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# A Data Analytics Perspective of Power Grid Analysis—Part 1: The Clarke and Related Transforms

Affordable, reliable, and readily accessible electric power underpins our modern society in a multitude of ways, and in this context, the smart grid is becoming an increasingly important factor in power generation, transmission, and distribution. Current analytical tools for the planning, operation, and circuit design in power systems derive from the antecedent technology area of circuit theory, which is both nonobvious for modern data analysts and assumes balanced conditions and a steady state, even though future power networks will routinely experience transient and steady-state unbalances. Next-generation analytical tools should therefore be fully equipped for dynamically unbalanced systems to approach the physical limits of power networks; data analytics is both well suited and necessary for this endeavor but is nonobvious for power engineers. Hence, to fully exploit their evident and promised advantages, an analysis of the smart grid requires close collaboration and convergence between power engineers and experts in signal processing and machine learning, whereby analytical tools expressed in a common language would be a natural step forward.

To this end, we revisit the Clarke and related transforms from subspace (see "Tribute to Edith Clarke, a Pioneer of Power Grid Analysis") latent component and spatial-frequency perspectives to establish fundamental relationships between the standard three-phase transforms and modern data analytics. We show that the Clarke transform can be physically interpreted as a "spatial-dimensionality reduction" technique, which is equivalent to principal component analysis (PCA) for balanced systems but is suboptimal for dynamically unbalanced smart grids, while the related Park transform performs further "temporal" dimensionality reduction.

Such a perspective opens up numerous new avenues for the use of signal processing and machine learning in

## Tribute to Edith Clarke, a Pioneer of Power Grid Analysis

Edith Clarke (1883–1959) was a true pioneer in the application of circuit theory and mathematical techniques to electrical power systems. In 1919, she became the first woman to obtain an M.S. degree in electrical engineering from the Massachusetts Institute of Technology and the first female professor of electrical engineering in the United States, having been appointed at the University of Texas at Austin in 1947. Her pivotal contributions were concerned with the development of algorithms for the simplification of the laborious computations involved in the design and operation of electrical power systems [1]. One of her early inventions was the Clarke calculator in 1921, a graphical device that solved power system equations 10 times faster than a "human computer" [S1]. In 1943, she introduced the Clarke transform, also known as the *$\alpha\beta$ transform*, which has since been established as a fundamental and indispensable tool for the analysis of three-phase power systems. Recent progress toward the smart grid has only reinforced the importance of the Clarke transform as a platform for the wider involvement of signal processing and machine learning in numerous applications related to state estimation, frequency tracking, and fault detection [S2], [2], the most important data analytics aspects in the development of the future smart grid.

**References**

[S1] E. Clarke, "Calculator," U.S. Patent 1552113, Sept. 1925.
[S2] M. Canteli, A. Fernandez, L. Eguiluz, and C. Estebanez, "Three-phase adaptive frequency measurement based on Clarke's transformation," *IEEE Trans. Power Del.*, vol. 21, no. 3, pp. 1101–1105, 2006.

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power grid research and paves the way for innovative optimization, transformation, and analysis techniques that are not accessible from the standard circuit theory principles. We expect to demonstrate this in part 2 of this "Lecture Notes" article, which will be published in a future issue of *IEEE Signal Processing Magazine.* In addition, the material may be useful for lecture courses in multidisciplinary research, from the smart grid to big data, or as interesting reading for the intellectually curious and generally knowledgeable reader. Teaching and supplementary material can be found at http://www.commsp.ee.ic.ac.uk/~mandic/DSP_ML_for_Power.htm.

## Relevance

There is substantial interest in transforming the way we both produce and use energy as current ways are not sustainable. For the electrical power grid, this involves fundamental paradigm shifts as we build a smart grid, adopt more renewable energy sources, and promote more energy-efficient practices [3]. A smart grid delivers electricity from suppliers to users using digital technology and has a number of properties, including various forms of energy generation and storage, reliance on sensor information, active participation by end users, security and reliability, and the use of optimization and control to make decisions (see the Energy Independence and Security Act 2007, Section 1304 [8]). This will require fundamental shifts in the way we analyze and design power systems, together with the prominent involvement of modern data analytics disciplines that are currently outside the standard power systems operation, such as those enabled by signal processing and machine learning.

Although we have just begun to investigate a whole host of signal processing issues for the smart grid strategy, these new technologies will undoubtedly be critical to the efficient use of limited and intermittent power resources in the future. The first fundamental step in this direction is to bridge the gap between the power systems and data analytics communities by establishing a common language for the understanding and interpretation of system behavior, the ultimate goal of this perspective.

Regarding the opportunities for data analytics research, three-phase systems can be inherently difficult to analyze because the electrical quantities involved are coupled by design, while also exhibiting redundancies. Early in her career as a "human computer" with General Electric, Edith Clarke routinely faced problems related to the simplification of analyses of three-phase circuits. Fast-forward a century, and three-phase systems pose another class of practical problems, essentially of a signal processing and machine-learning nature, including the following.

1) In smart grids, the effects arising from the on–off switching of various subgrids and the dual roles of generators/loads will produce transients and spurious frequency/phasor estimates; the analysis thus requires modern signal processing and machine-learning techniques.
2) Accurate rate of change of frequency trackers are a prerequisite for the operation of smart grid, while rapid frequency trackers are envisaged to be part of many appliances, but their design is beyond the remit of power systems engineering.

All in all, it is critical that the estimation of the frequency/phasor remains accurate during the various interconnections, transients, faults, and voltage sags (IEEE Standard 1159-2009), while at the same time having the intelligence to determine whether the system experienced a one-, two-, or three-phase fault; this "smart frequency" area has been the subject of recent patents [4] and ongoing research [2], [5], [6].

## Prerequisites

Basic knowledge of power system analysis, linear algebra, and PCA is necessary. It is also advantageous if the reader is familiar with complex algebra and the discrete Fourier transform (DFT).

## Problem statement and solution

### Problem statement

We set out to investigate the redundancy within information-bearing signals in three-phase systems to establish a link between the current circuit-theory-inspired dimensionality-reduction techniques and a more general latent-component-analysis viewpoint rooted in data analytics.

### Solution: Essential overview

We first explore the redundancy in three-phase signal representation. To this end, consider a sampled three-phase voltage measurement vector, $\boldsymbol{s}_k$, which at a discrete time instant $k$, is given by

$$\boldsymbol{s}_k = \begin{bmatrix} v_{a,k} \\ v_{b,k} \\ v_{c,k} \end{bmatrix} = \begin{bmatrix} V_a \cos(\omega k + \phi_a) \\ V_b \cos\left(\omega k + \phi_b - \frac{2\pi}{3}\right) \\ V_c \cos\left(\omega k + \phi_c + \frac{2\pi}{3}\right) \end{bmatrix}, \tag{1}$$

where $V_a$, $V_b$, and $V_c$ are the amplitudes of the phase voltages $v_{a,k}$, $v_{b,k}$, and $v_{c,k}$, while $\omega = 2\pi f T$ is the fundamental angular frequency, with $f$ as the fundamental power system frequency, and $T$ as the sampling interval. The phase values for phase voltages are denoted by $\phi_a$, $\phi_b$, and $\phi_c$, respectively.

#### Remark 1

The three-phase power system is considered to be in a balanced condition if

1) the magnitudes of the phase voltages in (1) are equal, that is, $V_a = V_b = V_c$
2) the phase angle separation between the phase voltages is uniform and equal to $2\pi/3$, i.e., $\phi_a = \phi_b = \phi_c$ across the phase voltages.

Early power engineers were able to effectively reduce the dimensionality in representing the three-phase voltage signal in (1) by changing its reference frame (or basis), the so-called voltage transformations. Figure 1 illustrates the effects of the three-phase transformations considered in this article, i.e., the Clarke transform and the closely related Park transform.

#### Remark 2

Figure 1 gives a modern interpretation of the operation of the Clarke and Park transforms, whereby the Clarke transform reduces the 3D "spatial information space" in three-phase power signals to the 2D $\alpha\beta$ space, whereas the Park transform employs a 2D time-varying basis in the form of a rotation matrix to "demodulate" the Clarke transform. The Park bases rotate at the fundamental power system frequency of 50–60 Hz, which further reduces the "temporal information space" to only two constants, $v_d$ and $v_q$.

We now offer a signal processing view of spatial redundancy in three-phase power systems, showing that the voltage signal in (1) is essentially overparameterized, thus paving the way for a data analytics perspective of the Clarke transform. To this end, consider the empirical covariance matrix of the three-phase voltage signal, $\boldsymbol{s}_k$ in (1), defined as $\text{cov}(\boldsymbol{s}_k) \stackrel{\text{def}}{=} \mathbf{R}_s$, which can be computed from $N$-consecutive samples of $\boldsymbol{s}_k$ as

$$\mathbf{R}_s = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \boldsymbol{s}_k \boldsymbol{s}_k^{\mathsf{H}}, \qquad (2)$$

where the symbol $(\cdot)^{\mathsf{H}}$ denotes the Hermitian-transpose operator [note that, in (3)–(8), $\mathbf{s}_k$ is complex valued; hence, for consistency, we use $\mathbf{s}_k^{\mathsf{H}}$ here instead of $\mathbf{s}_k^{\mathsf{T}}$].

From the three-phase voltage, $\boldsymbol{s}_k$ in (1), upon employing the identity $\cos(x) = (e^{jx} + e^{-jx})/2$ we arrive at its complex-valued phasor representation in the form

$$\boldsymbol{s}_k = \frac{1}{2}\begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \bar{V}_c \end{bmatrix} e^{j\omega k} + \frac{1}{2}\begin{bmatrix} \bar{V}_a^* \\ \bar{V}_b^* \\ \bar{V}_c^* \end{bmatrix} e^{-j\omega k}, \qquad (3)$$

where, for compactness, the time-independent phasors, $\bar{V}_a = (V_a/\sqrt{2})e^{j\phi_a}$, $\bar{V}_b = (V_b/\sqrt{2})e^{j(\phi_b - (2\pi/3))}$ and $\bar{V}_c = (V_c/\sqrt{2})e^{j(\phi_c + (2\pi/3))}$ can be comprised into the complex-valued phasor vector

$$\boldsymbol{v} \stackrel{\text{def}}{=} [\bar{V}_a, \ \bar{V}_b, \ \bar{V}_c]^{\mathsf{T}}, \qquad (4)$$

so that the three-phase voltage vector in (3) now becomes

$$\boldsymbol{s}_k = \frac{1}{2}(\boldsymbol{v}e^{j\omega k} + \boldsymbol{v}^* e^{-j\omega k}). \qquad (5)$$

To arrive at the final expression for the empirical covariance matrix, $\mathbf{R}_s$ in (2), observe from (5) that the individual outer products, $\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}}$ in (2), represent an average of four outer products, i.e.,

$$\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}} = \frac{1}{4}(\boldsymbol{v}\boldsymbol{v}^{\mathsf{H}} + \boldsymbol{v}^*\boldsymbol{v}^{\mathsf{T}} + \boldsymbol{v}\boldsymbol{v}^{\mathsf{T}} e^{2j\omega k} + \boldsymbol{v}^*\boldsymbol{v}^{\mathsf{H}} e^{-2j\omega k}). \qquad (6)$$

For $\omega \neq 0$ or $\omega \neq \pi$, and for a large enough $N$, the following holds:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} e^{\pm 2j\omega k} = 0, \qquad (7)$$

so that the last two outer products in (6) vanish and every individual outer product within the covariance matrix for a general three-phase power-voltage measurement becomes

$$\boldsymbol{s}_k\boldsymbol{s}_k^{\mathsf{H}} = \frac{1}{4}(\boldsymbol{v}\boldsymbol{v}^{\mathsf{H}} + \boldsymbol{v}^*\boldsymbol{v}^{\mathsf{T}}) = \frac{1}{2}\mathsf{Re}\{\boldsymbol{v}\boldsymbol{v}^{\mathsf{H}}\} = \frac{1}{2}(\boldsymbol{v}_r\boldsymbol{v}_r^{\mathsf{T}} + \boldsymbol{v}_i\boldsymbol{v}_i^{\mathsf{T}}), \qquad (8)$$

where $\boldsymbol{v}_r = \mathsf{Re}\{\boldsymbol{v}\}$ and $\boldsymbol{v}_i = \mathsf{Im}\{\boldsymbol{v}\}$ denote the real and imaginary part of the phasor vector $\boldsymbol{v}$ in (4).

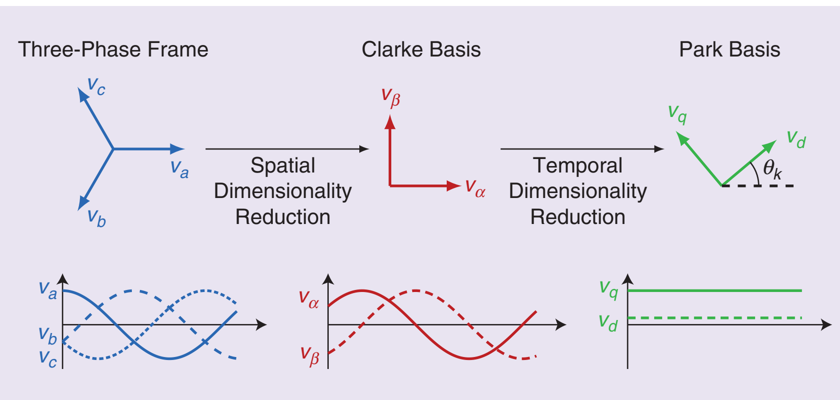


**FIGURE 1.** The geometric interpretation of "spatial" and "temporal" dimensionality reduction provided by the corresponding Clarke and Park transforms.

### Remark 3

Observe from (8) that the $3\times 3$ covariance matrix, $\mathbf{R}_s$ in (2), of the trivariate three-phase-voltage signal $\boldsymbol{s}_k$, is rank-deficient (rank-2), since it represents a sum of two rank-1 outer products, i.e., $\boldsymbol{v}_r\boldsymbol{v}_r^{\mathsf{T}}$ and $\boldsymbol{v}_i\boldsymbol{v}_i^{\mathsf{T}}$. In other words, without loss in information, the three-phase signal in (3) can be projected onto a 2D subspace spanned by $\boldsymbol{v}_r$ and $\boldsymbol{v}_i$. This implies that the use of all three data channels (system phases) is redundant in the analysis and offers a data analytics justification for the Clarke transform.

We next proceed with the formal definition of the Clarke transform and show that its dimensionality-reduction principle admits a PCA interpretation.

## *Clarke transform: A fundamental tool in power system analysis*

The Clarke transform, also known as the *αβ transform*, was introduced from a circuit theory viewpoint. It aims to change the basis of the original vector space where the three-phase voltage signal $\boldsymbol{s}_k$ in (1) resides, to a basis defined by the columns of the so-called Clarke matrix to yield the Clarke-transformed $v_{0,k}$, $v_{\alpha,k}$, and $v_{\beta,k}$ voltages in the form

$$\begin{bmatrix} v_{0,k} \\ v_{\alpha,k} \\ v_{\beta,k} \end{bmatrix} = \underbrace{\sqrt{\frac{2}{3}}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}_{\text{Clarke matrix}} \underbrace{\begin{bmatrix} v_{a,k} \\ v_{b,k} \\ v_{c,k} \end{bmatrix}}_{\boldsymbol{s}_k}. \qquad (9)$$

The quantities $v_{\alpha,k}$ and $v_{\beta,k}$ are referred to as the $\alpha$ and $\beta$ *sequences*, while the term $v_{0,k}$ is called the *zero-sequence* since it is zero when the three-phase signal $\boldsymbol{s}_k$ is balanced (see Remark 1).

### Remark 4

Under nominal conditions, only $v_{\alpha,k}$ and $v_{\beta,k}$ are used in the analysis since balanced phase voltages yield $v_{0,k} = 0$. The "standard" version of the Clarke transform thus employs only the last two rows of the Clarke matrix in (9), to project the three-phase voltage in (1) onto a 2D subspace spanned by its columns, i.e.,

$$\begin{bmatrix} v_{\alpha,k} \\ v_{\beta,k} \end{bmatrix} = \underbrace{\sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}_{\text{reduced Clarke matrix } \mathbf{C}} \begin{bmatrix} v_{a,k} \\ v_{b,k} \\ v_{c,k} \end{bmatrix}. \tag{10}$$

This is further visualized in Figure 2, which provides a geometric interpretation of the Clarke transform for balanced power systems. Observe the mutually orthogonal nature of the $v_{\alpha,k}$ and $v_{\beta,k}$ components, which allows for the compact complex-valued representation, $v_{\alpha\beta,k} = v_{\alpha,k} + jv_{\beta,k}$.

### Park transform

The Park transform (also known as the *dq transform*) is closely related to the Clarke transform and projects the three-phase signal $\boldsymbol{s}_k$ onto an orthogonal time-varying frame, which, by virtue of rotating at the fundamental power system frequency $f = \omega/2\pi$ (50–60 Hz), yields stationary constant outputs, $v_{d,k}$ and $v_{q,k}$. In other words, the Park voltages $v_{d,k}$ and $v_{q,k}$ are obtained from the Clarke $\alpha\beta$ voltages in (10) using a time-varying transformation given by [7]

$$\begin{bmatrix} v_{d,k} \\ v_{q,k} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta_k) & \sin(\theta_k) \\ -\sin(\theta_k) & \cos(\theta_k) \end{bmatrix}}_{\text{Park matrix: } \mathbf{P}_\theta} \begin{bmatrix} v_{\alpha,k} \\ v_{\beta,k} \end{bmatrix}, \tag{11}$$

where $\theta_k = \omega k$. Similar to the components of the Clarke transform, the orthogonal direct and quadrature components, $v_{d,k}$ and $v_{q,k}$, can be combined into a complex variable $v_{dq,k} = v_{d,k} + jv_{q,k}$.

### Remark 5

From the modern data analytics perspective, the Park matrix, $\mathbf{P}_\theta$, is a full-rank and time-varying clockwise rotation matrix, with the determinant $\det(\mathbf{P}_\theta) = 1$ and the unit-norm eigenvalues $|\lambda_{1,2}| = 1$. It therefore does not amplify the original Clarke vector $[v_{\alpha,k}, v_{\beta,k}]^T$ but performs a rotation, with the speed of rotation equal to the fundamental angular frequency of the power system, $\omega$.

### PCA

Modern data analytics often employs PCA, also known as the *Karhunen–Loeve transform*, to either separate meaningful data from noise or to reduce the dimensionality of the original signal space while maintaining the most important information-bearing latent components in data. Consider a general data vector, $\boldsymbol{x}_k \in \mathbb{R}^{M \times 1}$, for which the covariance matrix is defined as

$$\operatorname{cov}(\boldsymbol{x}_k) \overset{\text{def}}{=} \mathbf{R}_x = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \boldsymbol{x}_k \boldsymbol{x}_k^{\mathsf{T}}. \tag{12}$$

Then, this symmetric covariance matrix $\mathbf{R}_x$ admits the following eigenvalue decomposition

$$\mathbf{Q}^{\mathsf{T}} \mathbf{R}_x \mathbf{Q} = \boldsymbol{\Lambda}, \tag{13}$$

where the diagonal eigenvalue matrix, $\boldsymbol{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_M\}$, indicates the power of each component within $\boldsymbol{x}_k$, while the matrix of eigenvectors, $\mathbf{Q}_r = [\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_M]$, designates the principal directions in the data.

Suppose the signal $\boldsymbol{x}_k$ is to be transformed into a vector, $\boldsymbol{u}_k \in \mathbb{R}^{M \times 1}$, of the same dimensionality as the original signal $\boldsymbol{x}_k$, using a linear transformation matrix $\mathbf{W}$, to give

$$\boldsymbol{u}_k = \mathbf{W}\boldsymbol{x}_k, \quad \text{where} \quad \operatorname{cov}(\boldsymbol{u}_k) = \boldsymbol{\Lambda}. \tag{14}$$

PCA states that the transformation matrix $\mathbf{W}$ can be obtained from the eigenvector and eigenvalue matrices in (13) as $\mathbf{W} = \mathbf{Q}^{\mathsf{T}}$. In other words,

$$\begin{aligned} \operatorname{cov}(\boldsymbol{u}_k) &= \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \boldsymbol{u}_k \boldsymbol{u}_k^{\mathsf{T}} \\ &= \mathbf{W} \left( \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \boldsymbol{x}_k \boldsymbol{x}_k^{\mathsf{T}} \right) \mathbf{W}^{\mathsf{T}} \\ &= \mathbf{Q}^{\mathsf{T}} \mathbf{R}_x \mathbf{Q} = \boldsymbol{\Lambda}. \end{aligned} \tag{15}$$

This formulation admits a convenient dimensionality reduction by retaining only $r < M$ largest eigenvalues and the corresponding eigenvectors of $\mathbf{R}_x$. The so-obtained transformed data vector, $\boldsymbol{u}_{r,k} \in \mathbb{R}^{r \times 1}$, is now of a lower-dimension $r < M$ and is given by

$$\boldsymbol{u}_{r,k} = \mathbf{Q}_{1:r}^{\mathsf{T}} \boldsymbol{x}_k, \tag{16}$$

where $\mathbf{Q}_{1:r} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_r]$, and $r$ stands for the $r$-largest eigenvalues in $\boldsymbol{\Lambda}$. In other words, the PCA-based dimensionality-reduction scheme in (16) selects the directions along which the data expresses maximum variance, as designated by the principal eigenvectors of the data covariance matrix, $\mathbf{R}_x$.

### Clarke transform as a principal component analyzer

Without loss of generality, we consider normalized versions of the phasors, $\boldsymbol{v}$, relative to $\bar{V}_a$ and define $\delta_i \overset{\text{def}}{=} \bar{V}_i / \bar{V}_a$, $i \in \{a, b, c\}$, with $\delta_a = 1$, to yield

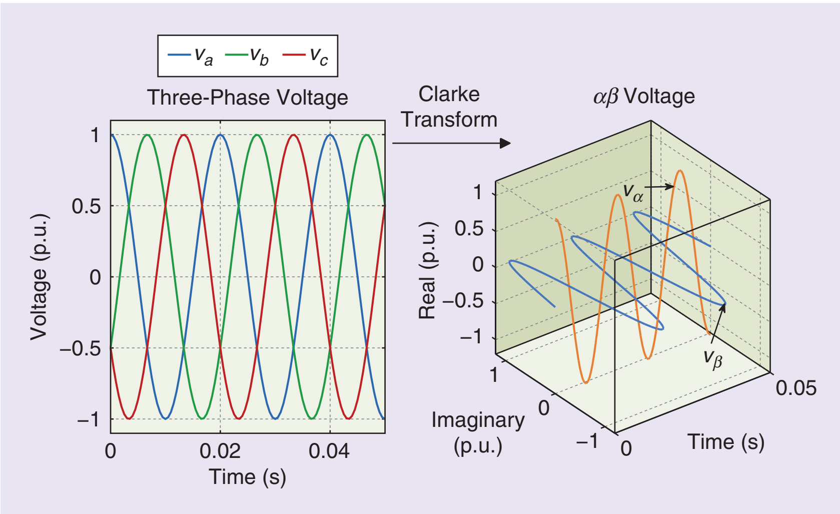


**FIGURE 2.** The waveforms of Clarke-transformed three-phase voltages. The Clarke voltages $v_\alpha$ and $v_\beta$ are orthogonal and admit a convenient complex-valued representation in the form $v_{\alpha\beta,k} = v_{\alpha,k} + jv_{\beta,k}$. p.u.: per unit.

$$\bar{\boldsymbol{v}} = [1,\ \delta_b,\ \delta_c]^{\mathsf{T}}. \quad (17)$$

For a balanced power system, the normalized phasor vector in (17) takes the form

$$\bar{\boldsymbol{v}}^b = \left[1,\ e^{-j\frac{2\pi}{3}},\ e^{j\frac{2\pi}{3}}\right]^{\mathsf{T}} \quad (18)$$

so that the covariance matrix of the normalized three-phase-voltage signal becomes

$$\mathbf{R}_s = \frac{1}{2}\mathrm{Re}\{\bar{\boldsymbol{v}}^b\bar{\boldsymbol{v}}^{b\mathsf{H}}\} = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad (19)$$

and thus admits the eigen decomposition in (13) to yield

$$\mathbf{R}_s = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathsf{T}}. \quad (20)$$

By inspecting $\mathbf{R}_s$ in (19) from the first eigenvector–eigenvalue pair, $(\boldsymbol{q}_1, \lambda_1)$, we have

$$\mathbf{R}_s\boldsymbol{q}_1 = \mathbf{0} \Rightarrow \boldsymbol{q}_1 = \frac{1}{\sqrt{3}}\mathbf{1},\ \lambda_1 = 0. \quad (21)$$

To find the remaining eigenvector–eigenvalue pairs, consider again the outer products within the covariance matrix, given in (8), and the normalized phasor vector, $\bar{\boldsymbol{v}}^b$ in (18). Notice that its real part, $\bar{\boldsymbol{v}}_r = \mathrm{Re}\{\bar{\boldsymbol{v}}^b\} = [1, -(1/2), -(1/2)]^{\mathsf{T}}$, and its imaginary part, $\bar{\boldsymbol{v}}_i = \mathrm{Im}\{\bar{\boldsymbol{v}}^b\} = [0, -(\sqrt{3}/2), (\sqrt{3}/2)]^{\mathsf{T}}$, are orthogonal, i.e., $\bar{\boldsymbol{v}}_r^{\mathsf{T}}\bar{\boldsymbol{v}}_i = 0$.

Therefore, the remaining two eigenvectors of $\mathbf{R}_s$ are $\boldsymbol{q}_2 = \bar{\boldsymbol{v}}_r / \|\bar{\boldsymbol{v}}_r\|$ and $\boldsymbol{q}_3 = \bar{\boldsymbol{v}}_i / \|\bar{\boldsymbol{v}}_i\|$, with the corresponding eigenvalues $\lambda_2 = (1/4)\|\bar{\boldsymbol{v}}_r\|$ and $\lambda_3 = (1/4)\|\bar{\boldsymbol{v}}_i\|$. The matrix of eigenvectors $\mathbf{Q}^{\mathsf{T}}$ and the diagonal matrix of eigenvalues $\boldsymbol{\Lambda}$ in (20) thus take the form

$$\mathbf{Q}^{\mathsf{T}} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}. \quad (22)$$

Inspection of the diagonal elements of $\boldsymbol{\Lambda}$ in (22) reveals only two nonzero eigenvalues. This verifies Remark 3, which states that the covariance matrix of a three-phase system voltage, $\mathbf{R}_s$, is of rank-2 and thus rank-deficient. The factor $\sqrt{2/3}$, which premultiplies $\mathbf{Q}^T$ in (22), serves to normalize the length of the eigenvectors to unity (orthonormality).

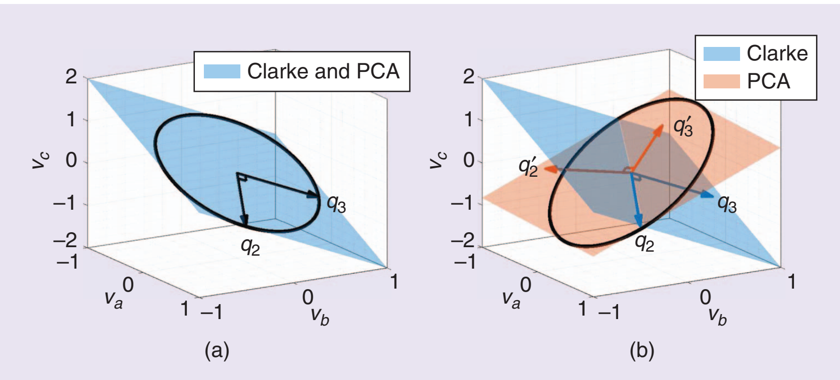


**FIGURE 3.** The dimensionality reduction by the Clarke matrix and PCA. (a) For a balanced system, the 2D subspaces (plane in light blue) for the Clarke matrix and PCA coincide and are spanned by the eigen/Clarke vectors $\mathbf{q}_2 = [1, -(1/2), -(1/2)]^{\mathsf{T}}$ and $\mathbf{q}_3 = [0, (\sqrt{3}/2), -(\sqrt{3}/2)]^{\mathsf{T}}$, as shown in black. The Clarke voltage $v_{\alpha\beta,k} = v_{\alpha,k} + jv_{\beta,k}$ describes a circle. (b) For an unbalanced system (in this case, phase mismatch $\phi_a = \phi_b = 0$ and $\phi_c = 2$), the fixed nature of the Clarke subspace means that it is still spanned by the original bases $\mathbf{q}_2$ and $\mathbf{q}_3$ (in blue) as in (a). Observe that the correct 2D subspace, identified by PCA, is designated by the plane in red, which is spanned by the correct eigenvectors $\mathbf{q}_2' = [-0.48, 0.61, 0.63]^{\mathsf{T}}$ and $\mathbf{q}_3' = [-0.88, -0.40, -0.28]^{\mathsf{T}}$ (in red). This exemplifies the limitations of the Clarke and related transforms in the analysis of modern, dynamically unbalanced power systems.

### Remark 6

The matrix of eigenvectors, $\mathbf{Q}^{\mathsf{T}}$ in (22), is identical to the Clarke transformation matrix defined in (9). Therefore, all of the variance in three-phase power system voltages can be explained by the two eigenvectors associated with the nonzero eigenvalues (principal axes) of the Clarke-transform matrix. This offers the modern, data analytics interpretation of Clarke transform as a principal component analyzer, which performs a projection of three-phase power system voltages that reside in $\mathbb{R}^3$ onto a 2D subspace spanned by the two largest orthogonal eigenvectors of the phase-voltage-correlation matrix, $[1, -(1/2), -(1/2)]^{\mathsf{T}}$ and $[0, \sqrt{3}/2,\ -(\sqrt{3}/2)]^{\mathsf{T}}$, as shown in Figure 3.

### Remark 7

Remark 6 and Figure 3 offer a modern interpretation of the Clarke transform from a PCA-based dimensionality-reduction viewpoint. This new perspective opens numerous new avenues for using data analytics (such as signal processing and machine learning) in power grid research and paves the way for innovative transformation and analysis techniques for the future smart grid that were not previously possible with the standard circuit theory principles.

We next provide a modern, spectral-analysis based interpretation of issues when using the Clarke and related transformations for the analysis of dynamically unstable smart grids.

## *Clarke and symmetrical component transforms as a three-point DFT*

The well-known limitations of current power system analysis techniques in unbalanced grid scenarios is difficult to explain from a circuit theory perspective. To help provide a more generic interpretation of this issue, we shall next examine a link with the well-known effects of incoherent sampling in spectral analysis of coarsely sampled data.

Here, we provide a spatial DFT interpretation of the symmetrical component transform (see "Dealing with Unbalanced Phasors: The Symmetrical Component Transform"). The vector of three-phase time-domain voltages, $\boldsymbol{s}_k$

in (1), is typically considered as a collection of three, univariate time-domain signals, i.e., $\mathbf{s}_k = [v_{a,k}, v_{b,k}, v_{c,k}]^{\mathsf{T}}$. It is interesting to note that, for balanced systems, due to the equal phase spacing between phase voltages of $2\pi/3$, the corresponding normalized phasor vector $\bar{\mathbf{v}}^b$ in (18) can be equally treated as a collection of samples of a monocomponent complex sinusoid, which rotates at a spatial frequency $\Omega = -(2\pi/3)$, to yield

$$\bar{\mathbf{v}}^b = [1,\ e^{j\Omega},\ e^{j2\Omega}]^{\mathsf{T}}, \tag{23}$$

thus offering a vehicle for spectral representation of this essentially time-domain phenomenon.

## Dealing with Unbalanced Phasors: The Symmetrical Component Transform

The symmetrical transform [S3] was introduced by Charles Fortesque in 1918 with the aim of converting a general (possibly unbalanced) phasor vector, $\mathbf{v} = [\bar{V}_a, \bar{V}_b, \bar{V}_c]^{\mathsf{T}}$ in (4), into three separate balanced components, $\bar{V}_0, \bar{V}_+$, and $\bar{V}_-$, which are referred to as the *zero-*, *positive-*, and *negative-sequence* phasor, respectively, and are given by

$$\begin{bmatrix} \bar{V}_0 \\ \bar{V}_+ \\ \bar{V}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}}_{\text{DFT matrix}} \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \bar{V}_c \end{bmatrix}, \tag{S1}$$

where $a = e^{-j(2\pi/3)}$. Unlike the Clarke and Park transforms, which are applied directly to the three-phase time-domain voltage, $\mathbf{s}_k$, the symmetrical transform operates on the phasors $\mathbf{v}_k$ and thus admits a spatial spectral domain interpretation, as elaborated upon in (24) and Remark 9. The Clarke transform can then be interpreted as the real part of the three-point DFT matrix in (S1) since the diagonalization of the eigenvector matrix for circulant matrices yields the DFT matrix.

### Reference


[S3] C. L. Fortescue, "Method of symmetrical coordinates applied to the solution of polyphase networks," *Trans. Amer. Inst. Electr. Engineers*, vol. 37, no. 2, pp. 1027–1140, 1918.


### Remark 8

Under unbalanced conditions, the elements of the normalized phasor vector will not have equal spacing of $2\pi/3$ in the spatial-frequency domain, due to different amplitudes and/or nonuniform phase separation of the individual phasors, as defined in (4). As a consequence, $\bar{\mathbf{v}}$ will no longer represent a single, complex-valued spatial sinusoid.

Consider now the DFT of the original phasor vector in (4), $\mathbf{v} = [v_0, v_1, v_2]^{\mathsf{T}} = [\bar{V}_a, \bar{V}_b, \bar{V}_c]^{\mathsf{T}} \in \mathbb{C}^{3\times 1}$, given by

$$X[k] = \frac{1}{\sqrt{3}} \sum_{n=0}^{2} v_n e^{-j\frac{2\pi}{3}nk}, \quad k = 0, 1, \text{and } 2,$$

which can be expressed in an equivalent matrix form as

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \\ \bar{V}_c \end{bmatrix}, \tag{24}$$

where $a = e^{-j(2\pi/3)}$. The three-point DFT in (24) therefore transforms the phasor vector $\mathbf{v}$ into a stationary (dc or zero) component $X[0]$ and the components $X[1]$ and $X[2]$, which rotate at their respective spatial frequencies $-(2\pi/3)$ and $2\pi/3$.

### Remark 9

The spatial DFT in (24) is identical to the symmetrical component transform in (S1). More specifically, the stationary DFT component, $X[0]$, corresponds to the zero-sequence phasor, $\bar{V}_0$, which is zero for balanced system conditions but nonzero for unbalanced ones, while the fundamental DFT components, $X[1]$ and $X[2]$, represent the positive- and negative-sequence phasors, respectively. This forms a basis for the treatment of three-phase component transforms from a spectral estimation perspective and offers an enhanced interpretation of the limitations of these transforms in smart grid problems together with new avenues for their mitigation.

In a signal processing interpretation, readers can observe that the spatial sampling in (23) represents coarse critical sampling, whereby the spatial system frequency is contained in the first component of the underlying three-point DFT with no provision for the interpretation of drifting frequencies, as required by the smart grid. This explains the well-known inability of the symmetrical component transform to deal with transients in three-phase power systems, together with the artefacts arising from incoherent sampling—a standard issue in coarsely sampled systems.

For example, the negative sequence phasor $\bar{V}_- = (1/\sqrt{3})[V_a e^{j\phi_a} + V_b e^{j(\phi_b - (2\pi/3))} + V_c e^{j(\phi_c + (2\pi/3))}]$ in (S1) vanishes in balanced system conditions with $V_a = V_b = V_c$ and $\phi_a = \phi_b = \phi_c = 0$, while its nonzero value indicates an unbalanced system and an inadequate spatial DFT representation in (23).

## What we have learned

The operation of the future and almost permanently dynamically unbalanced smart grids requires close cooperation and convergence between the power systems and data analytics communities, especially those working in signal processing and machine learning. A major prohibitive factor in this endeavor has been a lack of common language; for example, the most fundamental techniques, such as the Clarke and Park transforms introduced in 1943 and 1929, respectively, have been designed from a circuit theory perspective and only for balanced "nominal" system conditions. This renders such methodologies both awkward for linking up with data analytics communities and only partially suited for the demands of future dynamically unbalanced smart grids.

To help bridge this gap, we have provided modern interpretations of the Clarke and related transforms through the subspace and spatial DFT concepts. These have served as a mathematical lens into the inadequacies of current methodologies under unbalanced power system conditions and have enabled us to create a framework for the understanding and mitigation of the effects of off-nominal system frequency and dynamically unbalanced phase voltages and phases. All in all, such a conceptual insight helps demystify power system analysis for data analytics practitioners and permits the seamless migration of ideas between these typically disparate communities.

It is fitting to conclude with a quote from J.E. Brittain's article [1] on Clarke:

> She (Clarke) translated what many engineers found to be esoteric mathematical methods into graphs or simpler forms during a time when power systems were becoming more complex and when the initial efforts were being made to develop electromechanical aids to problem solving.

We hope that this modern perspective of the Clarke and related transforms will help extend their legacy well into the Information Age, in addition to empowering analysts with enhanced intuition and freedom in algorithmic design. An important additional feature of our perspective is that it opens up new possibilities in the otherwise prohibitive applications of Clarke-inspired transforms in future smart grids. We expect to address this aspect in part 2 of this "Lecture Notes" article. For an electronic supplement, please see http://www.commsp.ee.ic.ac.uk/~mandic/DSP_ML_for_Power.htm.

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