)$ is statistically independent of any other signal in the CLMS algorithm and $\widetilde{\mathbf{h}}(k)$ is statistically independent of the adaptive filter input $\mathbf{x}(k)$ [37], to yield

$$E[\widetilde{\mathbf{h}}(k+1)] = [\mathbf{I} - \mu\mathbf{R}]E[\widetilde{\mathbf{h}}(k)] + \mu\mathbf{a} \tag{13}$$

where $\mathbf{a} = E[(\mathbf{g}^{oT}\mathbf{x}(k))\mathbf{x}(k)] = [a_1, a_2, \ldots, a_N]^T$. The ith component, a_i, can be derived as

$$a_i = \sum_{j=1}^{N} g_j^o E[x_i(k)x_j(k)] = \sum_{j=1}^{N} g_j^o p_{ij} \tag{14}$$

where $p_{ij} = E[x_i(k)x_j(k)]$ is the $(i,\ j)$th element in the pseudo-covariance matrix $\mathbf{P}$.

Recall that $\lambda_{\max} \le \mathrm{tr}[\mathbf{R}]$, where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{R}$ and $\mathrm{tr}[\cdot]$ the trace operator. Hence, according to (13), the convergence in the mean of CLMS for WLE with second order noncircular input is guaranteed if the step-size μ satisfies

$$0 < \mu < \frac{2}{\mathrm{tr}[\mathbf{R}]} \tag{15}$$

Remark 1. The upper bound on μ for the mean stability of CLMS for WLE in (4) is identical to that derived for SLE in (3), for both second order circular and noncircular input signals [2,8,10,34,38].

At the steady state, from (13) we have

$$E[\widetilde{\mathbf{h}}(\infty)] = \mathbf{R}^{-1}\mathbf{a} \tag{16}$$

so that the steady state value of the weight vector of CLMS becomes

$$E[\mathbf{h}(\infty)] = \mathbf{h}^o + \mathbf{R}^{-1}\mathbf{a} \tag{17}$$

Remark 2. Eq. (17) indicates that for WLE with second order noncircular input signals, the CLMS yields a bias in the estimation of the optimal weight vector, $\mathbf{h}^o$, associated with input vector, $\mathbf{x}(k)$, quantified by $\mathbf{R}^{-1}\mathbf{a}$. The level of this bias depends upon the level of "undermodeling", that is, the dimensionality of $\mathbf{a}$, N, and the full second order statistics of the input vector $\mathbf{x}(k)$, that is, the covariance coefficients reflected in $\mathbf{R}^{-1}$ and the pseudo-covariance coefficients contained in the vector $\mathbf{a}$.

However, when either the input $\mathbf{x}(k)$ is second order circular (with $\mathbf{P} = \mathbf{0}$) or when $\mathbf{g}^o = \mathbf{0}$, which is the case when WSE in (4) reduces to SLE in (3), we have

$$\mathbf{a} = \mathbf{0} \quad \text{and} \quad E[\mathbf{h}(\infty)] = \mathbf{h}^o \tag{18}$$

Remark 3. For WLE of second order circular inputs and for SLE of the general second order noncircular inputs, the CLMS algorithm converges in the mean to the optimal weight coefficients associated with $\mathbf{x}(k)$, that is, $\mathbf{h}^{\mathrm{o}}$, and in an unbiased manner.

3. Mean square convergence analysis

The MSE of CLMS, denoted by $J(k)$, can be defined as

$$J(k) = E[|e(k)|^2] = E[e(k)e^*(k)] \tag{19}$$

From (10) to (11), and by employing the standard independence assumptions stated in Section 2, for the WLE performed by CLMS, the MSE in (19) can be evaluated as

$$\begin{aligned} J(k) &= E[\mathbf{g}^{\mathrm{o}T}\mathbf{x}(k)\mathbf{x}^H(k)\mathbf{g}^{\mathrm{o}*}] - E[\mathbf{g}^{\mathrm{o}H}\mathbf{x}^*(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)] \\ &\quad - E[\widetilde{\mathbf{h}}^H(k)\mathbf{x}(k)\mathbf{x}^T(k)\mathbf{g}^{\mathrm{o}}] + E[\widetilde{\mathbf{h}}^H(k)\mathbf{x}(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)] + \sigma_q^2 \\ &= \mathbf{g}^{\mathrm{o}T}\mathbf{R}\mathbf{g}^{\mathrm{o}*} - 2\Re\big[\mathbf{a}^H E[\widetilde{\mathbf{h}}(k)]\big] + \mathrm{tr}[\mathbf{R}\mathbf{K}(k)] + \sigma_q^2 \end{aligned} \tag{20}$$

where $\Re[\cdot]$ is the real part operator, and $\mathbf{K}(k) = E[\widetilde{\mathbf{h}}(k)\widetilde{\mathbf{h}}^H(k)]$ is the weight error covariance matrix. From the above, the MSE $J(k)$ can be explained through the contribution of three components: (i) a constant component $\mathbf{g}^{\mathrm{o}T}\mathbf{R}\mathbf{g}^{\mathrm{o}*} + \sigma_q^2$, in which the term $\mathbf{g}^{\mathrm{o}T}\mathbf{R}\mathbf{g}^{\mathrm{o}*}$ stems from the suboptimality of using CLMS for WLE, that is, from the inherent undermodelling; (ii) a component depending on the transient mean weight error vector $E[\widetilde{\mathbf{h}}(k)]$; (iii) a component depending on the weight error covariance matrix $\mathbf{K}(k)$.

The MSE of CLMS for SLE, denoted by $J^{\mathrm{SLE}}(k)$, can be directly deduced from (20) by setting $\mathbf{g}^{\mathrm{o}} = \mathbf{0}$, and consequently $\mathbf{a} = \mathbf{0}$, to give [2,3,6,10,38,39]

$$J^{\mathrm{SLE}}(k) = \mathrm{tr}[\mathbf{R}\mathbf{K}(k)] + \sigma_q^2 \tag{21}$$

Observe that the mean square analysis of CLMS rests upon the second order characteristics of the weight error vector, $\mathbf{K}(k)$, in (20). To analyse its evolution, we first apply the Hermitian operator $(\cdot)^H$ to both sides of (12), to yield

$$\begin{aligned} \widetilde{\mathbf{h}}^H(k+1) &= \widetilde{\mathbf{h}}^H(k)(\mathbf{I} - \mu\mathbf{x}(k)\mathbf{x}^H(k)) \\ &\quad + \mu(\mathbf{g}^{\mathrm{o}H}\mathbf{x}^*(k))\mathbf{x}^H(k) + \mu q(k)\mathbf{x}^H(k) \end{aligned}$$

Upon multiplying both sides of (12) by $\widetilde{\mathbf{h}}^H(k+1)$ and taking the statistical expectation, the evolution of the weigh error covariance matrix $\mathbf{K}(k)$ becomes

$$\begin{aligned} \mathbf{K}(k+1) &= \mathbf{K}(k) + \mu(E[\widetilde{\mathbf{h}}(k)]\mathbf{a}^H + \mathbf{a}E[\widetilde{\mathbf{h}}^H(k)] - \mathbf{R}\mathbf{K}(k) - \mathbf{K}(k)\mathbf{R}) \\ &\quad + \mu^2\big(\mathbf{C} - \mathbf{B}(k) - \mathbf{B}^H(k) + \sigma_q^2\mathbf{R} \\ &\quad + E[\mathbf{x}(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)\widetilde{\mathbf{h}}^H(k)\mathbf{x}(k)\mathbf{x}^H(k)]\big) \end{aligned} \tag{22}$$

where

$$\mathbf{B}(k) = E[(\mathbf{g}^{\mathrm{o}H}\mathbf{x}^*(k))\mathbf{x}(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)\mathbf{x}^H(k)] \tag{23}$$

$$\mathbf{C} = E[|\mathbf{g}^{\mathrm{o}T}\mathbf{x}(k)|^2\mathbf{x}(k)\mathbf{x}^H(k)] \tag{24}$$

The detailed expressions for $\mathbf{B}(k)$ and $\mathbf{C}$ are given respectively in Appendix A and Appendix B. We shall now examine the (i, j)th entry of the expectation matrix of the last term on the right hand side (RHS) of (22) under the independence assumptions, given by

$$\begin{aligned} &\big\{E[\mathbf{x}(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)\widetilde{\mathbf{h}}^H(k)\mathbf{x}(k)\mathbf{x}^H(k)]\big\}_{ij} \\ &= \sum_{l=1}^{N}\sum_{m=1}^{N} E[x_i(k)x_l^*(k)x_m(k)x_j^*(k)]E[\widetilde{h}_l(k)\widetilde{h}_m^*(k)] \end{aligned}$$

By employing the Gaussian fourth order moment factorising theorem, we obtain

$$E[x_i(k)x_l^*(k)x_m(k)x_j^*(k)] = r_{il}r_{mj} + p_{im}p_{lj}^* + r_{ij}r_{ml}$$

and hence [33,38,39],

$$\begin{aligned} &E[\mathbf{x}(k)\mathbf{x}^H(k)\widetilde{\mathbf{h}}(k)\widetilde{\mathbf{h}}^H(k)\mathbf{x}(k)\mathbf{x}^H(k)] \\ &\quad = \mathbf{R}\mathbf{K}(k)\mathbf{R} + \mathbf{P}\mathbf{K}^*(k)\mathbf{P}^* + \mathbf{R}\mathrm{tr}[\mathbf{R}\mathbf{K}(k)] \end{aligned}$$

Thus, the evolution of the weight error covariance matrix $\mathbf{K}(k)$ in (22) now becomes

$$\begin{aligned} \mathbf{K}(k+1) &= \mathbf{K}(k) + \mu(E[\widetilde{\mathbf{h}}(k)]\mathbf{a}^H + \mathbf{a}E[\widetilde{\mathbf{h}}^H(k)] - \mathbf{R}\mathbf{K}(k) - \mathbf{K}(k)\mathbf{R}) \\ &\quad + \mu^2\big(\mathbf{C} - \mathbf{B}(k) - \mathbf{B}^H(k) + \sigma_q^2\mathbf{R} \\ &\quad + \mathbf{R}\mathbf{K}(k)\mathbf{R} + \mathbf{P}\mathbf{K}^*(k)\mathbf{P}^* + \mathbf{R}\mathrm{tr}[\mathbf{R}\mathbf{K}(k)]\big) \end{aligned} \tag{25}$$

Eqs. (20) and (25), in which the convergence of $E[\widetilde{\mathbf{h}}(k)]$ is guaranteed by (13), now completely describe the MSE convergence behaviour of the CLMS algorithm for WLE of second order noncircular Gaussian input data. The impact of the degree of undermodelling on the mean-square evolution of CLMS, in the form of $\mathbf{a}$, $\mathbf{B}(k)$ and $\mathbf{C}$, can also be observed.

The evolution of the weight error covariance matrix of CLMS for SLE of second order noncircular inputs, denoted by $\mathbf{K}_{\mathrm{SLE}}(k)$, is obtained from (25) when the terms $\mathbf{a}$, $\mathbf{B}(k)$ and $\mathbf{C}$ vanish, and is given by

$$\begin{aligned} \mathbf{K}_{\mathrm{SLE}}(k+1) &= \mathbf{K}_{\mathrm{SLE}}(k) - \mu\big(\mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k) + \mathbf{K}_{\mathrm{SLE}}(k)\mathbf{R}\big) \\ &\quad + \mu^2\big(\sigma_q^2\mathbf{R} + \mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k)\mathbf{R} + \mathbf{P}\mathbf{K}_{\mathrm{SLE}}^*(k)\mathbf{P}^* \\ &\quad + \mathbf{R}\mathrm{tr}[\mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k)]\big) \end{aligned} \tag{26}$$

This update relation has been presented in [33,38,39]. By further assuming that the input data of the SLE is second order circular, i.e. by setting $\mathbf{P} = \mathbf{0}$ in (26), we arrive at the well-known recursive relation for the weight error covariance matrix of standard CLMS, presented in [2,3,10], and given by

$$\begin{aligned} \mathbf{K}_{\mathrm{SLE}}(k+1) &= \mathbf{K}_{\mathrm{SLE}}(k) - \mu(\mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k) + \mathbf{K}_{\mathrm{SLE}}(k)\mathbf{R}) \\ &\quad + \mu^2\big(\sigma_q^2\mathbf{R} + \mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k)\mathbf{R} + \mathbf{R}\mathrm{tr}[\mathbf{R}\mathbf{K}_{\mathrm{SLE}}(k)]\big) \end{aligned} \tag{27}$$

Remark 4. The mean square performance of CLMS for WLE of second order noncircular input data, given in (20) and (25), is dependent on its mean convergence behaviour, a situation that does not exist in SLE, as shown in (21) and (26).

4. Mean square stability

We shall now consider sufficient conditions for the mean square convergence of the weight error vector of CLMS for WLE with the general second order noncircular Gaussian input data. This work is challenging in the sense that a simultaneous diagonalisation of both the covariance matrix $\mathbf{R}$ and the pseudo-covariance matrix $\mathbf{P}$ in (25) is a prerequisite for a compact, closed-form, solution. The first attempt along this direction is given in [38], which applies the strong uncorrelating transform (SUT) [40,41] on $\mathbf{R}$ and $\mathbf{P}$. However, since SUT admits a single SVD for both matrices only for a doubly white and a special type of correlated second order noncircular signals, the analysis in [38] cannot be extended to the general second order noncircular signals for which the off-diagonal elements of both $\mathbf{R}$ and $\mathbf{P}$ contain nonzero elements. In order to both address the diagonalisation problem encountered by SUT and at the same time simplify the analysis, we employ the recently introduced approximate uncorrelating transform (AUT), which allows for a single singular value decomposition (SVD) of both the covariance and pseudo-covariance matrices, $\mathbf{R}$ and $\mathbf{P}$, within reasonable approximations [33,36,39].

The Takagi factorisation states that any complex symmetric matrix, like the pseudo-covariance matrix $\mathbf{P} = \mathbf{P}^T$, can be diagonalised as

$$\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}_{\mathbf{p}}\mathbf{Q}^T \tag{28}$$

where $\mathbf{Q}$ is a unitary matrix, that is $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, and $\boldsymbol{\Lambda}_\mathbf{p} = \text{diag}\{p_1, p_2, \ldots, p_N\}$ is a diagonal matrix of real-valued entries, where $p_1 \geq p_2 \geq \cdots \geq p_N$ are the nonnegative square roots of $\mathbf{P}\mathbf{P}^H$ [42].

The approximate uncorrelating transform (AUT) [36] states that the same matrix $\mathbf{Q}$ can be used to approximately diagonalise the covariance matrix $\mathbf{R}$, so that

$$\mathbf{R} \simeq \mathbf{Q}\boldsymbol{\Lambda}_\mathbf{r}\mathbf{Q}^H \tag{29}$$

and hence its inversion

$$\mathbf{R}^{-1} \simeq \mathbf{Q}\boldsymbol{\Lambda}_\mathbf{r}^{-1}\mathbf{Q}^H \tag{30}$$

where $\boldsymbol{\Lambda}_\mathbf{r} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, and λ_i are the real-valued eigenvalues. The approximations in (29) and (30) are valid for univariate data, and the equality is achieved when $\mathbf{x}(k)$ is real-valued, that is, maximum noncircular [33,36]. The benefits of using the AUT in obtaining the bound on the step-size μ in the mean square sense and the closed-form solutions for the steady state performance of CLMS are illustrated in the following.

With the AUT, we can now rotate the weight error vector and the input vector as

$$\hat{\mathbf{h}}(k) = \mathbf{Q}^H\tilde{\mathbf{h}}(k) \quad \text{and} \quad \hat{\mathbf{x}}(k) = \mathbf{Q}^H\mathbf{x}(k) \tag{31}$$

In a similar way, we can define

$$\hat{\mathbf{B}}(k) = \mathbf{Q}^H\mathbf{B}(k)\mathbf{Q} \quad \text{and} \quad \hat{\mathbf{C}} = \mathbf{Q}^H\mathbf{C}\mathbf{Q} \tag{32}$$

Therefore, the term $\text{tr}[\mathbf{R}\mathbf{K}(k)]$ on the RHS of (25) can be decomposed as

$$\text{tr}[\mathbf{R}\mathbf{K}(k)] = E[\tilde{\mathbf{h}}^H(k)\mathbf{R}\tilde{\mathbf{h}}(k)] = E[\hat{\mathbf{h}}^H(k)\boldsymbol{\Lambda}\hat{\mathbf{h}}(k)] = \boldsymbol{\lambda}^T\boldsymbol{\kappa}(k) \tag{33}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T$, and $\boldsymbol{\kappa}(k)$ is the $N \times 1$ second order moment vector, the components of which are the diagonal elements of $\hat{\mathbf{K}}(k) = E[\hat{\mathbf{h}}(k)\hat{\mathbf{h}}^H(k)]$. Then, based on (25), the evolution of $\boldsymbol{\kappa}(k)$ becomes

$$\boldsymbol{\kappa}(k+1) = \underbrace{\left(\mathbf{I} - 2\mu\boldsymbol{\Lambda}_\mathbf{r} + \mu^2\left(\boldsymbol{\Lambda}_\mathbf{r}^2 + \boldsymbol{\Lambda}_\mathbf{p}^2 + \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\right)}_{\mathbf{F}}\boldsymbol{\kappa}(k) + \mathbf{p}(k) + \mu^2\mathbf{c} + \mu^2\sigma_q^2\boldsymbol{\lambda} \tag{34}$$

with

$$\mathbf{p}(k) = 2\mu\Re\left[\mathbf{A}E[\hat{\mathbf{h}}^*(k)]\right] - 2\mu^2\Re[\mathbf{b}(k)] \tag{35}$$

$$\mathbf{A} = \text{diag}\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N\} \tag{36}$$

where $\hat{a}_i$ is the ith component of $\hat{\mathbf{a}} = \mathbf{Q}^H\mathbf{a}$. The vector $\mathbf{b}(k) = [\hat{B}_{11}(k), \hat{B}_{22}(k), \ldots, \hat{B}_{NN}(k)]^T$, in which $\hat{B}_{ii}(k)$ is the ith diagonal element of $\hat{\mathbf{B}}(k)$, and the vector $\mathbf{c} = [\hat{C}_{11}, \hat{C}_{22}, \ldots, \hat{C}_{NN}]^T$, in which $\hat{C}_{ii}$ is the ith diagonal element of $\hat{\mathbf{C}}$.

Similar to the analysis in Section 2, through the vanishing of the terms containing $\mathbf{g}^o$, that is, $\mathbf{p}(k)$ and $\mathbf{c}$ in (34), we obtain the evolution of $\boldsymbol{\kappa}(k)$ in the context of SLE with general second order noncircular inputs, as discussed in [33,38,39] and given by

$$\boldsymbol{\kappa}_{\text{SLE}}(k+1) = \underbrace{\left(\mathbf{I} - 2\mu\boldsymbol{\Lambda}_\mathbf{r} + \mu^2\left(\boldsymbol{\Lambda}_\mathbf{r}^2 + \boldsymbol{\Lambda}_\mathbf{p}^2 + \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\right)}_{\mathbf{F}}\boldsymbol{\kappa}_{\text{SLE}}(k) + \mu^2\sigma_q^2\boldsymbol{\lambda} \tag{37}$$

By further assuming that the input of SLE is second order circular, i.e. setting $\boldsymbol{\Lambda}_\mathbf{p} = \mathbf{0}$ in (37), we arrive at the well-known recursive relation for $\boldsymbol{\kappa}(k)$, provided by Horowitz and Senne [2], Fisher and Bershad [3], and Godavarti and Hero [10] and given by

$$\boldsymbol{\kappa}_{\text{SLE}}(k+1) = \left(\mathbf{I} - 2\mu\boldsymbol{\Lambda}_\mathbf{r} + \mu^2\left(\boldsymbol{\Lambda}_\mathbf{r}^2 + \boldsymbol{\lambda}\boldsymbol{\lambda}^T\right)\right)\boldsymbol{\kappa}_{\text{SLE}}(k) + \mu^2\sigma_q^2\boldsymbol{\lambda} \tag{38}$$

Remark 5. By comparing (34) and (37), observe that no matter whether CLMS is performing WLE or SLE with second order noncircular inputs, the mean square evolution of the weight error vector $\tilde{\mathbf{h}}(k)$ has the same transition matrix $\mathbf{F}$.

Convergence of the recursion in (34) is subject to two conditions: (1) all eigenvalues of $\mathbf{F}$ are less than unity; this is also the condition on the step-size μ of CLMS for SLE with second order noncircular inputs, as indicated by (37); (2) $\mathbf{p}(k)$ is bounded. First, observe that the boundedness of $\mathbf{p}(k)$ is guaranteed if $E[\hat{\mathbf{h}}(k)]$ is bounded. Since $E[\hat{\mathbf{h}}(k)] = \mathbf{Q}^H E[\tilde{\mathbf{h}}(k)]$, by considering (15), condition (2) is satisfied if $0 < \mu < 2/\text{tr}[\mathbf{R}]$. On the other hand, according to the analysis in [39], the upper bound on μ for the condition (1) to hold is given by

$$0 < \mu < \frac{1}{\sum_{i=1}^{N}\left(\lambda_i + \frac{p_i^2}{2\lambda_i}\right)} \tag{39}$$

while for second order circular inputs, $p_i = 0$, from (39), we have

$$0 < \mu < \frac{1}{\sum_{i=1}^{N}\lambda_i} = \frac{1}{\text{tr}[\mathbf{R}]} \tag{40}$$

Remark 6. Eqs. (39) and (40) are the respective bounds on μ for the mean square stability of CLMS for SLE with second order noncircular and circular inputs. For the case of WLE, since both the upper bounds in (39) and (40) are tighter than that governed by condition (2), in order to guarantee the boundedness of $\mathbf{p}(k)$ in (34), that is, $2/\text{tr}[\mathbf{R}]$, these are also the conditions on the step-size μ which ensure the mean square stability of the CLMS for WLE for second order noncircular and circular input data, respectively. From the above discussion, we are now able to draw the conclusion that the mean square stability bound on μ of CLMS is identical for both the cases of WLE and SLE, mainly due to the identical transition matrix $\mathbf{F}$.

5. Steady state analysis

To investigate the steady state performance of CLMS for WLE of general second order noncircular Gaussian signals, based on (33), we shall first rewrite the MSE $J(k)$ in (20) as

$$J(k) = \sigma_q^2 + \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*} - 2\Re[\mathbf{a}^H E[\tilde{\mathbf{h}}(k)]] + \boldsymbol{\lambda}^T\boldsymbol{\kappa}(k) \tag{41}$$

Suppose that μ is chosen such that the mean square stability is guaranteed; then using (16), (30), (33) and $\mathbf{a} = \mathbf{Q}\hat{\mathbf{a}}$, the steady state MSE (SSMSE), that is, $J(\infty)$, can be derived from (41) as

$$\begin{aligned} J(\infty) &= \sigma_q^2 + \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*} - 2\mathbf{a}^H\mathbf{R}^{-1}\mathbf{a} + \boldsymbol{\lambda}^T\boldsymbol{\kappa}(\infty) \\ &= \sigma_q^2 + \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*} - 2\sum_{j=1}^{N}\frac{|\hat{a}_j|^2}{\lambda_j} + \boldsymbol{\lambda}^T\boldsymbol{\kappa}(\infty) \end{aligned} \tag{42}$$

On the other hand, the minimum MSE (MMSE), denoted by $J_{\min}$, of the CLMS can be obtained substituting $E[\tilde{\mathbf{h}}(\infty)]$ for $\tilde{\mathbf{h}}(k)$ in (20), to give

$$J_{\min} = \sigma_q^2 + \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*} - \sum_{j=1}^{N}\frac{|\hat{a}_j|^2}{\lambda_j} \tag{43}$$

which provides an exact measure of the effect of the level of system undermodeling on the MMSE of CLMS for WLE.

By definition, the SSMSE of LMS-type adaptive algorithms is described by Widrow and Stearns [43]

$$J(\infty) = J_{\min} + J_{\text{ex}}(\infty) \tag{44}$$

where $J_{\text{ex}}(\infty)$ is the steady state excess MSE (SSEMSE). From (42) to (43), $J_{\text{ex}}(\infty)$ of the CLMS assumes the form

$$J_{\text{ex}}(\infty) = \boldsymbol{\lambda}^T\boldsymbol{\kappa}(\infty) - \sum_{j=1}^{N}\frac{|\hat{a}_j|^2}{\lambda_j} \tag{45}$$

where using (43) and (34) and considering that in the steady state $k \to \infty$, we arrive at

$$\begin{aligned}\boldsymbol{\kappa}(\infty) &= (\mathbf{I}-\mathbf{F})^{-1}\mathbf{s}(\infty)\\ &= (2\mu\boldsymbol{\Lambda}_{\mathbf{r}} - \mu^2\boldsymbol{\Lambda}_{\mathbf{r}}^2 - \mu^2\boldsymbol{\Lambda}_{\mathbf{p}}^2 - \mu^2\boldsymbol{\lambda}\boldsymbol{\lambda}^T)^{-1}\mathbf{s}(\infty)\end{aligned} \tag{46}$$

where

$$\begin{aligned}\mathbf{s}(\infty) &= \mathbf{p}(\infty) + \mu^2\mathbf{c} + \mu^2\sigma_q^2\boldsymbol{\lambda}\\ &= \mathbf{p}(\infty) + \mu^2\mathbf{c} + \mu^2\sum_{j=1}^{N}\frac{|\hat{a}_j|^2}{\lambda_j}\boldsymbol{\lambda}\\ &\quad - \mu^2\mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*}\boldsymbol{\lambda} + \mu^2 J_{\min}\boldsymbol{\lambda}\end{aligned} \tag{47}$$

Upon defining

$$\boldsymbol{\Lambda}_{\mathbf{1}} = 2\boldsymbol{\Lambda}_{\mathbf{r}} - \mu\boldsymbol{\Lambda}_{\mathbf{r}}^2 - \mu\boldsymbol{\Lambda}_{\mathbf{p}}^2$$

and employing the matrix inversion lemma, after some manipulations, from (46) we arrive at

$$\boldsymbol{\kappa}(\infty) = \left(\mathbf{I} + \frac{\mu\boldsymbol{\Lambda}_{\mathbf{1}}^{-1}\boldsymbol{\lambda}\boldsymbol{\lambda}^T}{1-\mu\boldsymbol{\lambda}^T\boldsymbol{\Lambda}_{\mathbf{1}}^{-1}\boldsymbol{\lambda}}\right)\mu^{-1}\boldsymbol{\Lambda}_{\mathbf{1}}^{-1}\mathbf{s}(\infty)$$

and hence,

$$\begin{aligned}\boldsymbol{\lambda}^T\boldsymbol{\kappa}(\infty) &= \frac{\mu^{-1}\boldsymbol{\lambda}^T\boldsymbol{\Lambda}_{\mathbf{1}}^{-1}\mathbf{s}(\infty)}{1-\mu\boldsymbol{\lambda}^T\boldsymbol{\Lambda}_{\mathbf{1}}^{-1}\boldsymbol{\lambda}}\\ &= \frac{\sum_{i=1}^{N}\frac{\mu^{-1}\lambda_i s_i(\infty)}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}}{1-\sum_{i=1}^{N}\frac{\mu\lambda_i^2}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}}\end{aligned} \tag{48}$$

where $s_i(\infty)$ is the ith element of $\mathbf{s}(\infty)$ in (47). Substituting (48) into (45) and after some mathematical manipulations, we obtain

$$J_{\text{ex}}(\infty) = \frac{\sum_{i=1}^{N}\frac{\mu(t_i+J_{\min}\lambda_i^2)}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}}{1-\sum_{i=1}^{N}\frac{\mu\lambda_i^2}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}} \tag{49}$$

where

$$\begin{aligned}t_i &= \left(\lambda_i + \frac{p_i^2}{\lambda_i}\right)|\hat{a}_i|^2 - 2\lambda_i\Re[\hat{B}_{ii}(\infty)] + \lambda_i\hat{C}_{ii}\\ &\quad + 2\sum_{j=1}^{N}\frac{|\hat{a}_j|^2}{\lambda_j}\lambda_i^2 - \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*}\lambda_i^2\end{aligned} \tag{50}$$

Remark 7. Eqs. (49) and (50) provide a detailed expression for the SSEMSE of CLMS for WLE with general second order noncircular inputs. A complete view on the SSMSE, $J(\infty)$, can be obtained by considering (43), (44), (49) and (50) together. It is clear that $J(\infty)$ is influenced by the values of the system impulse response associated to the input conjugate $\mathbf{x}^*(k)$, that is, $\mathbf{g}^o$ in (4), whose estimation is not considered by CLMS due to the system undermodelling. The SSMSE, $J(\infty)$, is also dependent on the full second order statistics of input data (both covariance and pseudo-covariance coefficients), the step-size μ and the noise variance σ_q^2.

The MMSE and SSEMSE performances of CLMS for WLE with second order circular inputs can be subsequently deduced from (43), (49) to (50) by setting the terms containing the pseudo-covariance information, that is, $\hat{a}_i$, $\hat{B}_{ii}(\infty)$ and p_i, all equal to zero, and are given by

$$J_{\min} = \sigma_q^2 + \mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*} \tag{51}$$

$$J_{\text{ex}}(\infty) = \frac{\sum_{i=1}^{N}\frac{\mu(\hat{C}_{ii}+\sigma_q^2\lambda_i)}{2-\mu\lambda_i}}{1-\sum_{i=1}^{N}\frac{\mu\lambda_i}{2-\mu\lambda_i}} \tag{52}$$

In a similar way, the MMSE and SSEMSE performances of CLMS for SLE with second order noncircular inputs, denoted by $J_{\min}^{\text{SLE}}$ and $J_{\text{ex}}^{\text{SLE}}(\infty)$, can be respectively deduced from (43), (49) to (50) by setting the terms containing $\mathbf{g}^o$, that is, $\mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*}$, $\hat{a}_i$, $\hat{B}_{ii}(\infty)$, and $\hat{C}_{ii}$, all equal to zero, and is given by

$$J_{\min}^{\text{SLE}} = \sigma_q^2 \tag{53}$$

$$J_{\text{ex}}^{\text{SLE}}(\infty) = \frac{\sum_{i=1}^{N}\frac{\mu\sigma_q^2\lambda_i^2}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}}{1-\sum_{i=1}^{N}\frac{\mu\lambda_i^2}{2\lambda_i-\mu\lambda_i^2-\mu p_i^2}} \tag{54}$$

Observe the agreement with the results in [33,38]. Furthermore, when the input of SLE becomes second order circular with $p_i = 0$, from (53) to (54), we have

$$J_{\min}^{\text{SLE}} = \sigma_q^2 \tag{55}$$

$$J_{\text{ex}}^{\text{SLE}}(\infty) = \frac{\sum_{i=1}^{N}\frac{\mu\sigma_q^2\lambda_i}{2-\mu\lambda_i}}{1-\sum_{i=1}^{N}\frac{\mu\lambda_i}{2-\mu\lambda_i}} \tag{56}$$

As expected, this is identical to the well-known results in [2,3], obtained by implicitly using a circular input assumption.

Remark 8. In all the cases considered, both the SSEMSE and the SSMSE of CLMS are monotonically increasing functions of the step-size μ and the system noise variance σ_q^2.

5.1. Steady state MSE (SSMSE) of CLMS v.s. degree of input noncircularity

It is of particular interest to find an explicit link between the degree of input noncircularity and the SSMSE performances of CLMS for both WLE and SLE. To simplify the analysis, we consider the case when the second order noncircular Gaussian input $\mathbf{x}(k)$ is doubly white (DW), for which

$$\mathbf{R} = \sigma_x^2\mathbf{I} \quad \text{and} \quad \mathbf{P} = \rho_x\mathbf{I} \tag{57}$$

where $\sigma_x^2 = E[x(k)x^*(k)]$ and $\rho_x = E[x^2(k)]$ are respectively the covariance and pseudo-covariance of $x(k)$ [17,18]. Note that in this case, we have[1]

$$\lambda_i = \sigma_x^2 \quad \text{and} \quad p_i = |\rho_x| \tag{58}$$

Consider a measure, η, of the degree of input noncircularity, defined as the ratio of the absolute value of the pseudo-covariance ρ_x to the variance σ_x^2, giving

$$\eta = \frac{|\rho_x|}{\sigma_x^2} \tag{59}$$

which is bounded to within [0, 1) [44,45]. Then, from (53), (54) to (44), the steady state performance of CLMS for SLE can be derived as [33]

$$\begin{aligned}J^{\text{SLE}}(\infty) &= J_{\min}^{\text{SLE}} + J_{\text{ex}}^{\text{SLE}}(\infty)\\ &= \sigma_q^2 + \frac{\frac{\mu N\sigma_q^2\sigma_x^4}{2\sigma_x^2-\mu\sigma_x^4-\mu|\rho_x|^2}}{1-\frac{\mu N\sigma_x^4}{2\sigma_x^2-\mu\sigma_x^4-\mu|\rho_x|^2}}\\ &= \sigma_q^2 + \frac{\mu N\sigma_q^2\sigma_x^2}{2-\mu(N+1)\sigma_x^2-\mu\sigma_x^2\eta^2}\end{aligned} \tag{60}$$

[1] This is because the diagonal elements $\{p_i\}$, $i = 1, 2, \ldots, N$, in $\boldsymbol{\Lambda}_p$ obtained by the Takagi factorisation are nonnegative square roots of $\mathbf{P}\mathbf{P}^H$. Therefore, for doubly white second order noncircular data, we have $\mathbf{P}\mathbf{P}^H = |\rho_x|^2\mathbf{I}$, and hence $p_i = |\rho_x|$ for $\forall i$.

Remark 9. For SLE, the SSMSE of CLMS $J^{\text{SLE}}(\infty)$ is a monotonically increasing function of the input noncircularity η, since $\partial J^{\text{SLE}}(\infty)/\partial\eta > 0$ for $\eta\in[0,\ 1)$. However, this relationship may be not applicable for a small step-size μ, a long filter length N, or a small input variance σ_x^2, since in such scenarios the third term in the denominator of the SSEMSE, $J_{\text{ex}}^{\text{SLE}}(\infty)$ in (60), is negligible as compared with the other two terms, making $J^{\text{SLE}}(\infty)$ almost independent on the input noncircularity η.

To investigate the relationship between the steady state performance of CLMS and the degree of input noncircularity η for WLE, first observe that for doubly white Gaussian input data, from (14), we have

$$\hat{a}_i = g_i^{\text{o}}\rho_x \tag{61}$$

Also note that

$$\mathbf{g}^{\text{o}T}\mathbf{R}\mathbf{g}^{\text{o}*} = \|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^2 \tag{62}$$

and hence, the MMSE of CLMS for WLE, that is, $J_{\min}$ in (43), now becomes

$$J_{\min} = \sigma_q^2 + \|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^2(1-\eta^2) \tag{63}$$

In a similar way, according to Appendix A and (32)

$$\hat{B}_{ii}(\infty) = \sigma_x^2\rho_x^*\sum_{l=1}^{N} g_l^{\text{o}*}E[\hat{h}_l(\infty)] + 2g_i^{\text{o}*}\sigma_x^2\rho_x^*E[\hat{h}_i(\infty)] \tag{64}$$

According to (16) and (31), we have

$$E[\hat{h}_l(\infty)] = \frac{g_l^{\text{o}}\rho_x}{\sigma_x^2}, \quad l = 1, 2, \ldots, N \tag{65}$$

Now, substitute this into (64) to yield

$$\begin{aligned}\hat{B}_{ii}(\infty) &= \|\mathbf{g}^{\text{o}}\|_2^2|\rho_x|^2 + 2|g_i^{\text{o}}|^2|\rho_x|^2\\ &= (\|\mathbf{g}^{\text{o}}\|_2^2 + 2|g_i^{\text{o}}|^2)\sigma_x^4\eta^2\end{aligned} \tag{66}$$

Similarly, according to Appendix B and (32), we have

$$\begin{aligned}\hat{C}_{ii} &= \|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^4 + |g_i^{\text{o}}|^2(|\rho_x|^2 + \sigma_x^4)\\ &= \|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^4 + |g_i^{\text{o}}|^2\sigma_x^4(1+\eta^2)\end{aligned} \tag{67}$$

By considering (58), (61), (66) and (67), after some mathematical manipulations, the term t_i in (50) can be simplified as

$$t_i = |g_i^{\text{o}}|^2\sigma_x^6(1-\eta^2)^2 \tag{68}$$

while based on (58), (63) and (68), the SSEMSE of CLMS for WLE, that is, $J_{\text{ex}}(\infty)$ in (49), can be obtained as

$$J_{\text{ex}}(\infty) = \frac{\mu\|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^4\big((1-\eta^2)^2 + N(1-\eta^2)\big) + \mu N\sigma_q^2\sigma_x^2}{2-\mu(N+1)\sigma_x^2 - \mu\sigma_x^2\eta^2} \tag{69}$$

Therefore, from (63) to (69), the SSMSE of CLMS for WLE becomes

$$\begin{aligned}J(\infty) &= J_{\min} + J_{\text{ex}}(\infty)\\ &= \sigma_q^2 + \|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^2(1-\eta^2)\\ &\quad + \frac{\mu\|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^4\big((1-\eta^2)^2 + N(1-\eta^2)\big) + \mu N\sigma_q^2\sigma_x^2}{2-\mu(N+1)\sigma_x^2 - \mu\sigma_x^2\eta^2}\\ &= \sigma_q^2 + \frac{2\|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^2(1-\eta^2)(1-\mu\sigma_x^2\eta^2) + \mu N\sigma_q^2\sigma_x^2}{2-\mu(N+1)\sigma_x^2 - \mu\sigma_x^2\eta^2}\end{aligned} \tag{70}$$

Remark 10. By comparing (60) and (70), observe that for a second order noncircular doubly white Gaussian input process $x(k)$, CLMS always yields a larger MSE for WLE as compared with SLE, since the term $2\|\mathbf{g}^{\text{o}}\|_2^2\sigma_x^2(1-\eta^2)(1-\mu\sigma_x^2\eta^2)$ in (70) is always positive for $\eta\in[0,\ 1)$ and the value of step-size μ is within the bound in (39) for the mean square stability of CLMS, that is, $0<\mu< 2/(N\sigma_x^2(2+\eta^2))$.

Remark 11. For WLE, the relationship between the steady state performance of CLMS, $J(\infty)$ in (70), and the input noncircularity η is more complicated as compared with that for SLE. Observe that the denominator and the numerator of the SSEMSE, $J_{\text{ex}}(\infty)$, are both monotonically decreasing functions of η, and hence, theoretically speaking, $J(\infty)$ has no monotonicity in the degree of input noncircularity η for $\eta\in[0,\ 1)$. However, since the numerator is a much sharper decreasing function of η than the denominator, $J(\infty)$ is very likely to decrease as the input noncircularity η increases. Simulations at the end of the next section illustrate these findings. Also observe that when the condition $\|\mathbf{g}^{\text{o}}\|_2^2 \ll \mu N\sigma_q^2/2$ holds, $J(\infty)$ behaves as an increasing function of η. This may happen when the WL system to be estimated is of long impulse response length N, very weakly WL with a small $\|\mathbf{g}^{\text{o}}\|_2^2$, or very noisy with a large σ_q^2. This may also be the case when a large step-size μ is used by the CLMS algorithm.

6. Simulations

Numerical examples were conducted in the MATLAB programming environment in order to evaluate the theoretical findings on both the mean and mean square convergence performance of the CLMS algorithm for both widely linear estimation (WLE) and strictly linear estimation (SLE) with second order noncircular and circular Gaussian input data. The experiments were performed in a system identification setting, where the strictly linear system to be identified was a strictly linear FIR channel, for which the weight coefficients $\mathbf{h}^{\text{o}}$ were drawn from a uniformly distributed complex-valued vector random variable. For the WLE task, the desired signal $d_{\text{WL}}(k)$ assumed the form [19],

$$\begin{aligned}d_{\text{WL}}(k) &= \mathbf{h}^{\text{o}H}\mathbf{x}(k) + \mathbf{g}^{\text{o}H}\mathbf{x}^*(k) + q(k)\\ &= 2\Re[\mathbf{h}^{\text{o}H}\mathbf{x}(k)] + q(k)\end{aligned} \tag{71}$$

where $\mathbf{g}^{\text{o}} = \mathbf{h}^{\text{o}*}$, while the system input $\mathbf{x}(k)$ was assumed to be a zero-mean complex-valued Gaussian process and $q(k)$ was zero-mean complex-valued doubly white circular Gaussian noise with $\sigma_q^2 = E[|q(k)|^2]$. The SLE task considered here is directly deduced from (71), through a vanishing $\mathbf{g}^{\text{o}}$, as given in (3). The Gaussian input $\mathbf{x}(k)$ to both the unknown system and the CLMS algorithm was second order noncircular, and obeys the following widely linear autoregressive (AR)(1) process

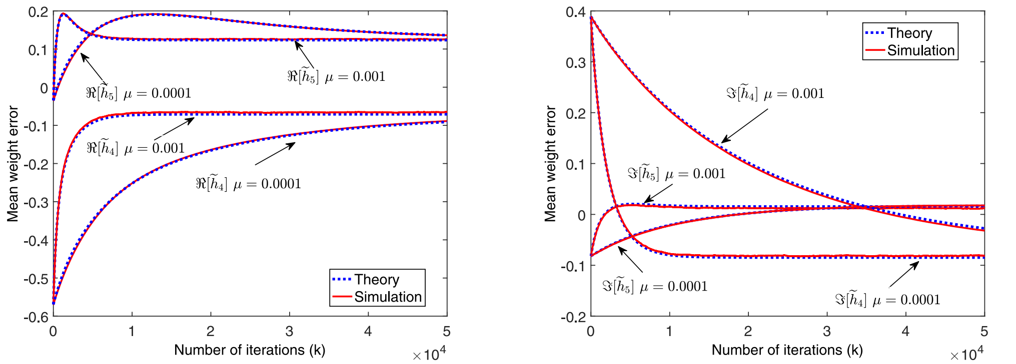
$$x(k) = 0.7x(k-1) + 0.2x^*(k-1) + u(k) \tag{72}$$

where $u(k)$ is a zero-mean doubly white circular Gaussian noise process with unit variance. In this case, the real and the imaginary parts of $x(k)$ obey different AR(1) processes. For the second order circular input considered, $\mathbf{x}(k)$ obeys the following strictly linear AR(1) process, given by

$$x(k) = 0.7x(k-1) + u(k) \tag{73}$$

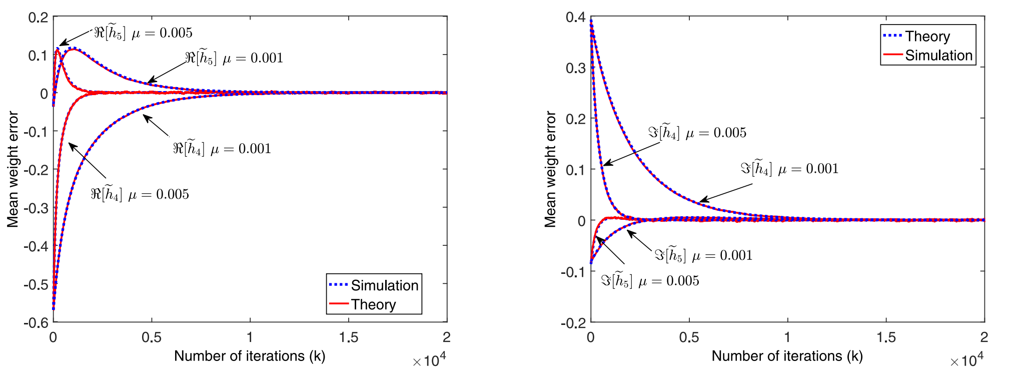
The weights within the CLMS algorithm were initialised with zeros, and simulation results were obtained by averaging over 10,000 independent trials. Note that the proposed performance analysis on second order noncircular Gaussian input data, together with the above WLE task represented in (71), are valid to theoretically investigate the suboptimalities of a CLMS based multiple access interference (MAI) suppressor in multiuser wireless communications over frequency selective channels for real-valued constellation schemes, such as amplitude-shift keying and binary phase-shift keying (BPSK) [20]. The present analysis is a missing piece in such scenarios, since just simulation results exist in [20] to reflect the performance deficiency of CLMS.

We first chose filter length $N=5$, step-size $\mu = 0.0001, 0.001$, and $\sigma_q^2 = 0.0001$. Fig. 1(a) and (b) illustrate the mean behaviour, in terms of real and imaginary parts of several weight error coefficients, of CLMS for WLE with second order noncircular Gaussian



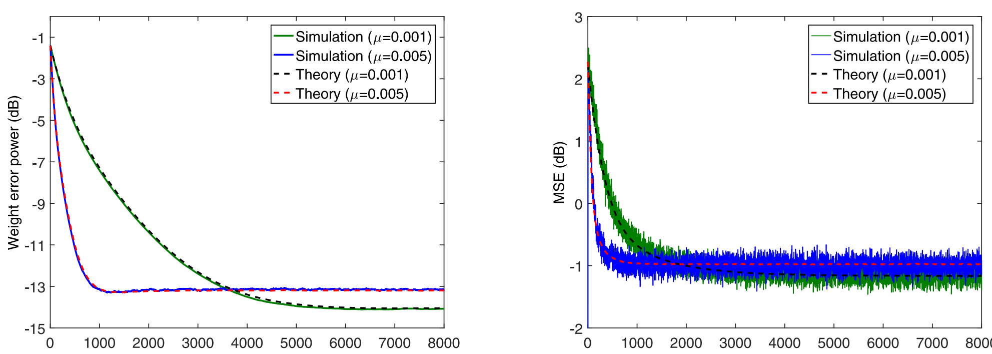
(a) Real part of the weight error coefficients (b) Imaginary part of the weight error coefficients

Fig. 1. Comparison of the theoretical and simulated curves of the mean evolution of representative weight error coefficients of the CLMS for WLE with second order noncircular Gaussian input signals, where $\mu \in \{0.0001, 0.001\}$, and $\sigma_q^2 = 0.0001$. (a) Real part, and (b) Imaginary part.



(a) Real part of the weight error coefficients (b) Imaginary part of the weight error coefficients

Fig. 2. Comparison of the theoretical and simulated curves of the mean evolution of representative weight error coefficients of the ACLMS for WLE with second order circular Gaussian input signals, where $\mu \in \{0.001, 0.005\}$, and $\sigma_q^2 = 0.005$. (a) Real part, and (b) Imaginary part.



(a) Weight error power (b) MSE

Fig. 3. Comparison of the theoretical and simulated curves for the mean square behaviours of the CLMS for WLE with second order noncircular Gaussian input data, with $N = 5$, $\mu \in \{0.001, 0.005\}$ and $\sigma_q^2 = 0.0001$. (a) Weight error power, and (b) MSE.

input signals. The results were obtained for both the simulations and theoretical analysis from (13). Due to its strictly linear nature, the CLMS algorithm was of 'deficient length' when used for WLE, as it does not consider the conjugate weight coefficients $\mathbf{g}^o$ associated with the input conjugate $\mathbf{x}^*(k)$ in (71), resulting in unavoidable bias in the estimation of $\mathbf{h}^o$, and hence, as shown in Fig. 1(a) and (b), the mean weight error vector $E[\widetilde{\mathbf{h}}(k)]$ was not able to converge to $\mathbf{0}$. However, as discussed in Remark 3, the case of circular input can remove this bias, as shown in Fig. 2(a) and (b).

In the next stage, we investigated the validity of the proposed mean square convergence analysis of CLMS for WLE with second order noncircular Gaussian input data. Fig. 3(a) and (b) show respectively the theoretical and simulated convergence behaviours of the weight error power $\text{tr}[\mathbf{K}(k)]$, where $\mathbf{K}(k) = E[\widetilde{\mathbf{h}}(k)\widetilde{\mathbf{h}}^H(k)]$, and the MSE $J(k)$, for which theoretical expressions are respectively given in (25) and (20). It can be observed that our theoretical analysis accurately describes both the transient and the steady state mean square behaviours of CLMS for different values of the step-size μ. Also note that the mean square convergence analysis provided in (25) and (20) is the most general one for the CLMS in terms of second order statistics, since other mean square statistical descriptions of CLMS on SLE in the literature [2,3,10,38] can be conveniently deduced as special cases, as discussed in Section 3.

The steady state performance of CLMS for both WLE and SLE with second order noncircular input data against different values of the step-size μ and system noise variance σ_q^2 is given in Table 1. In order to achieve the closed-form expressions of the steady state excess MSE (SSEMSE) of CLMS for WLE and SLE, re-

Table 1
Comparison of theoretical and simulated SSMSE of the CLMS algorithm for both WLE and SLE of second order noncircular Gaussian input data against different values of the step-size μ and the system noise variance σ_q^2.

μ	σ_q^2	Theoretical SSMSE (dB)		Simulated SSMSE (dB)	
		WLE	SLE	WLE	SLE
0.0005	0.0001	−1.263	−39.987	−1.252	−39.980
0.001	0.0001	−1.250	−39.973	−1.240	−39.971
0.005	0.0001	−1.124	−39.860	−1.034	−39.820
0.0005	0.001	−1.258	−29.987	−1.246	−29.985
0.001	0.001	−1.243	−29.973	−1.224	−29.965
0.005	0.001	−1.119	−29.860	−1.105	−29.799
0.0005	0.01	−1.206	−19.987	−1.191	−19.973
0.001	0.01	−1.191	−19.966	−1.179	−19.961
0.005	0.01	−1.067	−19.860	−0.987	−19.810
0.005	0.02	−1.010	−16.850	−0.977	−16.825
0.005	0.05	−0.843	−12.871	−0.805	−12.839

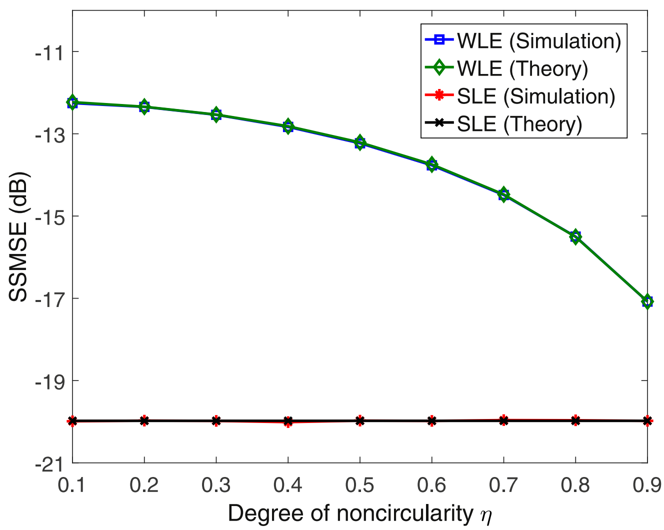


Fig. 4. Steady state MSE performances of CLMS for both WLE and SLE against varying degrees of input noncircularity η, with $N = 10$, $\mu = 0.001$, and $\sigma_q^2 = 0.01$.

spectively given in (49) and (54), the approximate uncorrelating transform (AUT) was used to diagonalise both the covariance and pseudo-covariance matrices **R** and **P** with a single SVD [33,36]. The steady state MSE (SSMSE) of CLMS, $J(\infty)$, for both cases was subsequently obtained by using (44). This conforms with the analysis in Remark 8, showing that in both the cases of WLE and SLE, $J(\infty)$ increases with an increase in μ or σ_q^2, and the good agreement between the theoretical results and the simulated ones can be observed. Also observe that CLMS yielded a much larger MSE for WLE as compared with that obtained for SLE for a specific setting. This is mainly because of the positive term $\mathbf{g}^{oT}\mathbf{R}\mathbf{g}^{o*}$ in the MMSE of CLMS for WLE, given in (43), that appears as a result of the undermodelling problem, which constituted the major part of the corresponding SSMSE.

By virtue of AUT, we are also able to build up an intuitive and explicit link between the theoretical SSMSE of the CLMS algorithm and the degree of noncircularity η of the doubly white Gaussian input data. As discussed in Remark 9, theoretically speaking, for SLE, the SSMSE of CLMS in (60) is a monotonically increasing function of the input noncircularity η, however, this relationship is not likely to be visible for a small step-size μ or a long filter length N, since the term involving η is negligible compared with other terms, making the SSMSE of CLMS largely independent on η in such scenarios. This phenomenon is illustrated in Fig. 4, in which the results were obtained by using a long filter length $N = 10$ and a small step-size $\mu = 0.001$. For a fair comparison, the doubly white input vector $\mathbf{x}(k)$ was fixed to have unit variance, and the different degrees of input noncircularity η were produced by varying the pseudo-covariance ρ_x. On the other hand, as discussed in Remark 11, the SSMSE of the CLMS algorithm for WLE theoretically has no monotonicity in η, however, since the numerator is a much sharper decreasing function of η as compared with the denominator, it is very likely to decrease as the input noncircularity η increases, which is the case shown in Fig. 4. The simulation results in Fig. 4 also support our analysis in Remark 10, which states that for a specific input noncircularity η, CLMS renders a larger MSE in the steady state for WLE as compared with SLE. Again, observe the good match between the simulated and theoretical SSMSE of CLMS in all the cases considered. In order to illustrate the theoretical links between the SSMSE of CLMS and η for WLE and SLE, we carefully designed the following experiment: the system to be identified in SLE was of short impulse response with $N = 2$, and a large step-size $\mu = 0.3$ was used by CLMS, so as to make the value of the term $\mu\sigma_x^2\eta^2$ in (60) closer to the other two terms within the denominator of the SSEMSE of CLMS. For WLE, the system to be identified was also of short impulse response, with $N = 2$, but very weakly WL with a small value of $\|\mathbf{g}^o\|_2^2 = 0.001$, and very noisy with $\sigma_q^2 = 0.2$. A large step-size $\mu = 0.25$ was used by CLMS, in order to introduce the condition $\|\mathbf{g}^o\|_2^2 \ll \mu N\sigma_q^2/2$. The simulation results for this experiment are given in Fig. 5, which conforms with our finding that for SLE, the MSE performance decreases with an increase in input noncircularity η, which can also be the case for WLE. However, we should emphasise that the discrepancy between the theoretical SSMSE of CLMS and the corresponding simulated one became obvious due to the use of a large step-size μ.

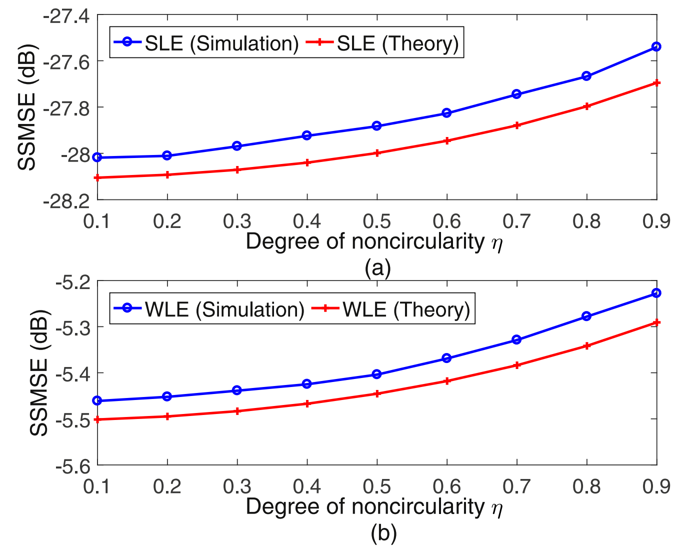


Fig. 5. Steady state MSE performances of CLMS, for both WLE and SLE, against varying degrees of input noncircularity η. For SLE, the system to be identified was generated with $N = 2$ and $\sigma_q^2 = 0.001$, and a large step-size $\mu = 0.3$ was used by CLMS. For WLE, the system to be identified was generated with $N = 2$, $\sigma_q^2 = 0.2$, a very small $\|\mathbf{g}^o\|_2^2 = 0.001$, and a large step-size $\mu = 0.25$ was used by CLMS.

7. Conclusions

The mean and mean square convergence behaviours of the complex least mean square (CLMS) adaptive filtering algorithm have been analysed for widely linear estimation (WLE) with second order noncircular Gaussian input data, in order to quantify its suboptimality in this scenario. This is achieved by considering CLMS as a special, deficient length, case of augmented CLMS (ACLMS) for WLE. The analysis has provided theoretical mean and mean square evolutions for the weight error coefficients, as well as the MSE performance, and has shown that CLMS exhibits different convergence behaviours for second order noncircular and circular inputs. It has also been illustrated that the existing statistical descriptions of CLMS for strictly linear estimation (SLE) can be conveniently obtained as special cases of this analysis. A unified bound on the step-size has been derived for mean square stability of CLMS when performing WLE, illustrating that CLMS has identical stability bound on the step-size for both WLE and SLE, but exhibits different bounds for second order circular and noncircular inputs. The closed-form expressions for the steady state MSE and excess MSE of CLMS for WLE have also been derived, enabling us to explicitly link the degree of input noncircularity with the steady state MSE for doubly white Gaussian input data. We have shown that, unlike the case of SLE where the steady state MSE of CLMS always increases with an increase in the degree of noncircularity, the steady state MSE for WLE has no monotonicity in the input noncircularity. We have also showed that in some scenarios, the steady state performance of CLMS for WLE increases as the input noncircularity increases. Simulations in system identification setting support the analysis. The proposed performance analysis, together with the simulation results, also provide physical insights into the suboptimalities in both the transient and steady state stages of a CLMS based multiple access interference (MAI) suppressor in multiuser wireless communications when real-valued constellation schemes are adopted. A detailed analysis of this application scenario is subject to our future work.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grant 61771124, the Natural Science Foundation of Jiangsu Province under Grant BK20140645, and the Fundamental Research Funds for the Central Universities (Grant no. 2242016K41050).

Appendix A. The Detailed Form of $B(k)$ in (23)

Using the independence assumptions stated in Section 2, $\mathbf{B}(k)$ in (23) can be written as

$$\mathbf{B}(k) = \sum_{l=1}^{N}\sum_{m=1}^{N} g_m^{o*} E[\widetilde{h}_l(k)] E[x_l^*(k) x_m^*(k) \mathbf{x}(k) \mathbf{x}^H(k)] \tag{74}$$

The (i, j)th term of $\mathbf{B}(k)$ is then given by

$$\begin{aligned} B_{ij}(k) &= \sum_{l=1}^{N}\sum_{m=1}^{N} g_m^{o*} E[\widetilde{h}_l(k)] E[x_l^*(k) x_m^*(k) x_i(k) x_j^*(k)] \\ &= \sum_{l=1}^{N}\sum_{m=1}^{N} g_m^{o*} E[\widetilde{h}_l(k)] (p_{lm}^* r_{ij} + r_{il} p_{mj}^* + p_{lj}^* r_{im}) \end{aligned} \tag{75}$$

The last step in (75) is achieved by using the Gaussian fourth order moment factorisation theorem.

Appendix B. The Detailed Form of C in (24)

First note that from (24), the term $|\mathbf{g}^{oT}\mathbf{x}(k)|^2$ can be decomposed as

$$\begin{aligned} |\mathbf{g}^{oT}\mathbf{x}(k)|^2 &= \left(\mathbf{g}^{oT}\mathbf{x}(k)\right)\left(\mathbf{g}^{oH}\mathbf{x}^*(k)\right) \\ &= \sum_{l=1}^{N}\sum_{m=1}^{N} g_l^o g_m^{o*} x_l(k) x_m^*(k) \end{aligned} \tag{76}$$

Using the independence assumptions stated in Section 2 and applying the Gaussian fourth order moment factorisation, we can write the (i, j)th term of $\mathbf{C}$ in (24) as

$$\begin{aligned} C_{ij} &= \sum_{l=1}^{N}\sum_{m=1}^{N} g_l^o g_m^{o*} E[x_l(k) x_m^*(k) x_i(k) x_j^*(k)] \\ &= \sum_{l=1}^{N}\sum_{m=1}^{N} g_l^o g_m^{o*} (r_{lm} r_{ij} + p_{li} p_{mj}^* + r_{lj} r_{im}) \end{aligned} \tag{77}$$

Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.sigpro.2018.03.009.

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