Applied Mathematics Letters 24 (2011) 1806–1809

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On the unitary diagonalisation of a special class of quaternion matrices

Clive Cheong Took [a,*], Danilo P. Mandic [a], Fuzhen Zhang [b]

[a] Department of Electrical and Electronic Engineering, Imperial College, London SW7 2AZ, United Kingdom
[b] Nova Southeastern University, 3301 College Ave, Davie Fort Lauderdale, FL 33314, USA

ARTICLE INFO

Article history:
Received 9 December 2010
Received in revised form 26 April 2011
Accepted 27 April 2011

Keywords:
Takagi factorisation
Quaternion involutions
Unitary diagonalisation
Augmented quaternion statistics

ABSTRACT

We propose a unitary diagonalisation of a special class of quaternion matrices, the so-called η-Hermitian matrices $\mathbf{A} = \mathbf{A}^{\eta H}$, $\eta \in \{\imath, \jmath, \kappa\}$ arising in widely linear modelling. In 1915, Autonne exploited the symmetric structure of a matrix $\mathbf{A} = \mathbf{A}^T$ to propose its corresponding factorisation (also known as the Takagi factorisation) in the complex domain $\mathbb{C}$. Similarly, we address the factorisation of an 'augmented' class of quaternion matrices, by taking advantage of their structures unique to the quaternion domain $\mathbb{H}$. Applications of such unitary diagonalisation include independent component analysis and convergence analysis in statistical signal processing.

1. Introduction

Quaternions have found a number of practical applications [1–3] and have been considered in various areas of Applied Mathematics [4–6], yet theoretical advances, especially in the linear algebra community [1], have been slow, mostly due to problems related to the noncommutativity of quaternion multiplication ($ab \neq ba$). An important example of this ambiguity is the spectrum of quaternions: the left (λ_l) and the right (λ_r) eigenvalues of a quaternion matrix $\mathbf{A}$ have to be treated independently, that is,

$$\text{Left spectrum:} \quad \mathbf{Ax} = \lambda_l \mathbf{x} \tag{1}$$

$$\text{Right spectrum:} \quad \mathbf{Ax} = \mathbf{x}\lambda_r. \tag{2}$$

The left eigenvalues are still a subject of ongoing research, whereas the right eigenvalues are well understood [1], and are the only eigenvalues considered in this work. For understanding theoretical performance bounds and the stability of quaternion-valued algorithms in practical applications, further studies on the eigenvalue decomposition of quaternion matrices are a prerequisite. To this end, we propose a novel set of factorisations for a recently introduced special class of 'augmented' quaternion matrices [3]; their symmetry structures are unique to the quaternion domain and cannot be treated using conventional notions borrowed from the complex domain, such as symmetric or Hermitian structures, as is often the case in the existing literature [1,7,8]. Prior to their formulation, we illustrate the need for novel factorisations by revisiting the Autonne (also known as Takagi) factorisation for complex symmetric matrices [9, p. 204]:

$$\mathbf{A} = \mathbf{A}^T = \mathbf{USU}^T \tag{3}$$

which is a special case of the singular value decomposition (SVD), that is, $\mathbf{A} = \mathbf{USV}^H$ when $\mathbf{U} = \mathbf{V}^*$. In this work, symbols $(\cdot)^*$, $(\cdot)^T$, and $(\cdot)^H$ denote respectively the conjugate, transpose and conjugate transpose operations.

* Corresponding author.
E-mail addresses: c.cheong-took@ic.ac.uk (C. Cheong Took), d.mandic@ic.ac.uk (D.P. Mandic), zhang@nova.edu (F. Zhang).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.04.038

The quaternion domain offers more degrees of freedom than the complex domain, and apart from the standard symmetric and Hermitian structures in $\mathbb{H}$, the set of real quaternions, there also exists a class of quaternion matrices which we refer to as the $\imath$-Hermitian $\mathbf{A} = \mathbf{A}^{\imath H}$, the $\jmath$-Hermitian $\mathbf{A} = \mathbf{A}^{\jmath H}$, and the κ-Hermitian $\mathbf{A} = \mathbf{A}^{\kappa H}$, which arise in widely linear modelling [3] due to the following quaternion involutions[1] of a quaternion $q = q_a + \imath q_b + \jmath q_c + \kappa q_d$, defined as [10]

$$\begin{aligned} q^{\imath} &= -\imath q \imath = q_a + \imath q_b - \jmath q_c - \kappa q_d \\ q^{\jmath} &= -\jmath q \jmath = q_a - \imath q_b + \jmath q_c - \kappa q_d \\ q^{\kappa} &= -\kappa q \kappa = q_a - \imath q_b - \jmath q_c + \kappa q_d. \end{aligned} \tag{4}$$

Useful properties of these involutions can be found in the Appendix (where $\mathbf{A}^{\eta} = -\eta A \eta$). In the context of matrices, an η-Hermitian matrix $\mathbf{A} = \mathbf{A}^{\eta H}$ can be defined as $\mathbf{A} = -\eta(\mathbf{A}^{H})\eta,\ \eta \in \{\imath, \jmath, \kappa\}$. To the best of our knowledge, the unitary diagonalisation of this new class of quaternion matrices has not yet been considered and is addressed in this work, by demonstrating that they admit the factorisation of the form

$$\mathbf{A} = \mathbf{A}^{\eta H} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\eta H} \quad \eta \in \{\imath, \jmath, \kappa\}$$

where $\mathbf{U}$ and $\boldsymbol{\Lambda}$ are respectively a unitary and a diagonal matrix. Its existence is implicitly implied in Lemma 8 on p. 404 in [11]; however, it does not state explicitly what the quaternion sesquilinear forms[2]should be. Prior to the proof, as a motivation we next illustrate an application in statistics.

1.1. Application in statistics: diagonalisation of complementary covariance matrices

For a given quaternion-valued vector $\mathbf{x}$, the so-called complementary covariance matrices $\mathcal{C}_{\mathbf{x}}^{\eta}$ can be concisely expressed as (for more detail, see [3])

$$\mathcal{C}_{\mathbf{x}}^{\eta} = E\{\mathbf{x}\mathbf{x}^{\eta H}\} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\eta H} \quad \forall\, \eta \in \{\imath, \jmath, \kappa\} \tag{5}$$

and are η-Hermitian. We can then apply a unitary transform $\mathbf{y} = \mathbf{U}^{H}\mathbf{x}$ to diagonalise the covariance matrix given by

$$\mathcal{C}_{\mathbf{y}}^{\eta} = E\{\mathbf{y}\mathbf{y}^{\eta H}\} = E\{\mathbf{U}^{H}\mathbf{x}\mathbf{x}^{\eta H}\mathbf{U}^{\eta}\} = \mathbf{U}^{H}(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\eta H})\mathbf{U}^{\eta} = \boldsymbol{\Lambda}. \tag{6}$$

Potential applications of the proposed diagonalisation in $\mathbb{H}$ include independent component analysis [12] and convergence analysis in statistical signal processing [13].

2. The main results

Prior to giving the main results, we first give an important lemma which is an essential tool for subsequent analysis. The proof of the lemma can be straightforwardly obtained as in [14, p. 411].

Lemma 2.1 (*which does not assume that A is η-Hermitian*) *states that*

Let $\mathbf{A}$ be an n-square quaternion matrix. There exists a unitary matrix $\mathbf{X}$, a diagonal matrix $\boldsymbol{\Delta}$ with non-negative entries, and a matrix $\mathbf{Y}$ with orthonormal rows such that $\mathbf{A} = \mathbf{X}\boldsymbol{\Delta}\mathbf{Y}$. The columns of the matrix $\mathbf{X}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{H}$. If $\mathbf{A}\mathbf{A}^{H}$ has distinct eigenvalues, then $\mathbf{X}$ is determined up to a right diagonal factor[3] $\mathbf{D} = \mathrm{diag}(\mathrm{e}^{i_{r,1}\theta_1}, \ldots, \mathrm{e}^{i_{r,n}\theta_n})$ with all $\theta_i \in \mathbb{R}$ and $|d_i| = |\mathrm{e}^{i_{r,i}\theta_i}| = 1$; that is, if $\mathbf{A} = \mathbf{X}_1\Delta\mathbf{Y}_1 = \mathbf{X}_2\Delta\mathbf{Y}_2$, then $\mathbf{X}_2 = \mathbf{X}_1\mathbf{D}$.

Proposition 2.1. *Let $\mathbf{A}$ be an n-square quaternion-valued η-Hermitian matrix, i.e., $\mathbf{A} = \mathbf{A}^{\eta H}$, where $\eta \in \{\imath, \jmath, \kappa\}$. If $\mathbf{A}$ has distinct singular values, then $\mathbf{A}$ admits the following factorisations:*

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\eta H} \tag{7}$$

$$= \mathbf{V}^{\eta}\boldsymbol{\Sigma}\mathbf{V}^{H} \tag{8}$$

$$= \mathbf{Q}\mathbf{S}\mathbf{Q}^{\eta H} \tag{9}$$

where the matrices $\mathbf{U}, \mathbf{S}$ and $\mathbf{V}$ are obtained from the singular value decomposition (SVD) of $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{H}$, while $\boldsymbol{\Lambda}$ and $\boldsymbol{\Sigma}$ are quaternion-valued diagonal matrices whose diagonal elements have vanishing η-imaginary parts, and $\mathbf{Q}$ is a unitary matrix.

---

1 These involutions can be used to extract the four components of a quaternion $q = q_a + \imath q_b + \jmath q_c + \kappa q_d$ as

$$q_a = \frac{1}{2}(q + q^{*}) \qquad q_b = \frac{1}{2\imath}(q - q^{\imath *}) \qquad q_c = \frac{1}{2\jmath}(q - q^{\jmath *}) \qquad q_d = \frac{1}{2\kappa}(q - q^{\kappa *}).$$

2 The sesquilinear forms for obtaining the unitary diagonalisation are $\phi(x, y) = x^{H}y$ and $\psi(x, y) = x^{\eta H}Ay$.

3 Where the imaginary unit $i_{r,i} = (\imath q_{b,i} + \jmath q_{c,i} + \kappa q_{d,i})/\sqrt{q_{b,i}^2 + q_{c,i}^2 + q_{d,i}^2}$.

Proof. Based on the SVD of $\mathbf{A} = \mathbf{USV}^H$, the matrix product $\mathbf{AA}^H$ can be expressed as

$$\begin{aligned}\mathbf{AA}^H &= (\mathbf{USV}^H)(\mathbf{USV}^H)^H \\ &= \mathbf{US}^2\mathbf{U}^H.\end{aligned} \tag{10}$$

Using the η-Hermitian property, $\mathbf{A} = \mathbf{A}^{\eta H}$, the matrix product $\mathbf{AA}^H$ can also be written as

$$\begin{aligned}\mathbf{AA}^H &= \mathbf{A}^{\eta H}\mathbf{A}^{\eta} = (\mathbf{USV}^H)^{\eta H}(\mathbf{USV}^H)^{\eta} \\ &= (\mathbf{V}^{\eta}\mathbf{SU}^{\eta H})(\mathbf{U}^{\eta}\mathbf{SV}^{\eta H}) = \mathbf{V}^{\eta}\mathbf{S}^2\mathbf{V}^{\eta H}.\end{aligned} \tag{11}$$

According to Lemma 2.1, the unitary matrices $\mathbf{U}$ and $\mathbf{V}^{\eta}$ in (10) and (11) are related by a diagonal matrix $\mathbf{D}$ as

$$\mathbf{U} = \mathbf{V}^{\eta}\mathbf{D}. \tag{12}$$

To derive the matrix factorisation in (7), we use the result $\mathbf{U} = \mathbf{V}^{\eta}\mathbf{D}$ as follows:

$$\begin{aligned}\mathbf{A} &= \mathbf{USV}^H = (\mathbf{USV}^H)\mathbf{U}^{\eta}\mathbf{U}^{\eta H} \\ &= \mathbf{US}(\mathbf{V}^H\mathbf{VD}^{\eta})\mathbf{U}^{\eta H} = \mathbf{U}(\mathbf{SD}^{\eta})\mathbf{U}^{\eta H} \\ &= \mathbf{U\Lambda U}^{\eta H}\end{aligned}$$

where $\mathbf{\Lambda}$ is the product of the two diagonal matrices $\mathbf{S}$ and $\mathbf{D}^{\eta}$. Noting the unitary property of the diagonal matrix $\mathbf{D}^{-1} = \mathbf{D}^H$ in Lemma 2.1, the factorisation in (8) can be obtained similarly as

$$\begin{aligned}\mathbf{A} &= \mathbf{USV}^H = \mathbf{V}^{\eta}\mathbf{V}^{\eta H}(\mathbf{USV}^H) \\ &= \mathbf{V}^{\eta}(\mathbf{DU}^H\mathbf{U})\mathbf{SV}^H = \mathbf{V}^{\eta}(\mathbf{DS})\mathbf{V}^H \\ &= \mathbf{V}^{\eta}\mathbf{\Sigma V}^H. \quad \square\end{aligned}$$

Remark 1. Observe that the diagonal matrices in (7) and (8) have the relationship $\mathbf{\Sigma} = \mathbf{\Lambda}^{\eta}$, since $\mathbf{\Sigma} = \mathbf{DS}$ and $\mathbf{\Lambda} = \mathbf{SD}^{\eta} = \mathbf{D}^{\eta}\mathbf{S}$.

Remark 2. The η-imaginary components of the diagonal elements of $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ vanish, since $\mathbf{\Sigma} = \mathbf{\Sigma}^{\eta H}$ and $\mathbf{\Lambda} = \mathbf{\Lambda}^{\eta H}$.

Remark 3. The magnitude of each diagonal element of $\mathbf{\Sigma} = \mathrm{diag}[\sigma_1, \ldots, \sigma_n]$ equals the magnitude of its corresponding diagonal element of $\mathbf{\Lambda} = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$, that is, $|\sigma_i| = |s_i e^{i_{r,i}\theta_i}| = |-\eta s_i e^{i_{r,i}\theta_i}\eta| = |\lambda_i|\ \forall\, i = 1, \ldots, n$, where s_i and $e^{i_{r,i}\theta_i}$ denote respectively the ith element of diagonal matrices $\mathbf{S}$ and $\mathbf{D}$. Furthermore, the magnitudes of these diagonal elements are equal to their corresponding singular values of $\mathbf{A}$.

Remark 4. One benefit of the factorisations in (7) or (8) is that in order to factorise this class of matrices, either the left eigenvectors $\mathbf{U}$ or the right eigenvectors $\mathbf{V}$ need to be determined, and *not both* as in the case of the SVD, thus offering lower computational cost; for computing the SVD, a practical algorithm can be found in [15].

Remark 5. All the results in this work hold not only for $\eta \in \{\imath, \jmath, \kappa\}$, but more generally for any pure imaginary unit quaternion η.

Remark 6. The condition that $\mathbf{A}$ has distinct singular values plays a role in the proof of Proposition 2.1. Similar results are expected to hold for a general square quaternion matrix. Possible approaches to resolving the problem may be by a continuity argument (see, e.g., [14, p. 416]) or by the uniqueness of the SVD (see, e.g., [16, p. 334] or [9, p. 144]) or through consimilarity.

Noting that $\mathbf{D} = \mathbf{D}^{\eta H}$ or $\mathbf{D}^{\eta} = \mathbf{D}^H$ from Remark 2 and its unitary property, the factorisation in (9) can be verified as

$$\begin{aligned}\mathbf{A} &= \mathbf{U\Lambda U}^{\eta H} = \mathbf{U}(\mathbf{SD}^{\eta})\mathbf{U}^{\eta H} \\ &= \mathbf{U}(\mathbf{D}^{\eta})^{1/2}\mathbf{S}(\mathbf{D}^{\eta})^{1/2}\mathbf{U}^{\eta H} = \mathbf{QSQ}^{\eta H}\end{aligned}$$

where the unitary matrix is

$$\mathbf{Q} = \mathbf{U}(\mathbf{D}^{\eta})^{1/2} = \mathbf{U}(\mathbf{V}^H\mathbf{U}^{\eta})^{1/2} = \mathbf{V}^{\eta}\mathbf{D}(\mathbf{D}^{\eta})^{1/2}. \tag{13}$$

Computation. The unitary matrices $\mathbf{U}$ and $\mathbf{V}$ in (7) and (8) can be calculated from the SVD of $\mathbf{A} = \mathbf{USV}^H$, and the unitary matrix $\mathbf{Q}$ in (9) can be obtained from (13). The diagonal matrices in (7)–(9) can be computed respectively as $\mathbf{\Lambda} = \mathbf{U}^H\mathbf{AU}^{\eta}$, $\mathbf{\Sigma} = \mathbf{V}^{\eta H}\mathbf{AV}$, and $\mathbf{S} = \mathbf{Q}^H\mathbf{AQ}^{\eta}$.

Appendix. Properties of quaternion involutions

A.1. Quaternion products

For any two quaternion variables q and p, the properties of the quaternion operations with η are given by [10]

p1. $(p^\eta)^\eta = p$,
p2. $(p^\eta)^* = (p^*)^\eta$,
p3. $(pq)^\eta = p^\eta q^\eta$,
p4. $(pq)^{\eta *} = q^{\eta *} p^{\eta *}$,
p5. $(p^\delta)^\eta = (p^\eta)^\delta = p^\alpha$ for all distinct $\eta, \delta, \alpha \in \{\imath, \jmath, \kappa\}$.

It is crucial for practical applications to consider how these properties can be used in the context of quaternion matrices elementwise.

A.2. Quaternion matrices

For two quaternion matrices $\mathbf{A}$ and $\mathbf{B}$, some useful matrix operations are listed below:

P1. $(\mathbf{A}^*)^T = (\mathbf{A}^T)^*$,
P2. $(\mathbf{A}^*)^\eta = (\mathbf{A}^\eta)^*$,
P3. $(\mathbf{A}^\eta)^T = (\mathbf{A}^T)^\eta$,
P4. $(\mathbf{A}^\eta)^H = (\mathbf{A}^H)^\eta$,
P5. $(\mathbf{A}^\delta)^\eta = (\mathbf{A}^\eta)^\delta = \mathbf{A}^\alpha$ for all distinct $\eta, \delta, \alpha \in \{\imath, \jmath, \kappa\}$,
P6. $(\mathbf{AB})^* \neq \mathbf{A}^*\mathbf{B}^*$,
P7. $(\mathbf{AB})^T \neq \mathbf{B}^T\mathbf{A}^T$,
P8. $(\mathbf{AB})^\eta = \mathbf{A}^\eta\mathbf{B}^\eta$,
P9. $(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H$,
P10. $(\mathbf{AB})^{\eta H} = \mathbf{B}^{\eta H}\mathbf{A}^{\eta H}$.

Observe that P8 can be proved by using property p3 of Appendix A.1 and that $(p+q)^\eta = p^\eta + q^\eta$. Property P10 can be proved by using P4, P8 and P9.

References

[1] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra and its Applications 251 (1997) 21–57.
[2] C. Cheong Took, D.P. Mandic, The quaternion LMS algorithm for adaptive filtering of hypercomplex real world processes, IEEE Transactions on Signal Processing 57 (2009) 1316–1327.
[3] C. Cheong Took, D.P. Mandic, Augmented second-order statistics of quaternion random signals, Signal Processing 91 (2011) 214–224.
[4] M. Özdemir, The roots of a split quaternion, Applied Mathematics Letters 22 (2009) 258–263.
[5] M. Dehghan, M. Hajarian, A lower bound for the product of eigenvalues of solutions to matrix equations, Applied Mathematics Letters 22 (2009) 1786–1788.
[6] S. De Leo, Quaternionic analyticity, Applied Mathematics Letters 16 (2003) 1077–1081.
[7] F. Zhang, Gersgorin type theorem for quaternionic matrices, Linear Algebra and its Applications 424 (2007) 139–153.
[8] L. Autonne, Sur les Matrices Hypohermitiennes et sur les Matrices Unitaires, in: University of Lyon, series I, vol. 38, 1915, pp. 1–77.
[9] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, First ed., Cambridge University Press, Cambridge, 1991.
[10] T. Ell, S.J. Sangwine, Quaternion involutions and anti-involutions, Computers and Mathematics with Applications 53 (2007) 137–143.
[11] S.A. Andersson, H.K. Brons, S.T. Jensen, Distribution of eigenvalues in multivariate statistical analysis, The Annals of Statistics 11 (2) (1983) 392–415.
[12] J. Eriksson, V. Koivunen, Complex random vectors and ICA models: identifiability, uniqueness, and separability, IEEE Transactions on Information Theory 52 (2006) 1017–1029.
[13] S. Douglas, D. Mandic, Performance analysis of the conventional complex LMS and augmented complex LMS algorithms, in: Proceedings of IEEE International conference on Acoustics, Speech and Signal Processing, Dallas USA, March 2010, pp. 3794–3797.
[14] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1990.
[15] S.J. Sangwine, N. Le Bihan, Quaternion singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion Householder transformations, Applied Mathematics and Computation 182 (1) (2006) 727–738.
[16] N.J. Higham, Functions of Matrices: Theory and Computation, SIAM, Philadelphia, 2008.