A Real Time Tracker of Complex Circularity

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***Abstract*—A new insight into the relationship between the complex circularity quotient and the coefficient of a linear minimum mean square estimator (LMMSE) which estimates a complex random variable from its complex conjugate is established. An adaptive version of this estimator, suitable for real time tracking of the degree of complex circularity is proposed based on the complex least mean square (CLMS) algorithm. For Gaussian data, the mean and mean square analyses show the effect of data non-circularity on the stability bounds of the real time circularity tracker. The concept is verified over simulations on both synthetic single- and multi-channel data and on real world wind signals with varying degrees of circularity.**

***Index Terms*—Complex random variables, complex circularity, circularity coefficient, complex least mean square (CLMS), multichannel signals.**

I. Introduction

Complex-valued signals are common in communications, signal and array processing and can also be used as a more convenient way to represent bivariate real-valued signals [1], [2]. Although statistical tools for complex random variables are traditionally treated as generic extensions of their real-valued counterparts [3], this approach is valid only for second order circular (proper) random variables which are uncorrelated with their complex conjugates [2]. For example, the covariance, $c = \mathsf{E}\left\{|z|^2\right\}$, of a complex random variable (r.v.), $z = x + jy \in \mathbb{C}$, is obtained by replacing the operator, $(\cdot)^2$, in $\mathbb{R}$ with the magnitude squared, $|\cdot|^2$, in $\mathbb{C}$ but it only reflects the total power in the x and y channels. To ensure a sufficient number of degrees of freedom in the processing of general improper data, the pseudocovariance $p = \mathsf{E}\left\{z^2\right\}$, which measures the correlation between the r.v. z and its conjugate, z^*, needs also to be considered. Signals with vanishing pseudocovariance (i.e. $p = 0$) are proper.

Early methods to test the impropriety of a signal have been based on hypothesis tests that use block estimates of the covariance and the pseudocovariance. The authors in [4] developed a hypothesis test for the impropriety of Gaussian signals using a generalized likelihood ratio (GLR) test which was further studied in [5], [6]. This method was extended for signals from a general class of complex elliptically symmetric (CES) distributions in [7]. To deal with measurement errors, a robust circularity coefficient estimator was developed in [8] based on solving M-estimation equations with a novel weighting scheme.

Impropriety can be measured using the circularity quotient, ρ, given by

$$\rho \triangleq \frac{p}{c} \in \mathbb{C} \tag{1}$$

which is the ratio between the pseudocovariance and the covariance [9]. The magnitude of the circularity quotient is referred to as the the circularity coefficient, $|\rho| \triangleq \frac{|p|}{c} \in \mathbb{R}$.

A real time algorithm for tracking the impropriety of a signal was introduced in [10] based on a convex combination of a strictly and widely linear estimators. The impropriety of a signal is then tracked by observing the evolution of the adaptive convex mixing parameter which indicates which sub-filter (strictly or widely linear) is a better match to the nature of the data.

The limitations of current approaches in identifying the second order circularity are:

(a) Block-based estimators are accurate, but are not suitable for real time applications, or for non-stationary data;
(b) Hypothesis tests are limited since they are only able to reveal whether the signal is proper or improper, however, they cannot assess the degree of impropriety;
(c) The value of the convex mixing parameter in [10] depends also on the filter settings and its relationship to the circularity quotient is yet to be established rigorously.

We provide a solution for real-time circularity tracking by firstly establishing that the circularity quotient is effectively the optimal coefficient of an LMMSE estimator that estimates the complex conjugate of a signal from the original signal itself. The proposed algorithm then utilises an adaptive filter weight to track the circularity quotient of a signal in real time and overcomes the issues mentioned in (a) – (c) above.

II. Relationship Between A Complex Variable and Its Conjugate

Deterministic case. Consider the problem of finding a linear mapping that relates a deterministic variable, $\tilde{z} = |\tilde{z}|e^{j\angle\tilde{z}} \in \mathbb{C}$, with magnitude $|\tilde{z}|$ and phase $\angle\tilde{z}$, and its complex conjugate, $\tilde{z}^* = |\tilde{z}|e^{-\angle\tilde{z}}$. This mapping has the form

$$\tilde{z}^* = w^*\tilde{z} \tag{2}$$

The solution for the coefficient w is thus

$$w = \frac{\tilde{z}}{\tilde{z}^*} = \frac{\tilde{z}^2}{|\tilde{z}|^2} = e^{j2\angle\tilde{z}} \tag{3}$$

Physically, the coefficient w in (2) rotates the complex variable $\tilde{z}$ by an angle of $-2\angle\tilde{z}$.

Stochastic case. Now, consider the problem of using a zero-mean r.v. $z \in \mathbb{C}$ to estimate its complex conjugate, that is

$$\hat{z}^* = w^* z \tag{4}$$

where $\hat{z}^*$ denotes the estimate of the complex conjugate of z and w is the coefficient that relates the two variables. Unlike (3), every realization of the r.v. in the data stream has a different phase and we require a stochastic solution.

Our aim is therefore, to find an estimate of w that minimizes the estimation error, $e = \hat{z}^* - z^*$, across all realizations of z. To this end, we propose to employ minimum mean square error (MMSE) estimation whereby the optimal value of w is found by minimising the cost function

$$J_{\text{MSE}} = \mathsf{E}\left\{|e|^2\right\} = \mathsf{E}\left\{|z^* - \hat{z}^*|^2\right\} \tag{5}$$

The optimal value of w, denoted by w_{opt}, that minimizes the cost function J_{MSE} is given by the Wiener solution

$$w_{\text{opt}} = c^{-1} r \tag{6}$$

where $c = \mathsf{E}\{zz^*\}$ is the covariance of the input data and $r = \mathsf{E}\{z(z^*)^*\}$ is the cross-covariance between the desired signal z^* and the input z. Thus, the Wiener solution becomes

$$w_{\text{opt}} = c^{-1} r = \frac{\mathsf{E}\left\{z^2\right\}}{\mathsf{E}\left\{|z|^2\right\}} \tag{7}$$

Given that for our single-tap case, $p = \mathsf{E}\left\{z^2\right\}$ and $c = \mathsf{E}\left\{|z|^2\right\}$, we finally have

$$w_{\text{opt}} = \frac{p}{c} = \rho \tag{8}$$

Remark #1: From (1) and (8), the circularity quotient can be interpreted as the LMMSE solution for estimating the complex conjugate of a random variable from the original random variable itself.

Finding the Wiener solution requires the knowledge of the true statistics of the data (in our case, c and p) which is typically not available. As block based estimators for the Wiener solution are inadequate for non-stationary signals, we next develop an adaptive estimator.

III. Proposed Algorithm

Interpreting the circularity quotient as the optimal Wiener solution for estimating the complex conjugate of a r.v. from the original r.v. enables us to configure an adaptive filter, in our case, the complex least mean square (CLMS), to track the circularity quotient in real time. The proposed circularity tracking algorithm is

$$\hat{z}_k^* = w_k^* z_k \tag{9a}$$

$$e_k = z_k^* - \hat{z}_k^* \tag{9b}$$

$$w_{k+1} = w_k + \mu e_k^* z_k \tag{9c}$$

where the parameter μ in (9c) is the step-size which governs the convergence of the algorithm [11], while the degree of second order circularity is represented by the weight w_k.

As CLMS only uses instantaneous estimates of the data statistics, the weights can only asymptotically approach their optimal value, and we need to analyse the contributions of the bias and variance of parameter estimates to the total mean square error.

Mean behaviour. By substituting the error $e_k = z_k^* - w_k^* z_k$ into the CLMS recursion in (9c), we have

$$w_{k+1} = w_k + \mu(z_k^2 - w_k|z_k|^2) \tag{10}$$

so that the weight error, $v_k \triangleq \frac{p}{c} - w_k$, becomes

$$v_{k+1} = v_k - \mu v_k|z_k|^2 - \mu z_k^2 + \mu\frac{p}{c}|z_k|^2 \tag{11}$$

Given that $\mathsf{E}\left\{z_k^2\right\} = p$ and $\mathsf{E}\left\{|z_k|^2\right\} = c$, taking the expectation of (11) and using the independence assumption [12] yields

$$\mathsf{E}\{v_{k+1}\} = (1 - \mu c)\mathsf{E}\{v_k\} \tag{12}$$

so that the convergence condition $|1 - \mu c| < 1$, leads to the stability bound

$$0 < \mu < \frac{2}{c} \tag{13}$$

where c is the covariance (power) of the input signal z_k. Thus, the algorithm is asymptotically unbiased and its stability in the mean is identical to that of the CLMS algorithm.

Remark #2: The mean behaviour of the proposed circularity tracker is not affected by the degree of circularity of the input signal.

Mean square behaviour. The covariance of the weight error of our circularity tracker is given by

$$\begin{aligned} \mathcal{K}_{k+1} &\triangleq \mathsf{E}\left\{|v_{k+1}|^2\right\} \\ &= \mathsf{E}\left\{\left|v_k - \mu v_k|z_k|^2 - \mu z_k^2 + \mu\frac{p}{c}|z_k|^2\right|^2\right\} \end{aligned} \tag{14}$$

Assuming Gaussian data, the fourth order moments in (14) can be decomposed into a combination of second order moments as $\mathsf{E}\left\{|z_k|^4\right\} = 2c^2 + |p|^2$, $\mathsf{E}\left\{z_k^2|z_k|^2\right\} = 3pc$ and $\mathsf{E}\left\{z_k^{*2}|z_k|^2\right\} = 3p^*c$. After some algebraic manipulations, the weight error covariance becomes

$$\mathcal{K}_{k+1} = \theta\mathcal{K}_k + \beta\mathsf{Re}\left\{p^*\bar{v}_k\right\} + \alpha \tag{15}$$

where the non-recursive terms are $\theta = (2c^2+|p|^2)\mu^2 - 2c\mu + 1$, $\beta = 2\mu^2 c[1 - \frac{|p|^2}{c^2}]$ and $\alpha = \mu^2 c^2[\frac{|p|^4}{c^4} - 3\frac{|p|^2}{c^2} + 2]$. The term $\bar{v}_k \triangleq \mathsf{E}\{v_k\}$ refers to the mean of the weight error at time instant k, while the operator $\mathsf{Re}\{\cdot\}$ denotes the real part of a complex number.

The recursion for the weight error covariance, $\mathcal{K}_k$, involves an additional time varying term, $\mathsf{Re}\{p^*\bar{v}_k\}$, and its evolution can be analysed by multiplying Equation (12) by p^* to give

$$\mathsf{Re}\left\{p^*\bar{v}_{k+1}\right\} = (1 - \mu c)\mathsf{Re}\left\{p^*\bar{v}_k\right\} \tag{16}$$

This allows us to formulate the following vector recursion involving (15) and (16)

$$\begin{bmatrix} \mathcal{K}_{k+1} \\ \mathsf{Re}\left\{p^* \bar{v}_{k+1}\right\} \end{bmatrix} = \underbrace{\begin{bmatrix} \theta & \beta \\ 0 & \gamma \end{bmatrix}}_{\triangleq \mathbf{A}} \begin{bmatrix} \mathcal{K}_{k} \\ \mathsf{Re}\left\{p^* \bar{v}_{k}\right\} \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \quad (17)$$

where $\gamma = (1-\mu c)$. For the mapping in (17) to be contractive, all the eigenvalues of $\mathbf{A}$ should lie within the unit circle. Since $\mathbf{A}$ is an upper triangular matrix, its eigenvalues are θ and γ. The convergence condition $|\gamma| = |1 - \mu c| < 1$ was already addressed in the analysis of the convergence in the mean in (13). The second stability condition, $\theta < 1$, is satisfied when

$$(2c^2 + |p|^2)\mu^2 - 2c\mu < 0 \quad (18)$$

which leads to the bound on the learning rate μ

$$0 < \mu < \frac{2}{c(2 + \frac{|p|^2}{c^2})} = \frac{2}{c(2 + |\rho|^2)} \quad (19)$$

Remark $\#3$: Unlike the condition for stability in the mean, the mean square stability depends on the degree of circularity of the signal. For proper data, $\rho = 0$, and the condition in (19) becomes $\mu \leq 1/c$. Since $|p| \leq c$ [9], for highly non-circular data we have $\mu \leq \frac{2}{3c}$.

Steady state mean square behaviour. Assuming that the step-size μ satisfies the condition in (19), the covariance of the weight error, $\mathcal{K}_k = \mathsf{E}\left\{|v_k|^2\right\}$, then converges to a steady-state value of

$$\lim_{k\to\infty} \mathcal{K}_k = \frac{\alpha}{1-\theta} = \mu c \frac{\left(1 - \frac{|p|^2}{c^2}\right)\left(2 - \frac{|p|^2}{c^2}\right)}{2 - \mu c\left(2 + \frac{|p|^2}{c^2}\right)} \quad (20)$$

Thus, the estimate of the circularity quotient has a steady-state error power that is approximately proportional to μ for small step-sizes. The weight error covariance depends on the degree of impropriety of the signal and is lower for signals that are less proper. Moreover, we can see that for small μ

$$\lim_{k\to\infty} \mathcal{K}_k \quad \leq \quad \mu c + \mathcal{O}(\mu^2) \quad (21)$$

where equality holds when z_k is proper.

IV. Simulations

For all the simulations, the CLMS was configured with a filter length $L = 1$, step-size $\mu = 0.01$, input data z_k and desired signal z_k^*. The filter weight was initialized as $w_0 = 0.5 + 0.5j$.

Single-channel data. In the first set of simulations, we demonstrate the circularity tracking ability of the proposed algorithm on a synthetically generated signal that was constructed by concatenating three segments of zero-mean white Gaussian signals, $z_{i,k}$, with different properties, where

$$z_{i,k} = x_{i,k} + jy_{i,k} \qquad z_i \sim \mathcal{N}(0, c, p_i), \qquad i = \{1, 2, 3\} \quad (22)$$

These segments had the same covariance, $c = 1$, but different pseudocovariances, p_i, and thus different degrees of circularity, $|\rho_i|$ (see Table I).

Sample, k	$1-1000$	$1001-2000$	$2001-3000$
p_i	$0.8j$	$0.6+0.4j$	0

TABLE I: Pseudocovariances, p_i, of the Gaussian signals.

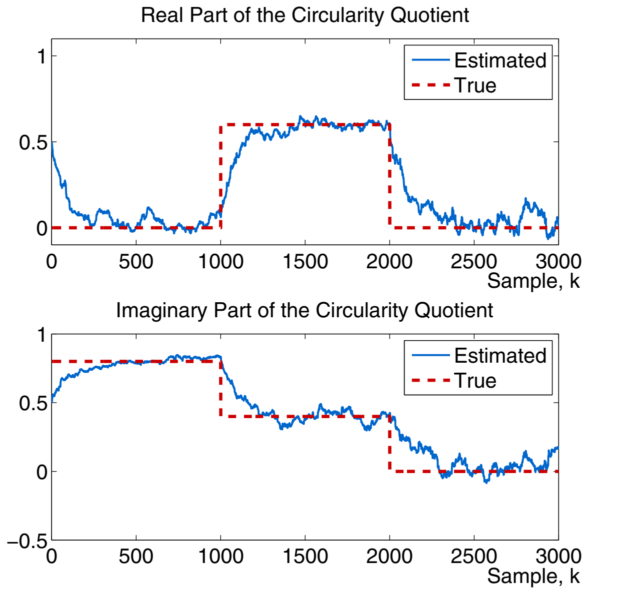

Fig. 1: The real and imaginary parts of the evolution of the CLMS weights when tracking circularity.

Figure 1 shows the evolution of the weight estimates within the proposed algorithm. Observe that the algorithm was able to converge to the accurate circularity quotients within 300 samples.

Next, we evaluated our algorithm on complex wind data which was sampled at 50 Hz and measured as a bivariate signal of wind speeds in the East-West and North-South directions, denoted respectively by s_{E} and s_{N}. The complex wind representation is therefore given by $s = s_{\mathrm{E}} + js_{\mathrm{N}}$ [13].

We considered three wind regimes of different dynamics: the "low", "medium" and "high" regimes. Figure 2 shows the circularity diagram of empirical distributions of the real and imaginary parts of the wind signal for these regimes, where 98.8% (2.5 standard deviations) of the samples are contained within the ellipses. The circularity diagram suggests that the higher the wind speeds, the higher the degree of non-circularity. This is physically meaningful, as high winds are highly directional, with dominant power in a narrow direction, resulting in a greater degree of non-circularity.

Figure 3 shows the estimates of the circularity coefficient, $|\rho|$, under low, medium and high wind regimes. The circularity tracker verifies the observation that the higher the wind speed, the greater its degree of non-circularity. Moreover, the advantage of the proposed algorithm compared to existing block based algorithms is obvious – it is able to track the degree of impropriety in a non-stationary environment.

Multi-channel data. Consider a multiple input single output (MISO) channel with two transmitters and one receiver. The transmitters transmit the same binary sequence

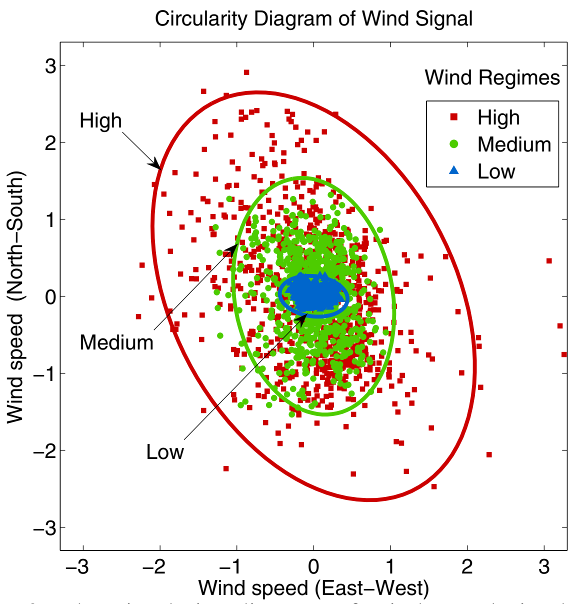

Fig. 2: The circularity diagram of wind speeds in the low, medium and high dynamic regimes.

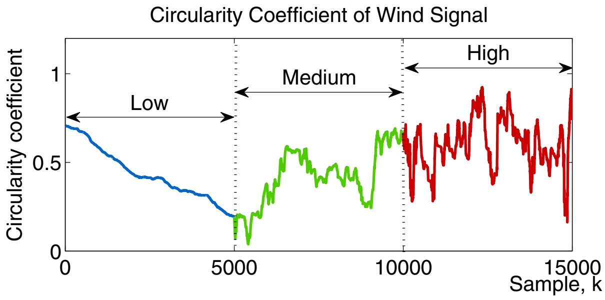

Fig. 3: The estimate of circularity coefficient for wind signals in the low, medium and high dynamic regimes using the proposed algorithm.

$b_k \in \{\pm 1\}$ and the channel gain from a transmitter ℓ to the receiver is given by $h_{\ell,k} = |h_{\ell,k}|e^{j\phi_{\ell,k}}$ for $\ell \in \{1, 2\}$. The received signal r_k is corrupted by zero mean circular white Gaussian noise η_k, and is given by

$$r_k = b_k|h_{1,k}|e^{j\phi_{1,k}} + b_k|h_{2,k}|e^{j\phi_{2,k}} + \eta_k \tag{23}$$

The objective is to identify whether the received signal r_k is circular or non-circular, which in turn can be used deduce if any phase information can be extracted from r_k [4]. To assess the performance of the circularity tracker in detecting a change in the rotation invariance property of the signal, we constructed two rotationally invariant channels ($\phi_{1,k}$ and $\phi_{2,k}$ are drawn from a uniform distribution $\mathcal{U}(0, 2\pi)$) for samples $k = 1, \ldots, 1500$. For samples $k = 1501, \ldots, 3000$, while $\phi_{1,k} \in \mathcal{U}(0, 2\pi)$, the phase of channel two, $\phi_{2,k}$, was drawn from a zero mean Gaussian distribution with variance $\sigma_\phi^2 = 0.04$, thus making the channel (and the received signal) non-circular. Figure 4 shows that the proposed circularity tracker indeed converges to a new steady state value from time instant 1501, thus correctly indicating a change in the properties of the channel. For this application, only the change

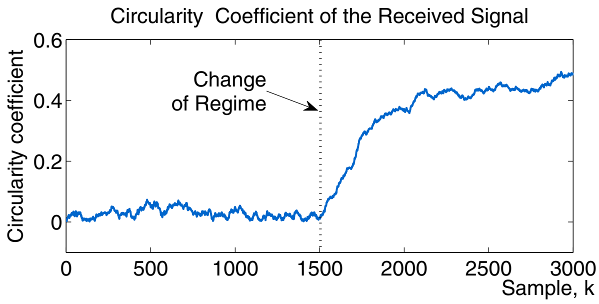

Fig. 4: The circularity coefficient estimate of the received statistic indicates a change in the channel properties.

point and not the actual values of the circularity coefficient is of interest.

V. Conclusion

A novel algorithm for the estimation of the degree of circularity of a complex-valued signal has been proposed. This has been achieved by tracking the evolution of the CLMS weight that estimates the complex conjugate of a signal from the original signal itself. The conditions for the stability in the mean and mean square have been derived, and their relationship with the degree of circularity has been established. The proposed circularity estimator has been verified on complex-valued wind data, synthetically generated Gaussian data, and a multi-channel communication example.

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