SOME POTENTIAL PITFALLS WITH S TO Z–PLANE MAPPINGS

Richard Harvey, Danilo P. Mandic

School of Information Systems,
University of East Anglia,
Norwich, UK.
{rwh,mandic}@sys.uea.ac.uk

Djemal H. Kolonic

Department of Electrical Engineering,
University of Banjaluka,
Banjaluka, Bosnia–Herzegovina.
kolónic@etf-bl.rstel.net

ABSTRACT

Design of digital Infinite Impulse Response (IIR) filters is a compulsory topic in most signal processing courses. Most often, it is taught by using the bilinear transform to map an analogue counterpart into the corresponding digital filter. The usual approach is to define a mapping between the complex variables s and z, and hence, by substitution, derive a mapping between ω, analogue frequency, and θ, sampled frequency. This is rather elliptical, since the real aim is to establish the correspondence between the frequency response of a prototype analogue system $H(j\omega)$, and $H(e^{j\theta})$, the response of the sampled system. Here we provide a rigorous analysis for the mutual invertibility between the analogue frequency ω, and the digital frequency θ for this case. Based upon the definition of the tan and arctan functions, conditions of existence, uniqueness and continuity of such a mutually inverse mapping are derived. Based upon these results, simple proofs for the mutually inverse mappings $\omega \rightarrow \theta$ and $\theta \rightarrow \omega$ are given. This is supported by appropriate diagrams. This problem arose as a student question while teaching DSP.

1. INTRODUCTION

Infinite Impulse Response (IIR) filter synthesis is an important topic in Digital Signal Processing (DSP) education. However, the way of teaching signal processing has always been a matter of discussion, even nowadays. Some authors propose teaching the analog signal processing first [1], while the others suggest teaching DSP first [2]. So far, the most commonly used procedure for teaching synthesis of IIR digital filters has been as follows [3, 4, 5]:

1. Design an analogue filter that matches the specified filter requirements, whose transfer function $H(s)$ in the s–plane is

$$H(s) = \frac{\sum_{i=0}^{P} c_i s^i}{1+\sum_{j=1}^{Q} d_j s^j}, \quad P,Q \in \mathbb{N}, \ c_i, d_i \in \mathbb{R} \tag{1}$$

2. Map the transfer function of an analog filter $H(s)$ from the s–plane into the transfer function $H(z)$ in the z–plane

$$H(z) = \frac{\sum_{i=0}^{M} a_i z^i}{1+\sum_{j=1}^{N} b_j z^j}, \quad M,N \in \mathbb{N}, \ a_i, b_i \in \mathbb{R} \tag{2}$$

   using one of the s–z transforms [6, 5, 7].

The idea is that in order to preserve stability of the desired digital filter, a mapping procedure from the s into the z plane should satisfy the following conditions

i) The imaginary $\Im(s)$ axis from the s–plane ($s = j\omega$, $-\infty < \omega < +\infty$) is mapped onto the unit circle in the z–plane ($|z| = 1 \Leftrightarrow z = e^{j\theta}, \ -\pi < \theta < \pi$), where ω is the analog frequency and θ is the digital frequency,

ii) The left open half–plane from the s–plane ($\Re(s) < 0$) is mapped into the unit circle in the z–plane ($|z| < 1$).

These two conditions are necessary to allow a stable analog filter to be mapped into a stable digital filter. The most popular s–z mapping that preserves this requirement is the bilinear transform [3, 7, 8], which is defined by [6, 5]

$$s = f(z) = C\frac{1-z^{-1}}{1+z^{-1}} \tag{3}$$

where the constant C is mostly taken as $C = \frac{2}{T}$, and T is the sampling period of the discrete filter. The inverse transform is defined as [5]

$$z = \frac{2+sT}{2-sT} \tag{4}$$

However, s and z are complex variables, and the whole idea behind the synthesis of IIR digital filters from its analogue counterparts is to match the frequency responses $H(j\omega)$ and $H(e^{j\theta})$. In addition, although it is easy to derive the mutually inverse relationships for the complex variables s and z,

0-7803-6293-4/00/$10.00©2000 IEEE

it is not trivial to obtain these inverse relationships in the frequency domain, and this issue is often neglected when teaching the topic.

Here, for rigor we obtain these $\omega \to \theta$ and $\theta \to \omega$ relationships, based upon the existence, uniqueness, and continuity of such a mapping, and identify an easy trap when deriving these relationships. As visualisation is an important part of presenting information to students [9], this is supported by appropriate graphical representation of the problem.

2. FREQUENCY DOMAIN RELATIONSHIPS FOR THE BILINEAR TRANSFORM

Consider first the frequency response of a digital filter in the z plane. With $s = j\omega$, (4) becomes [5]

$$\begin{aligned} z &= \frac{2+j\omega T}{2-j\omega T} = \frac{(2+j\omega T)^2}{4+(\omega T)^2} = \frac{4-(\omega T)^2+4j\omega T}{4+(\omega T)^2} \\ &= \frac{\sqrt{[4-(\omega T)^2]^2+(\omega T)^2}}{4+(\omega T)^2} e^{j\arctan\left[\frac{4\omega T}{4-(\omega T)^2}\right]} \\ &= e^{j\arctan\left[\frac{4\omega T}{4-(\omega T)^2}\right]} = e^{j\theta(\omega)} \end{aligned} \tag{5}$$

where

$$\theta(\omega) = \arctan\left[\frac{4\omega T}{4-(\omega T)^2}\right] \tag{6}$$

This means that the second condition required for a stable s–z mapping is indeed satisfied.

From (6), it is clear that the relationship between the frequency mappings $\omega \to \theta$ is nonlinear. To shed further light onto this relationship, it is necessary to find an inverse function to (6). It can be done from (6), but in the literature, it is always undertaken by starting from (3)

$$s = j\omega = \frac{2}{T}\frac{1-e^{-j\theta}}{1+e^{-j\theta}} = \frac{2}{T}\frac{e^{j\frac{\theta}{2}}-e^{-j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}}+e^{-j\frac{\theta}{2}}} = \frac{2}{T}\frac{j\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \tag{7}$$

which yields

$$\omega = \frac{2}{T}\tan\frac{\theta}{2} \tag{8}$$

From (6) and (8), it cannot be seen that $\omega = \omega(\theta)$ and $\theta = \theta(\omega)$ are indeed inverse functions.

Let us therefore prove that the frequency domain bilinear transforms as given by (6) and (8) are inverse functions.

2.1. The $\theta \to \omega$ Relationship

There are a number of ways to prove the mutually inverse relationship between ω and θ due to the bilinear transform. In this section we provide a rather cumbersome one, but which serves as an example of an easy trap when deriving equivalence of trigonometry–based relationships. Expressing θ from (8), we have

$$\theta = 2\arctan\frac{\omega T}{2} \tag{9}$$

which is supposed to be identical to $\theta(\omega) = \arctan\frac{4\omega T}{4-(\omega T)^2}$ (6). In order to prove this, recall the trigonometric identity [10]

$$\arctan a + \arctan b = \arctan\frac{a+b}{1-ab} \tag{10}$$

Replacing now $\arctan a$ by x and $\arctan b$ by y, and using the identity (10), we obtain

$$\begin{aligned} &\arctan a + \arctan b = \arctan(\tan(x+y)) \\ = \; &\arctan\frac{\tan x+\tan y}{1-\tan x\tan y} = \arctan\frac{a+b}{1-ab} \end{aligned} \tag{11}$$

Now, letting $a = b = \frac{\omega T}{2}$ yields

$$\theta = \arctan\frac{2\frac{\omega}{T}}{1-\left(\frac{\omega T}{2}\right)^2} = \arctan\frac{\omega T}{4-(\omega T)^2} \tag{12}$$

which proves that the analogue frequency ω and digital frequency θ given respectively by (8) and (6), are mutually inverse functions when using the bilinear transform.

2.2. Graphical representation

However, the analysis is not as simple and straightforward as it might seem from above. An insight into function (6) shows that this function has critical points at $\omega = \pm\frac{2}{T}$. This implies that the tan function changes it sign at these points, which in turn affects the existence, uniqueness, and continuity of the relationships between the analogue frequency ω and the digital frequency θ. Figure 1 shows the graphs of functions (6) and (9), which are supposed to be identical. As seen from the Figure, the curves are identical for $-2 < \omega T < 2$, whereas for $\omega T < -2$ and $\omega T > 2$, the difference between the amplitudes of the function shown in Figure 1 is $\pm\pi$. The curve given by (9) represents the shape that we actually want to obtain, whereas the curve (6) is not continuous, and does not follow the desired shape. As the area of interest is the whole analogue frequency axis $-\infty < \omega < \infty$, a further light has to be shed on the relationships between ω and θ given by the bilinear transform.

2.3. Why is it like this?

We now provide a rigorous analysis of the underlying trigonometric problem. Let is recall a definition of the function tan [10].

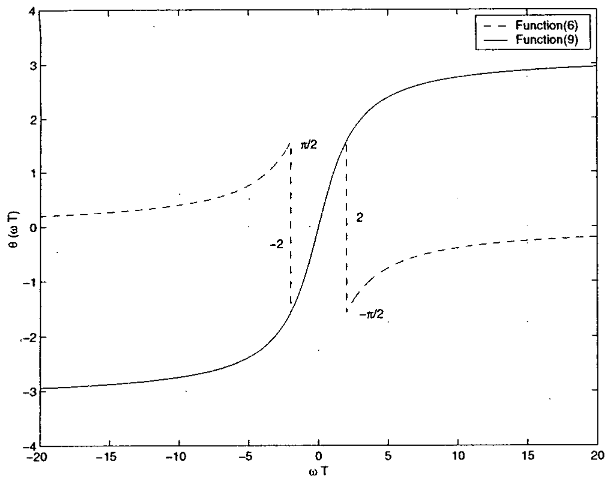

Figure 1: The curves from equations (6) and (8)

Definition 1 *Function* $\arctan : \mathbb{R} \to \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$ *is an inverse restriction of the function* $\tan$ *on the interval* $\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$, *i.e. an inverse restriction of the function* $\tan : \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \to \mathbb{R}$

Let us now consider the compositions of functions tan and arctan. As the composition of two mutually inverse function is an identity, we have

$$\begin{aligned} \tan \arctan x &= x, \ x \in \mathbb{R} \\ & \textit{and} \\ \arctan \tan x &= x, \ x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \end{aligned} \tag{13}$$

It is important to notice that

$$\arctan \tan x \neq x, \ \ for \ \ x \notin \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \tag{14}$$

However, since $\tan(x - k\pi) = \tan x, \ k \in \mathbb{Z}$, we have $\arctan \tan x = \arctan \tan(x - k\pi)$. If we chose k such that $x - k\pi \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$, then

$$\arctan \tan x = \arctan \tan(x - k\pi) = x - k\pi \tag{15}$$

which is well known from complex analysis. So, we have

$$\begin{aligned} \tan \arctan x &= x \\ \arctan \tan x &= x - k\pi, \ k = \left[\frac{x}{\pi} + \frac{1}{2} \right] \end{aligned} \tag{16}$$

which provides the *existence* of the desired inverse mapping.
Next recall that $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$ for $\cos \alpha \neq 0$. At this point it is important to notice that if we take $\alpha = \arctan \frac{\omega T}{2}$, then $\cos \alpha \neq 0$, since $\alpha \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$.

Now, from (8), (6), and the aforementioned discussion, we have

$$\begin{aligned} \tan \theta &= \tan 2 \arctan \tan \frac{\omega T}{2} = \tan 2\alpha \\ &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2 \tan \arctan \frac{\omega T}{2}}{1 - \left(\tan \arctan \frac{\omega T}{2} \right)^2} \\ &= \frac{2 \frac{\omega T}{2}}{1 - \left(\frac{\omega T}{2} \right)^2} = \frac{4 \omega T}{4 - (\omega T)^2} \end{aligned} \tag{17}$$

Now, as we have $\tan \theta = \frac{4\omega T}{4 - (\omega T)^2}$, it follows that

$$\begin{aligned} \arctan \frac{4\omega T}{4 - (\omega T)^2} &= \arctan \tan \theta \\ = \theta - k\pi = 2 \arctan \frac{\omega T}{2} - k\pi, & \quad k = \left[\frac{\theta}{\pi} + \frac{1}{2} \right] \end{aligned} \tag{18}$$

Finally

$$\begin{aligned} \arctan \frac{4\omega T}{4 - (\omega T)^2} &= 2 \arctan \frac{\omega T}{2} \\ &+ \begin{cases} -\pi, & \theta \in \langle \frac{\pi}{2}, \pi \rangle \\ 0, & \theta \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle \\ \pi, & \theta \in \langle -\pi, -\frac{\pi}{2} \rangle \end{cases} \end{aligned} \tag{19}$$

or, in terms of ω

$$\begin{aligned} \arctan \frac{4\omega T}{4 - (\omega T)^2} &= 2 \arctan \frac{\omega T}{2} \\ &+ \begin{cases} -\pi, & \omega > \frac{2}{T} \\ 0, & -\frac{2}{T} \leq \omega \leq \frac{2}{T} \\ \pi, & \omega < -\frac{2}{T} \end{cases} \end{aligned} \tag{20}$$

which provides the *uniqueness* of the solution.
This, due to the periodicity of the function tan, with the period π proves that

$$\omega = \frac{2}{T} \tan \frac{\theta}{2} \tag{21}$$

which provides the *continuity* of the inverse mapping, as desired. This analysis is graphically supported in Figure 2, for $\theta = \theta(\omega)$. As previously shown, due to the "bins" of the tan function, function θ given in (7) "borrows" the portion of the curve from one period (bin) of the tan function ahead for $\omega < -\frac{2}{T}$ and from one period before for $\omega > \frac{2}{T}$. This actually makes the whole mapping continuous when periodically going around the unit circle in the z plane. In other words it accounts for the periodicity of the θ function on the unit circle in the $\mathcal{Z}$ plane.

2.4. More Simple Proofs

Now, when we know the conditions of existence, uniqueness and continuity of the underlying problem, let us pro-

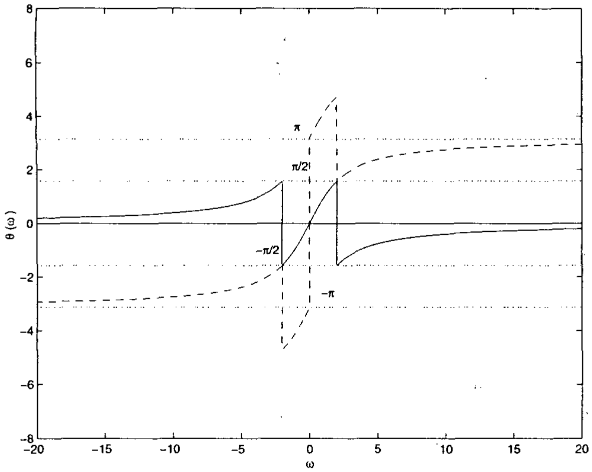

Figure 2: The exact curves from equations (6) and (8)

vide sketch of another couple of simple proofs of the mutually inverse relationship between ω and θ when the mapping from s into the z plane is undertaken using the bilinear transform.

- We start from the relationship $\tan\theta = \frac{4\omega T}{4-(\omega T)^2}$. Solving this relation for ω, we obtain

$$(\tan\theta)(\omega T)^2 + 4(\omega T) - 4\tan\theta = 0$$
$$(\omega T)_{1,2} = \frac{-4 \pm \sqrt{16 + 16\tan^2\theta}}{2\tan\theta} \quad (22)$$

  After simple trigonometric manipulation, we obtain $\omega T = 2\tan\frac{\theta}{2}$, which is the desired result.

- Start from $\tan\theta = \frac{4\omega T}{4-(\omega T)^2} = \frac{\omega T}{1-(\frac{\omega T}{2})^2}$. Now, recall that $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$, and rearrange the relation between θ and ω as

$$\tan 2\frac{\theta}{2} = \frac{\frac{\omega T}{2} + \frac{\omega T}{2}}{1 - \frac{\omega T}{2}\frac{\omega T}{2}} \quad (23)$$

From (23), we directly obtain the desired relationship $\theta = 2\arctan\frac{\omega T}{2}$ which is the inverse of $\omega = \frac{2}{T}\tan\frac{\theta}{2}$.

3. SUMMARY

An analysis of the transformation pairs in the frequency domain for Infinite Impulse Response (IIR) digital filters realized via the bilinear mapping is provided. The standard approach to this problem establishes the mutually inverse mapping in the domain of the complex variables s and z. Due to the need to match the frequency responses of the corresponding analogue and digital filters, we have derived these relationships starting from the analogue and digital frequencies ω and θ. Although the task initially seems to be an easy one, there are traps along the way due to the underlying transcendental functions. Hence, for rigor, the existence, uniqueness and continuity conditions of the solution are provided, as well as the proofs for the mutually inverse mappings between ω and θ. This is supported graphically. The motivation for this analysis was a student question while teaching the topic at our schools.

4. REFERENCES

[1] D. C. Munson and D. L. Jones, "Analog signal processing first," *IEEE Signal Processing Magazine*, vol. 16, no. 5, pp. 24–39, 1999.

[2] J. H. McClellan and R. W. Schafer, "Digital signal processing first," *IEEE Signal Processing Magazine*, vol. 16, no. 5, pp. 29–34, 1999.

[3] A. V. Oppenheim, J. R. Buck, and R. W. Schafer, *Discrete–time signal processing*. Prentice Hall, 1999.

[4] R. W. Hamming, *Digital Filters*. Prentice Hall, third ed., 1989.

[5] H. Y.-F. Lam, *Analog and digital filters: design and realization*. Englewood Cliffs: Prentice-Hall, 1979.

[6] S. Y. Hwang, "Realisation of canonical digital networks," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP–22, pp. 27–38, 1974.

[7] C. S. Williams, *Designing digital filters*. Englewood Cliffs London: Prentice-Hall, 1986.

[8] A. V. Oppenheim and A. S. Willsky, *Signals and Systems*. Prentice Hall, second ed., 1997.

[9] A. V. Oppenheim, "A personal view of education," *IEEE Signal Processing Magazine*, vol. 9, no. 2, pp. 69–72, 1992.

[10] C. B. Douthitt and J. A. McMillian, *Trigonometry*. McGraw–Hill, 1978.