Communications II

Lecture 7: Performance of digital modulation

Professor Kin K. Leung
EEE and Computing Departments
Imperial College London

Outline

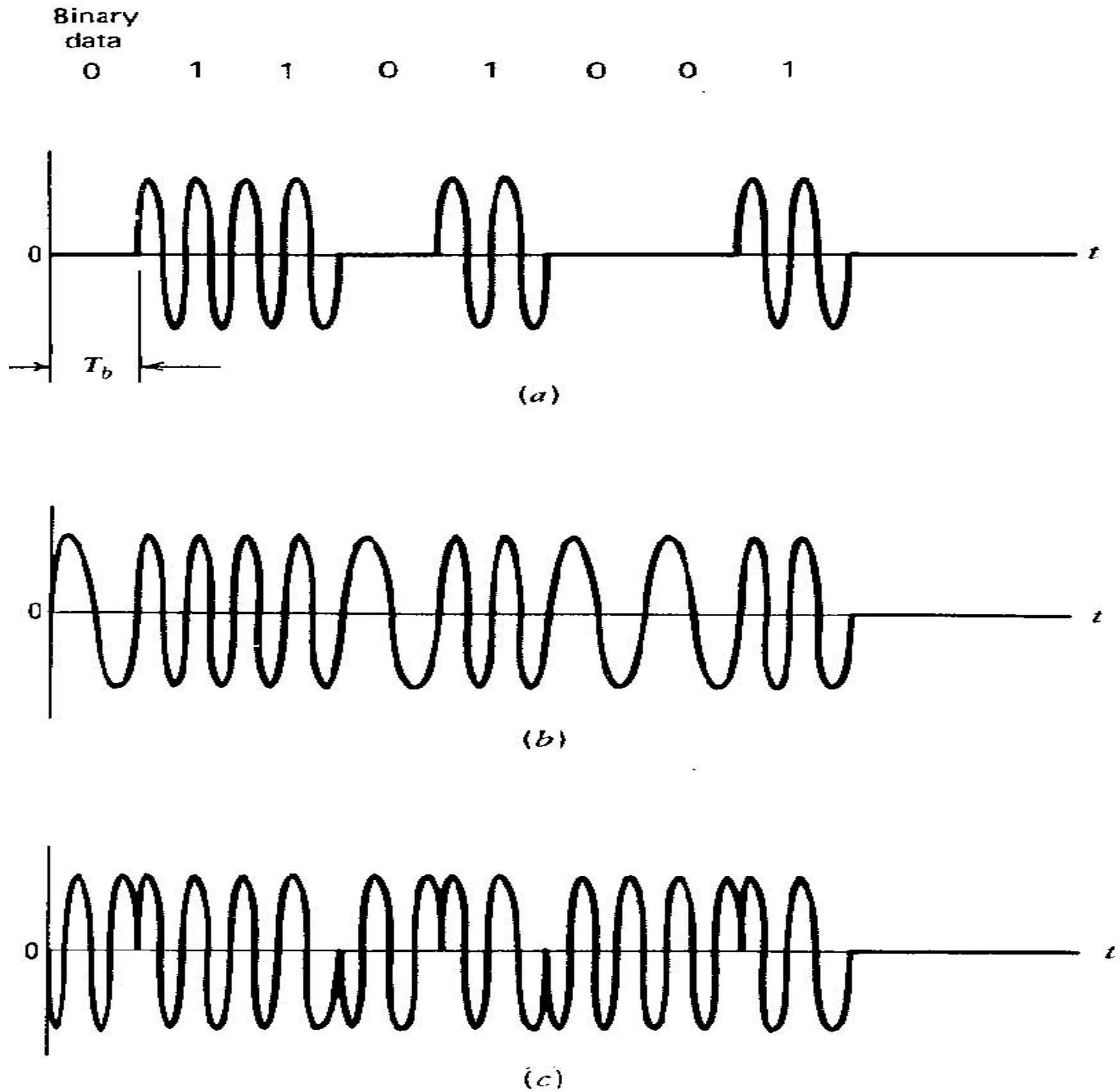
- Digital modulation and demodulation
- Error probability of ASK
- Error probability of FSK
- Error probability of PSK
- Noncoherent demodulation
- Reference: Lathi, Chap. 13.

Bandpass Data Transmission

How do we transmit symbols 1 and 0?

- Amplitude Shift Keying (ASK)
- Frequency Shift Keying (FSK)
- Phase Shift Keying (PSK)

ASK, FSK and PSK



How do we recover the transmitted symbol?

- Coherent (synchronous) detection

□ Use a BPF to reject out-of-band noise

□ Multiply the incoming waveform with a cosine of the carrier frequency

□ Use a LPF

□ Requires carrier regeneration (both frequency and phase synchronization by using a phase-lock loop)

- Noncoherent detection (envelope detection etc.)

□ Makes no explicit efforts to estimate the phase

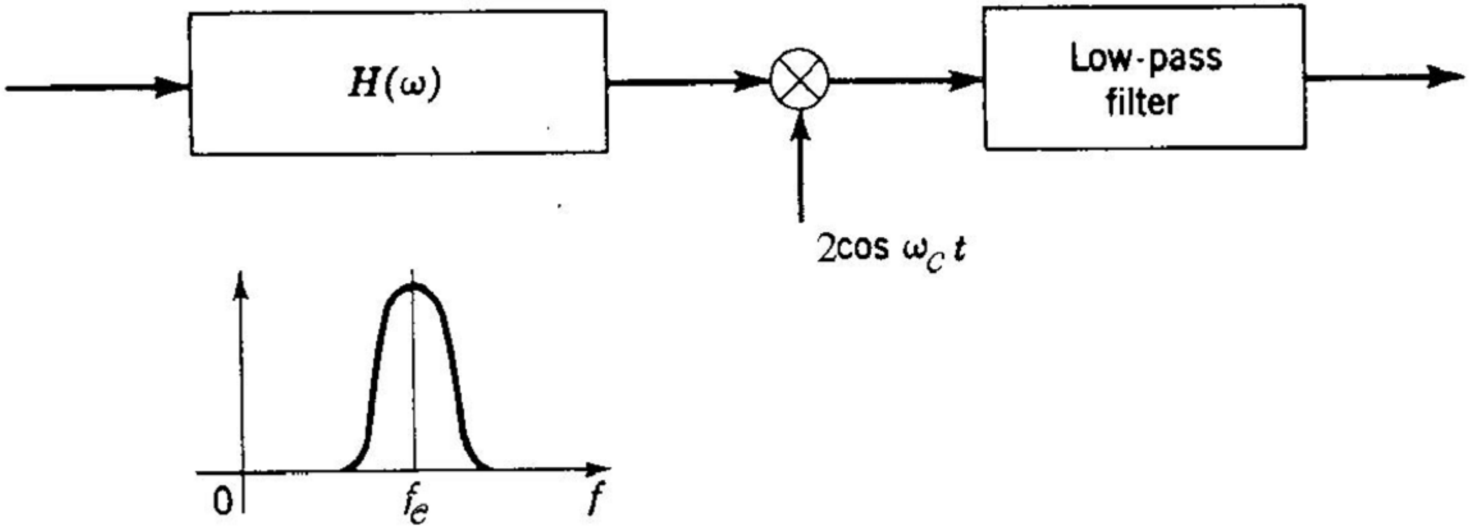
Amplitude shift keying (ASK) = on-off keying (OOK)

$$
\begin{aligned}
s_0(t) &= 0 \\
s_1(t) &= A\cos(2\pi f_c t)
\end{aligned}
$$

Or

$$s(t) = A(t)\cos(2\pi f_c t), \qquad A(t) \in \{0, A\}$$

Coherent Detection



Assume an ideal band-pass filter with unit gain on $[f_c - W, f_c + W]$. For a practical band-pass filter, $2W$ should be interpreted as the equivalent bandwidth.

Pre-detection signal:

$$
\begin{aligned}
x(t) &= s(t) + n(t) \\
&= A\cos(2\pi f_C t) + n_C(t)\cos(2\pi f_C t) - n_S(t)\sin(2\pi f_C t) \\
&= [A(t) + n_C(t)]\cos(2\pi f_C t) - n_S(t)\sin(2\pi f_C t)
\end{aligned}
$$

- After multiplication with $2\cos(2\pi f_c t)$:

$$y(t) = [A(t) + n_C(t)]2\cos^2(2\pi f_C t) - n_S(t)2\sin(2\pi f_C t)\cos(2\pi f_C t)$$
$$= [A(t) + n_C(t)](1 + \cos(4\pi f_C t)) - n_S(t)\sin(4\pi f_C t)$$

- After low-pass filtering:

$$\tilde{y}(t) = A(t) + n_C(t) \qquad (166)$$

Reminder: The in-phase noise component $n_c(t)$ has the same variance as the original band-pass noise $n(t)$

$\Rightarrow$ The received signal (166) is identical to the received signal (147) for baseband digital transmission

$\Rightarrow$ The sample values of $\tilde{y}(t)$ will have PDFs that are identical to those of the baseband case

$\Rightarrow$ For ASK the statistics of the receiver signal are identical to those of a baseband system

$\Rightarrow$ **The probability of error for ASK is the same as for the baseband case**

Assume equiprobable transmission of 0s and 1s.

Then the decision threshold must be $A/2$ and the probability of error is given by:

$$P_{e,ASK} = Q\left(\frac{A}{2\sigma}\right)$$

Phase shift keying (PSK)

$$s(t) = A(t)\cos(2\pi f_c t), \qquad A(t) \in \{-A, A\}$$

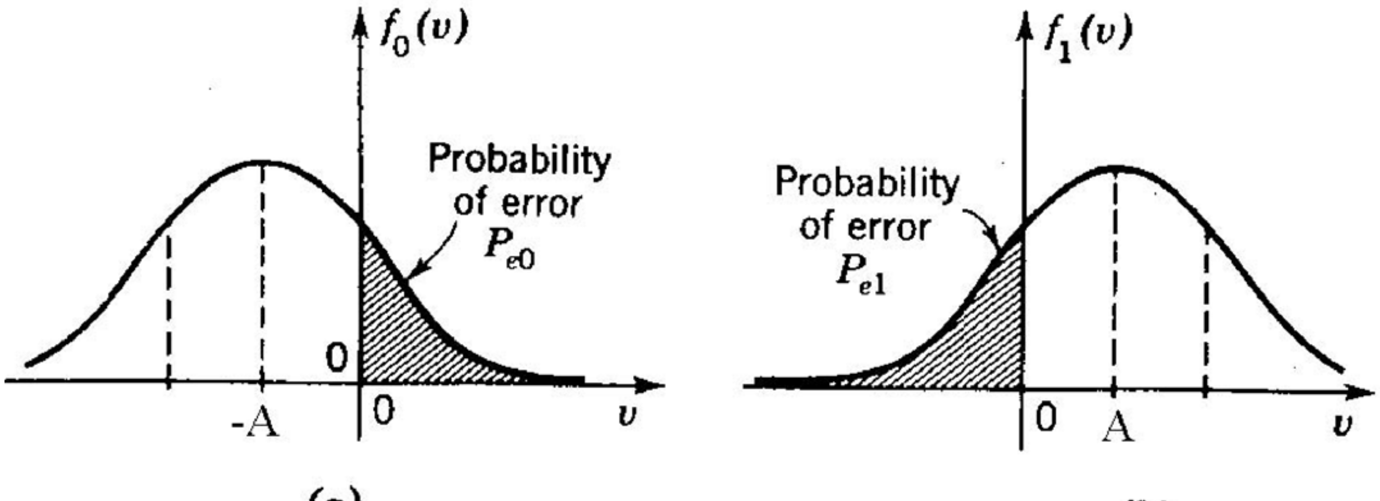
Use coherent detection again, to eventually get the detection signal:

$$\tilde{y}(t) = A(t) + n_C(t)$$

Probability density functions for PSK for equiprobable 0s and 1s in noise:

(a): symbol 0 transmitted

(b): symbol 1 transmitted



(a) (b)

Use threshold 0 for detection

Conditional error probabilities:

$$P_{e0} = \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(n+A)^2}{2\sigma^2} \right) dn$$

$$P_{e1} = \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(n-A)^2}{2\sigma^2} \right) dn$$

In the **first** set $\tilde{n} \equiv n + A \Rightarrow dn = d\tilde{n}$ and when $n = 0, \tilde{n} = A$

$$P_{e0} = \frac{1}{\sigma\sqrt{2\pi}} \int_A^{\infty} \exp\left(-\frac{\tilde{n}^2}{2\sigma^2} \right) d\tilde{n}$$

In the **second** set $\tilde{n} \equiv -(n - A) = -n + A \Rightarrow dn = -d\tilde{n}$ and when

$$n = 0, \tilde{n} = A, \; and \;\; when \; n = -\infty, \tilde{n} = +\infty :$$

$$P_{e1} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{A} \exp\left(-\frac{\tilde{n}^2}{2\sigma^2} \right) (-1) d\tilde{n}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_A^{\infty} \exp\left(-\frac{\tilde{n}^2}{2\sigma^2} \right) d\tilde{n}$$

So:

$$P_{e0} = P_{e1} = P_{e,PSK} = \int_A^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n^2}{2\sigma^2}\right) dn$$

Change variable of integration to $z \equiv n/\sigma \Rightarrow dn = \sigma dz$ and when $n = A$, $z = A/\sigma$. Then:

$$P_{e,PSK} = \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sigma}}^{\infty} e^{-z^2/2} dz = Q\left(\frac{A}{\sigma}\right) \qquad (172)$$

Remember that

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-t^2/2) dt$$

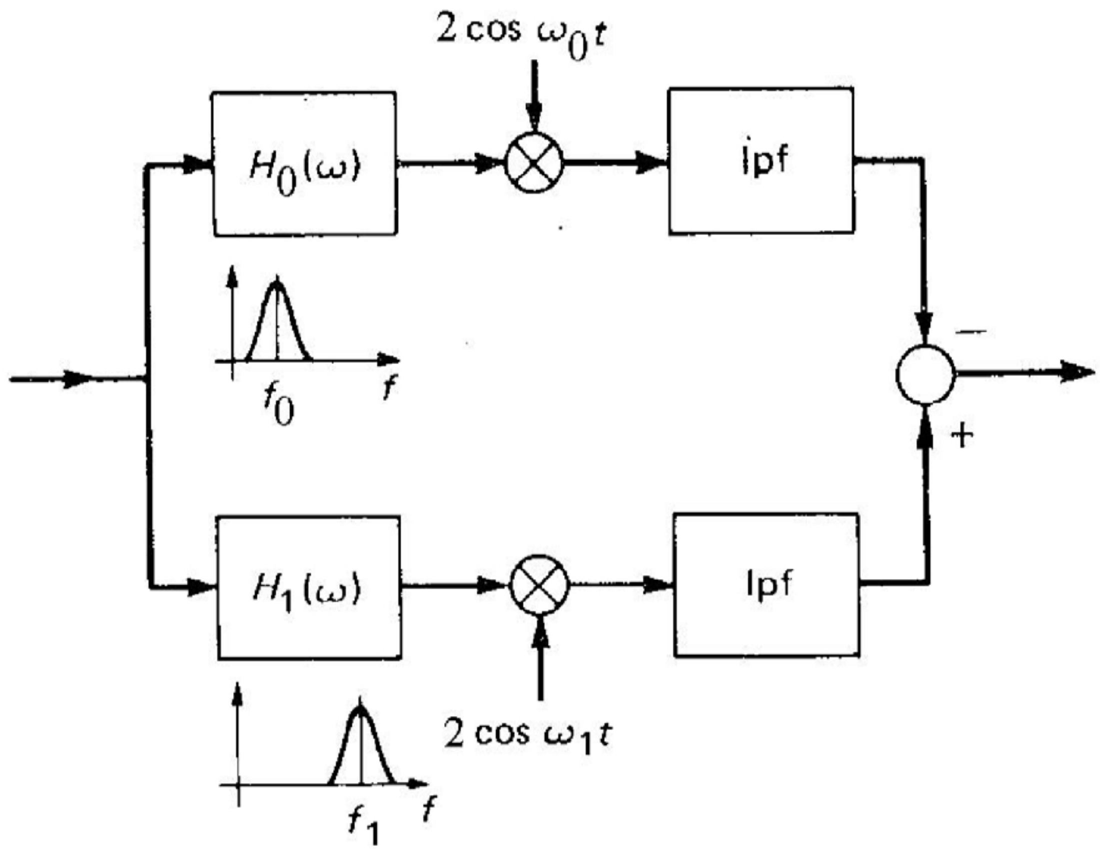
Frequency Shift Keying (FSK)

$$
\begin{aligned}
s_0(t) &= A\cos(2\pi f_0 t), \qquad \text{if symbol 0 is transmitted} \\
s_1(t) &= A\cos(2\pi f_1 t), \qquad \text{if symbol 1 is transmitted}
\end{aligned}
$$

Symbol recovery:

Use two sets of coherent detectors, one operating at a frequency f_0 and the other at f_1.

Coherent detector for FSK



The two BPF's are non-overlapping in frequency spectrum

$$\text{Each branch} = \text{an ASK detector}$$

$$\text{LPF output on each branch} = \begin{cases} \text{A} + \text{noise} & \text{if symbol present} \\ \text{noise} & \text{if symbol not present} \end{cases}$$

$n_0(t)$: the noise output of the top branch
$n_1(t)$: the noise output of the bottom branch

Each of these noise terms has identical statistics to $n(t)$.

Output if a symbol 1were transmitted

$$y_1(t) = A + [n_1(t) - n_0(t)]$$

Output if a symbol 0 were transmitted

$$y_0(t) = -A + [n_1(t) - n_0(t)]$$

Set detection threshold to 0

Difference from PSK: the noise term is now $n_1(t) - n_0(t)$.

The noises in the two channels are independent because their spectra are non-overlapping.

$\Rightarrow$ the variances add

$\Rightarrow$ the noise variance has doubled!

Probability of error for FSK:

Replace σ^2 in (172) by $2\sigma^2$ (or σ by $\sqrt{2}\,\sigma$)

$$P_{e,FSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right)$$

Noise of the sum or difference of two independent zero mean random variables:

x_1: a random variable with variance $\sigma_1{}^2$

x_2: a random variables with variance $\sigma_2{}^2$

What is the variance of $y \equiv x_1 \pm x_2$?

By definition, the variance of $y \equiv x_1 \pm x_2$ is

$$\begin{aligned}\sigma_y^2 &= E\{y^2\} - E\{y\}^2 \\ &= E\{(x_1 \pm x_2)^2\} \\ &= E\{x_1^2 \pm 2x_1x_2 + x_2^2\} \\ &= E\{x_1^2\} \pm E\{x_1x_2\} + E\{x_2^2\}\end{aligned}$$

For independent variables: $E\{x_1x_2\} = E\{x_1\}E\{x_2\}$

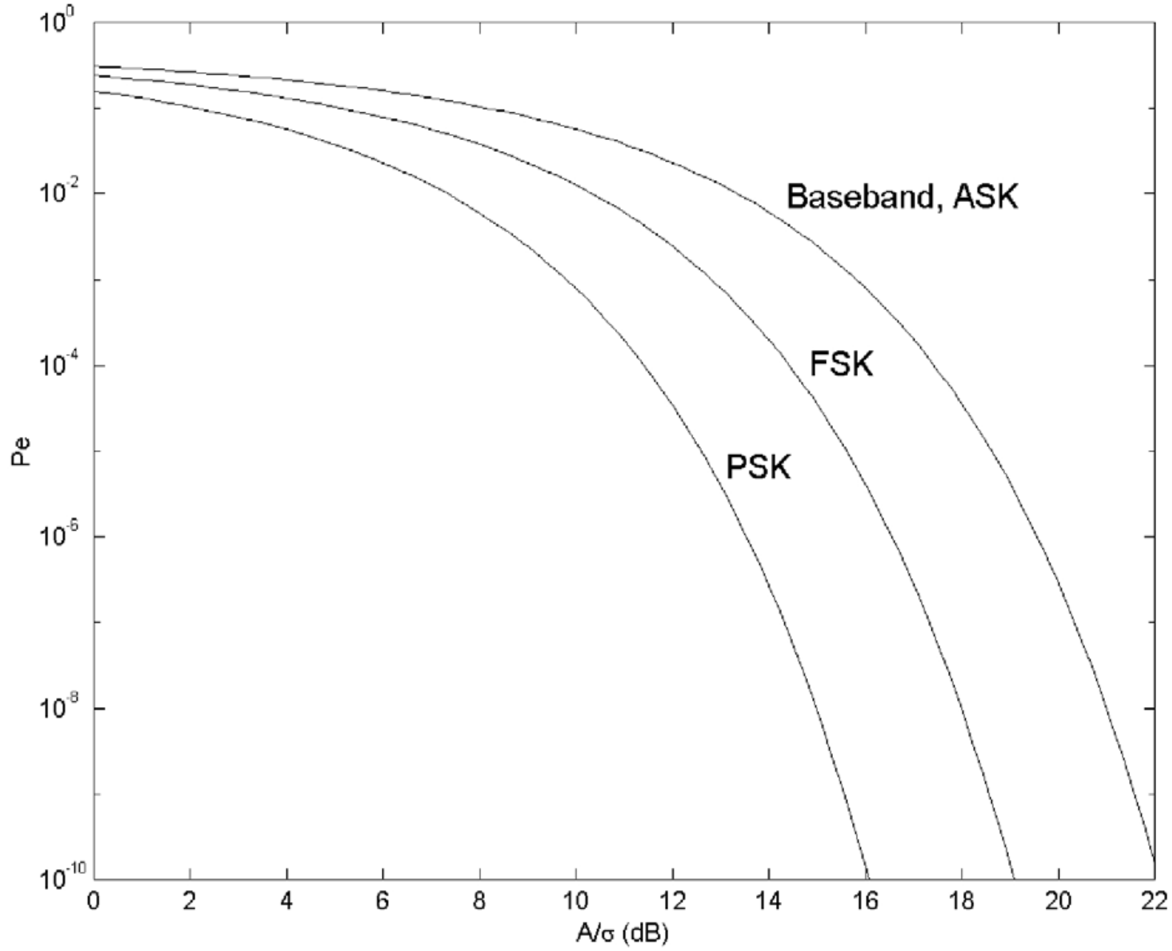
For zero-mean random variables:

$E\{x_1\} = E\{x_2\} = 0 \Rightarrow E\{x_1x_2\} = 0$

So

$$\begin{aligned}\sigma_y^2 &= E\{x_1^2\} + E\{x_2^2\} \\ &= \sigma_1^2 + \sigma_2^2\end{aligned}$$

Comparison of the three schemes



To achieve the same error probability (fixed P_e):

PSK can be reduced by 6 dB compared with a baseband or ASK system (a factor of 2 reduction in amplitude)

FSK can be reduced by 3 dB compared with a baseband or ASK (a factor of $\sqrt{2}$ reduction in amplitude)

Caution: The comparison is based on *peak* SNR. In terms of average SNR, PSK only has a 3 dB improvement over ASK, and FSK has the same performance as ASK

Properties of the *Q*-function

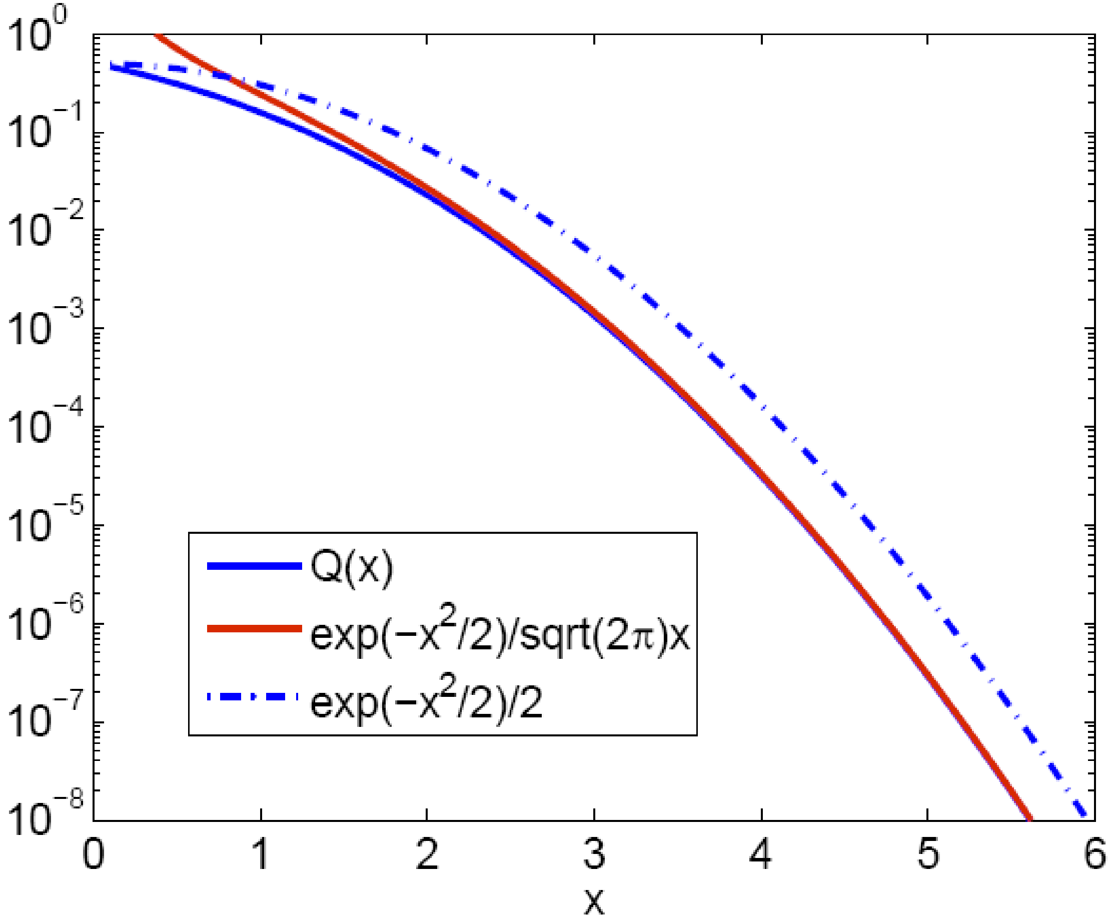
Upper bounds and good approximations:

$$Q(x) \le \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, \ x \ge 0$$

Becomes tighter for large *x*.

$$Q(x) \le \frac{1}{2} e^{-x^2/2}, \ x \ge 0$$

A better upper bound for small *x*.



Q(x)
exp(−x2/2)/sqrt(2π)x
exp(−x2/2)/2
x

Non-coherent detection

Accurate phase synchronization may be difficult in a dynamic channel.

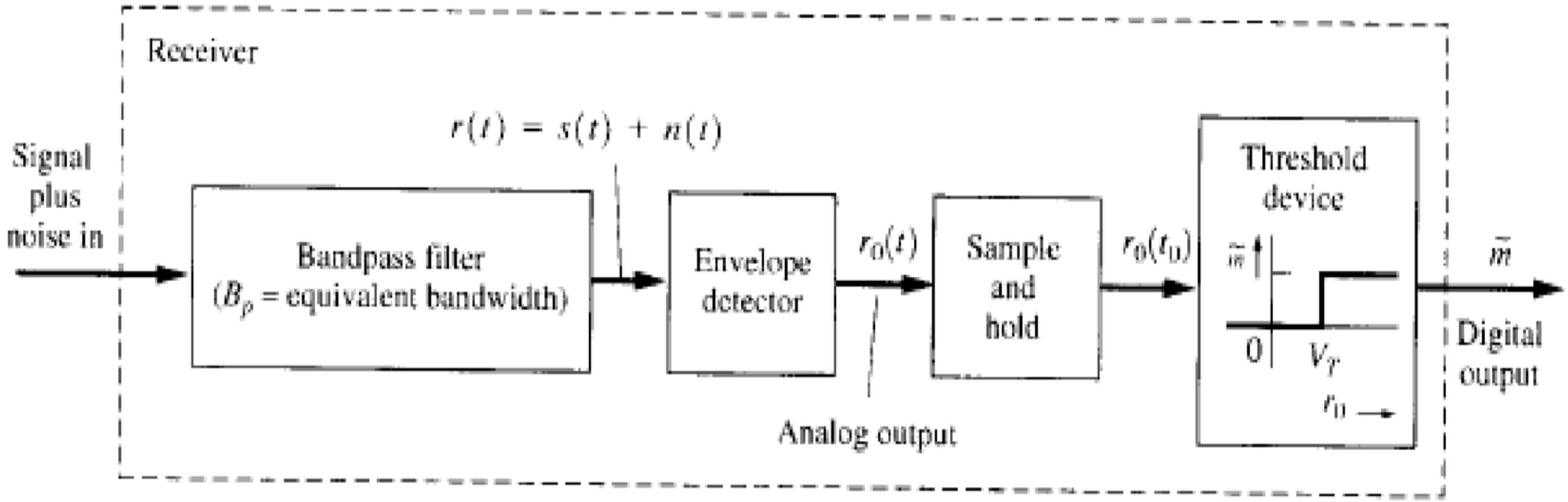
Phase synchronization error is due to frequency drift, instability of the local oscillator, effects of strong noise ...

When the carrier phase is unknown, one must rely on non-coherent detection.

The phase θ is assumed to be uniformly distributed on $[0, 2\pi]$.

Circuitry is simpler, but analysis is more difficult!

ASK



Receiver
$r(t) = s(t) + n(t)$
Signal plus noise in
Bandpass filter (B_p = equivalent bandwidth)
Envelope detector
$r_0(t)$
Sample and hold
$r_0(t_0)$
Threshold device
$\tilde{m}$
0
V_T
r_0
$\tilde{m}$
Digital output
Analog output

Error analysis

When symbol 0 is sent, the envelope (noise alone) has Rayleigh distribution

$$f(r) = \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)}, \quad r \geq 0 \qquad (180)$$

When symbol 1 is sent, the envelope (signal + noise) has Rician distribution

$$f(r) = \frac{r}{\sigma^2} e^{-(r^2+A^2)/(2\sigma^2)} I_0\left(\frac{Ar}{\sigma^2}\right), \quad r \geq 0$$

(180) dominates the error probability when $A/\sigma >> 1$.

Let the threshold be $A/2$ for simplicity.

The error probability (dominated by symbol 0) is given by

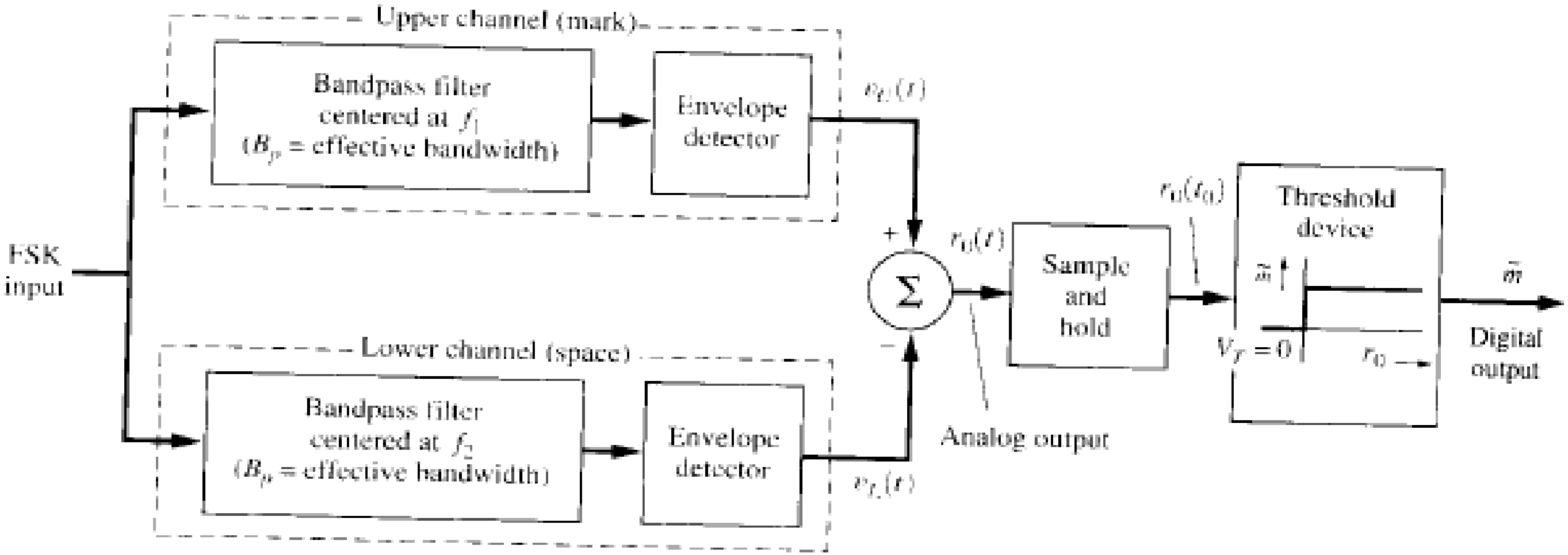
$$P_e \approx \frac{1}{2}\int_{A/2}^{\infty} \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)} dr, \quad r \geq 0$$

$$= \frac{1}{2} e^{-A^2/(8\sigma^2)}$$

Cf. coherent demodulation

$$P_{e,ASK} = Q\left(\frac{A}{2\sigma}\right) \leq \frac{1}{2} e^{-A^2/(8\sigma^2)}$$

FSK



Upper channel (mark)
Bandpass filter
centered at f_1
(B_p = effective bandwidth)
Envelope
detector
$v_U(t)$
FSK
input
+
−
Σ
$r_0(t)$
Sample
and
hold
$r_0(t_0)$
Threshold
device
$\tilde{m}$
$V_T = 0$
r_0
$\tilde{m}$
Digital
output
Analog output
Lower channel (space)
Bandpass filter
centered at f_2
(B_p = effective bandwidth)
Envelope
detector
$v_L(t)$

Error probability

When a symbol 1 is sent, one branch has Rayleigh distribution, the other has Rice distribution.

Error occurs if Rice < Rayleigh, and it can be shown that

$$P_e = \frac{1}{2} e^{-A^2/(4\sigma^2)}$$

Cf. coherent demodulation

$$P_{e,FSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right) \le \frac{1}{2} e^{-A^2/(4\sigma^2)}$$

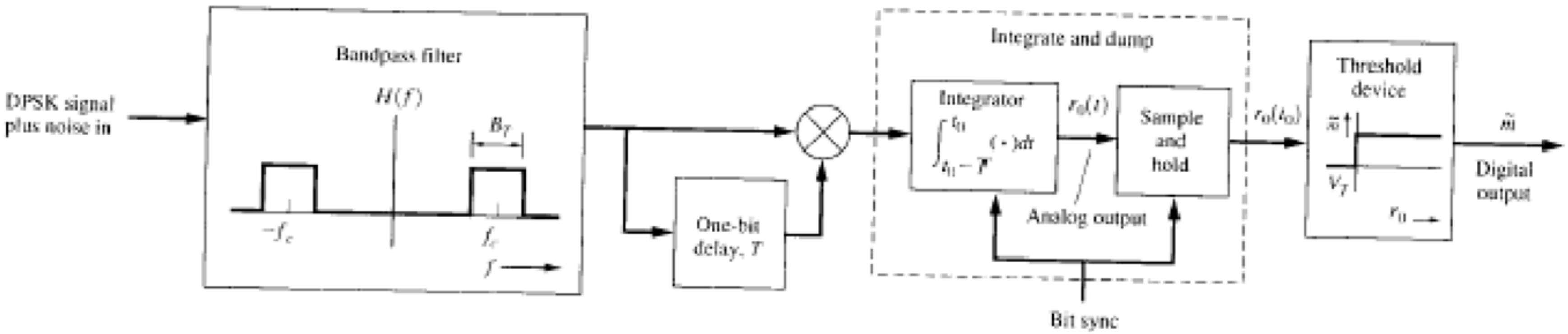
DPSK: Differential PSK

It is impossible to demodulate PSK with an envelop detector, since PSK signals have the same frequency and amplitude.

We can demodulate PSK differentially, where phase reference is provided by a delayed version of the signal in the previous interval.

Differential encoding is essential: $b_n = b_{n-1} \times a_n$, where $a_n, b_n \in \pm 1$.

Differential detection



Error probability

$$P_{e,DPSK} = \frac{1}{2} e^{-A^2/(2\sigma^2)}$$

Cf. coherent demodulation

$$P_{e,PSK} = Q\left(\frac{A}{\sigma}\right) \le \frac{1}{2} e^{-A^2/(2\sigma^2)}$$

Conclusions

Non-coherent demodulation retains the hierarchy of performance.

Non-coherent demodulation has error performance slightly worse than coherent demodulation, but approaches coherent performance at high SNR.

Non-coherent demodulators are considerably easier to build.