EE1 and EIE1: Introduction to Signals and Communications

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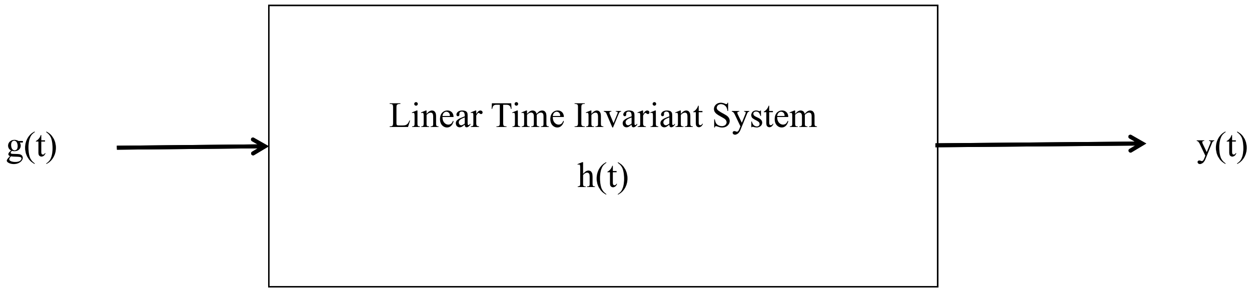
Lecture seven

Lecture Aims

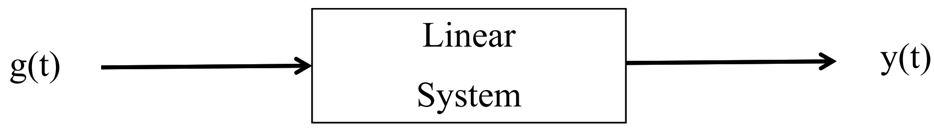
- To introduce linear systems
- To introduce convolution
- To give examples of real and ideal filters

Linear Systems

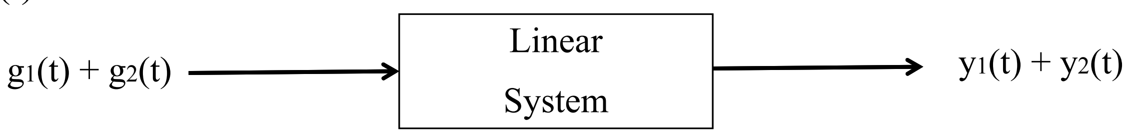


Linear Systems (continued)

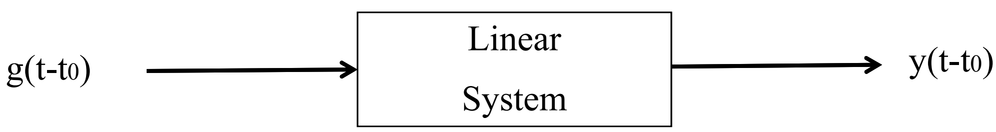
- A system converts an input signal $g(t)$ in an output signal $y(t)$.



- Assume the output for an input signal $g_1(t)$ is $y_1(t)$ and the output for an input $g_2(t)$ is $y_2(t)$. The system is linear if the output for input $g_1(t)$ + $g_2(t)$ is $y_1(t)$ + $y_2(t)$.



- A system is time invariant if its properties do not change with the time. That is, if the response to $g(t)$ is $y(t)$, then the response to $g(t - t_0)$ is going to be $y(t - t_0)$



Unit impulse response of a LTI system

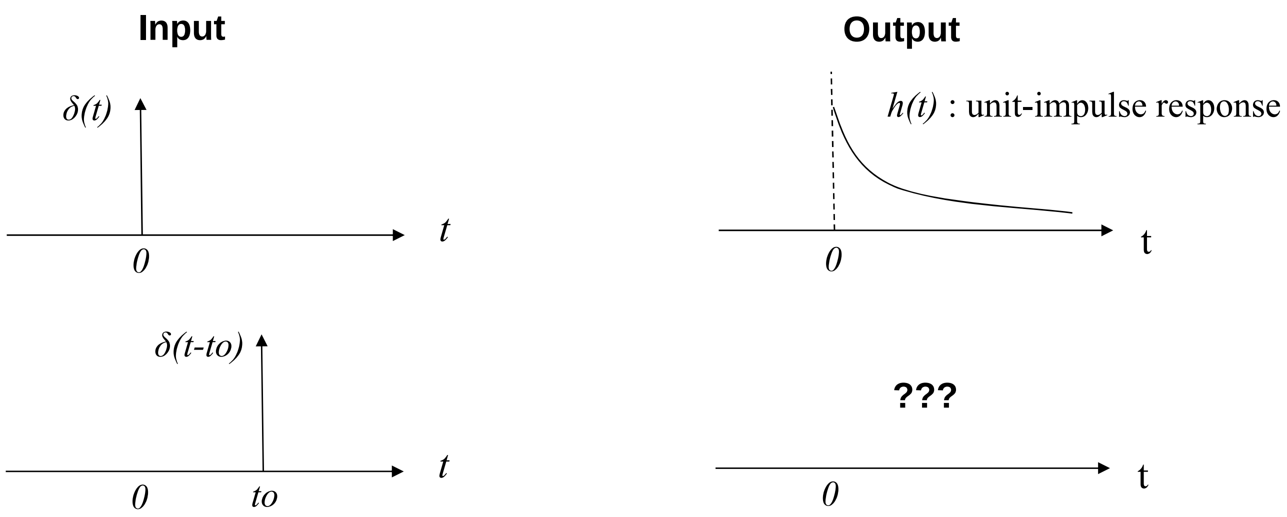
Consider a linear time invariant (LTI) system. Assume the input signal is a Dirac function $\delta(t)$. Call the observed output $h(t)$.

- $h(t)$ is called the **unit impulse response** function.

- With $h(t)$, we can relate the input signal to its output signal through the convolution formula:

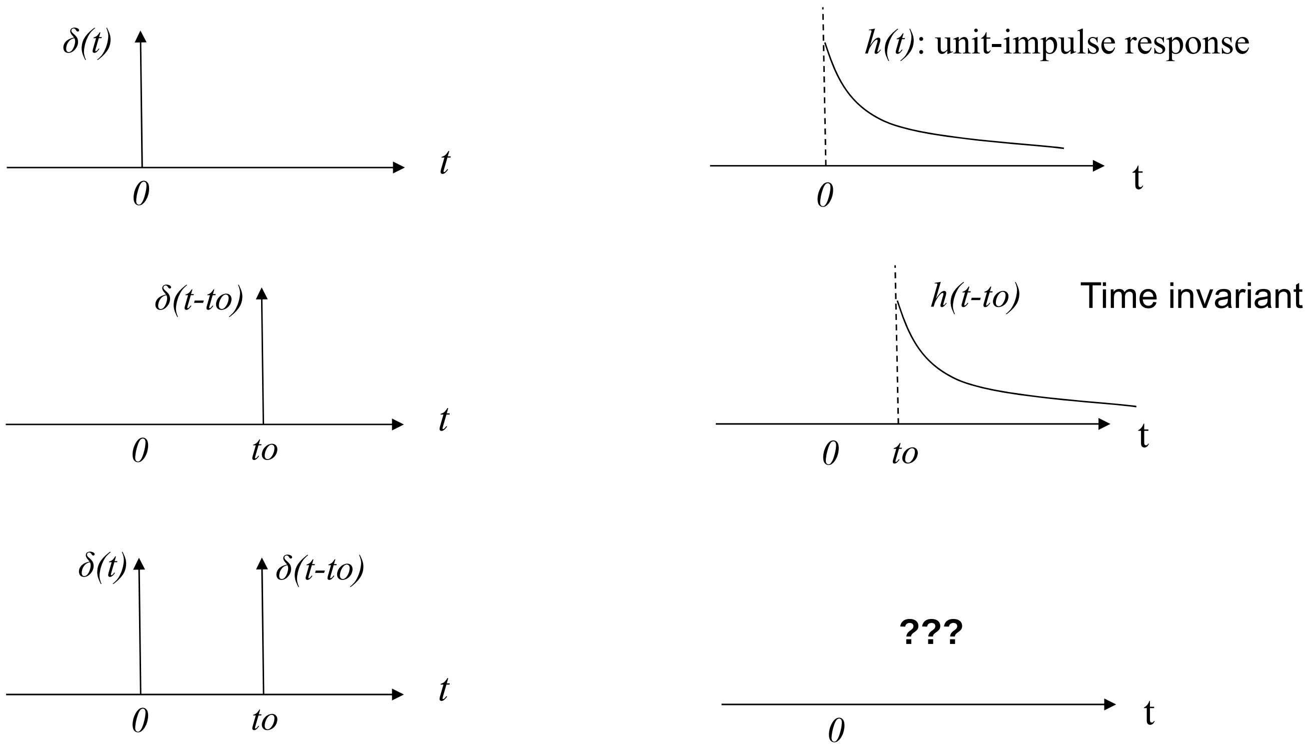
$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(\tau) g(t-\tau) d\tau.$$

Physical interpretation of linear system response



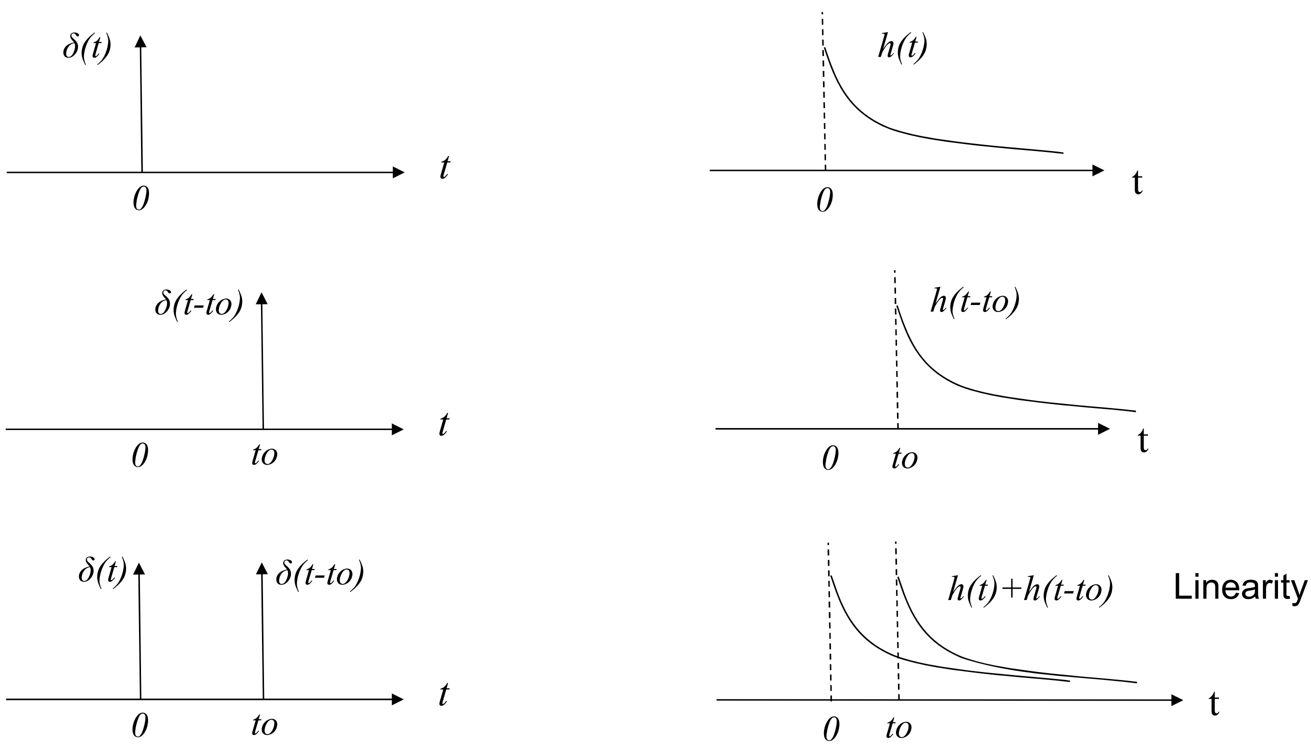
Physical interpretation of linear system response

Input **Output**

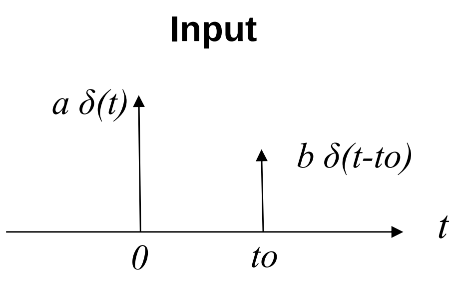


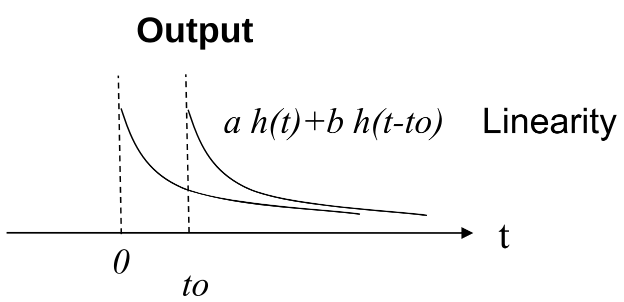
Physical interpretation of linear system response

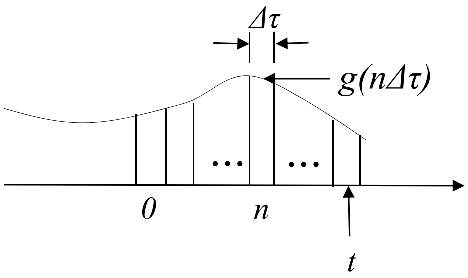
Input **Output**



Physical interpretation of linear system response





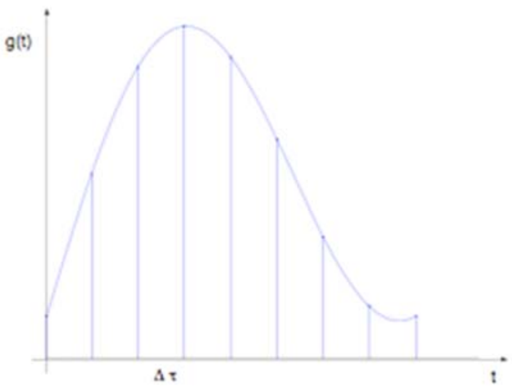


input $g(n\Delta\tau)$: output $g(n\Delta\tau)\Delta\tau\ h(t\text{-}n\Delta\tau)$

$$y(t) = \sum g(n\Delta\tau)\Delta\tau\ h(t\text{-}n\Delta\tau)$$

$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau$$

Intuitive explanation of the convolution formula

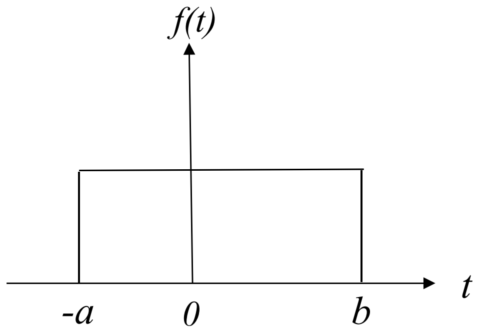


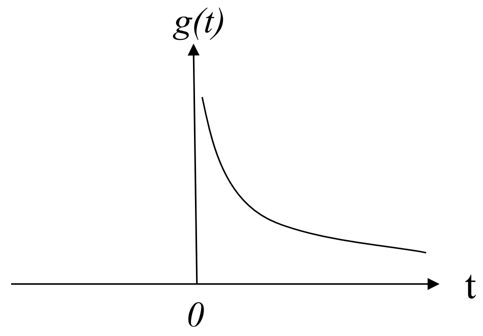
- $g(t)$ can be approximated as $g(t) \cong \Sigma_n g(n\Delta\tau)\Delta\tau\delta(t-n\Delta\tau)$.
- In the limit as $\Delta\tau \to 0$ this approximation approaches the true function $g(t)$.
- The response $\hat{y}(t)$ of the LTI system to the input as $\Sigma_n g(n\Delta\tau)\Delta\tau\delta(t-n\Delta\tau)$ is going to be $\Sigma_n g(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau$.
- Thus, $y(t) = \lim_{\Delta\to 0}\Sigma_n g(n\Delta\tau)h(t-n\Delta\tau)\Delta\tau =$

$$\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau.$$

Graphical Interpretation of Convolution (1)

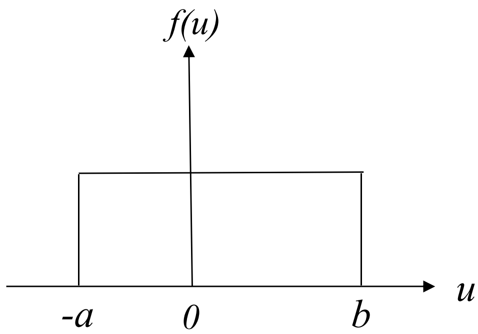
$$f(t) * g(t) = \int_{u=-\infty}^{\infty} f(u) g(t-u) du$$



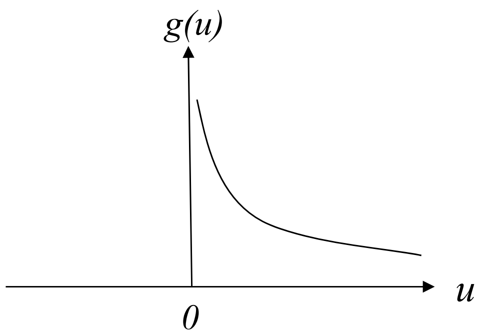


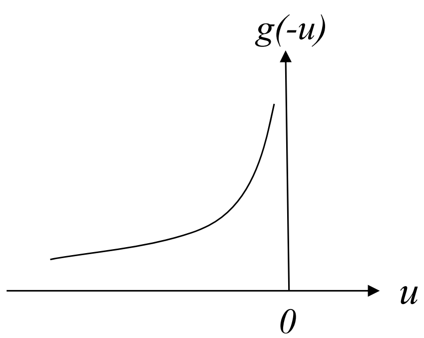
Graphical Interpretation of Convolution (2)

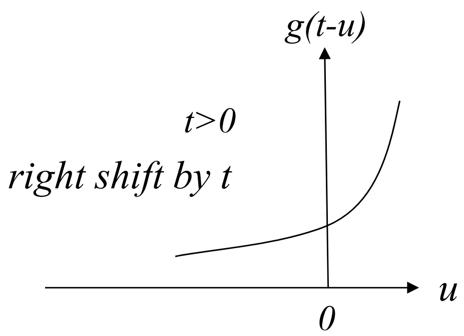
$$f(t) * g(t) = \int_{u=-\infty}^{\infty} f(u) g(t-u) du$$

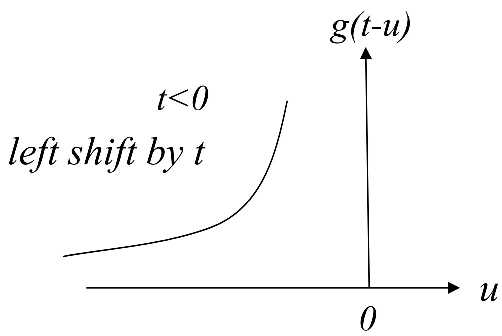


a, b: positive



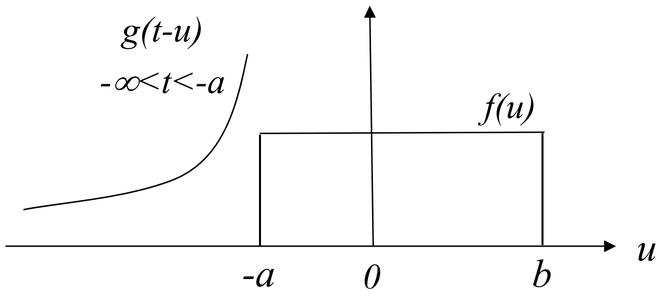


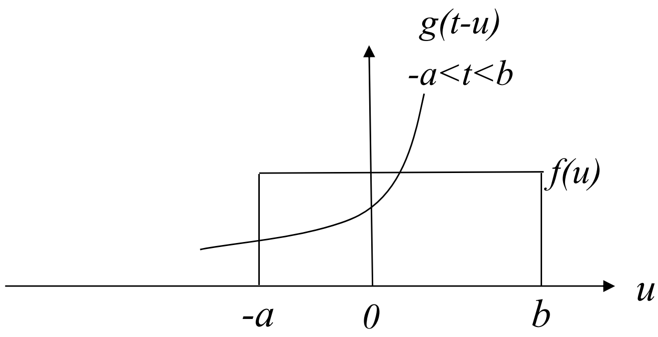


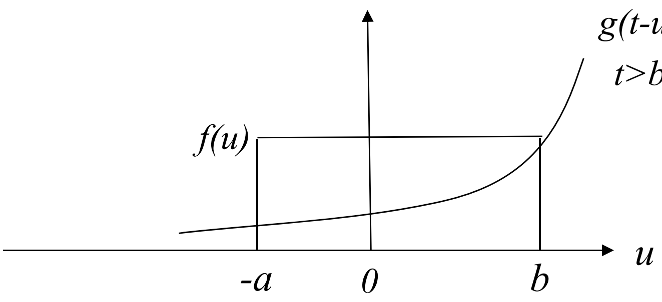


Graphical Interpretation of Convolution (3)

$$f(t) * g(t) = \int_{u=-\infty}^{\infty} f(u) g(t-u) du$$







Depending on *t*, the convolution integral is the area under *f(u)g(t-u)*.

Search “Convolution” on the Wikipedia site for an animation of convolution.

Convolution in the frequency domain

The convolution of two functions $g(t)$ and $h(t)$, denoted by $g(t) * h(t)$, is defined by the integral

$$y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(x) g(t-x) dx.$$

If $g(t) \Leftrightarrow G(\omega)$ and $h(t) \Leftrightarrow H(\omega)$ then the convolution reduces to a product in the Fourier domain

$$y(t) = h(t) * g(t) \Leftrightarrow Y(w) = H(\omega) G(\omega).$$

$H(\omega)$ is called the **system transfer function or the system frequency response** or the **spectral response**.

Notice that, for symmetry, a product in the time domain corresponds to a convolution in frequency domain. That is

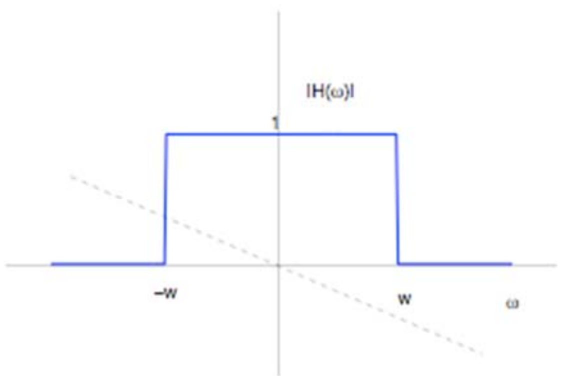
$$g_1(t) g_2(t) \Leftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega).$$

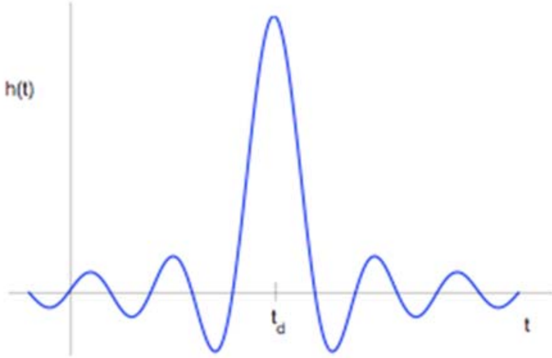
Bandwidth of the product of two signals

If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively.

The bandwidth of $g_1(t)\, g_2(t)$ is $B_1 + B_2$ Hz.

Ideal Low-Pass Filter





Ideal low-pass filter response

$$H(\omega) = \mathrm{rect}\left(\frac{\omega}{2W}\right) e^{-j\omega t_d}$$

Ideal low-pass filter impulse response

$$h(t) = \frac{W}{\pi} \mathrm{sinc}\left[W\left(t - t_d\right)\right]$$

Ideal High-Pass and Band-pass filters

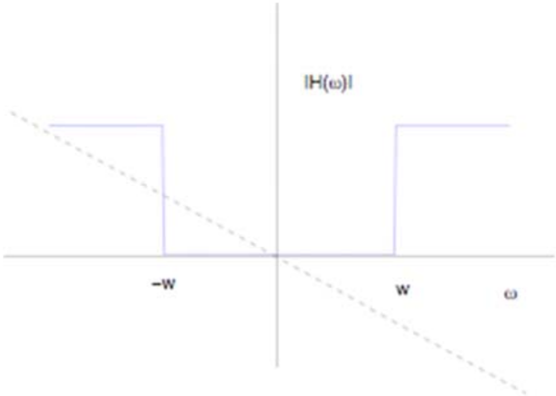


Figure 1: Ideal high-pass filter

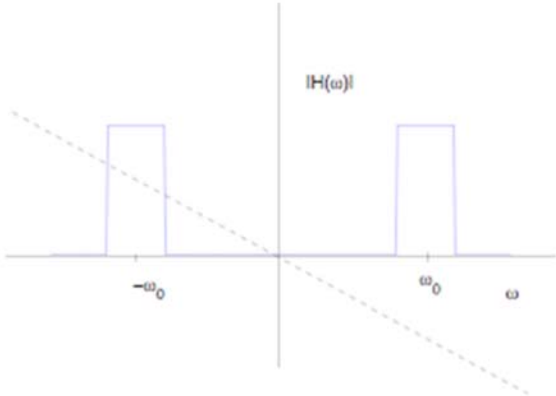


Figure 2: Ideal band-pass filter

Practical filters

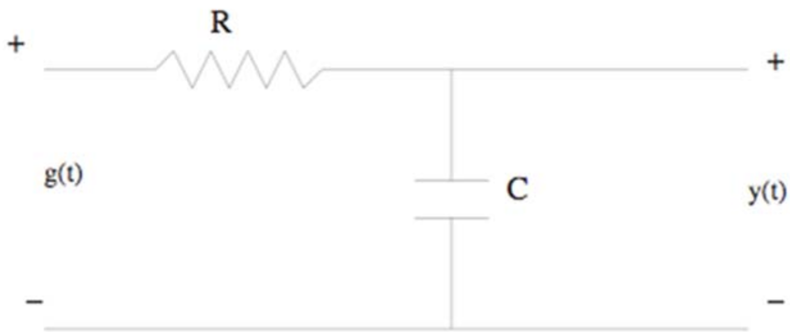
- The filters in the previous examples are ideal filters.

- They are not realizable since their unit impulse responses are everlasting (Think of the sinc function).

- Physically realizable filter impulse response $h(t) = 0$ for $t < 0$.

- Therefore, we can only obtain approximated version of the ideal low-pass, high-pass and band-pass filters.

Example of a linear system: RC circuit



Example: RC circuit (continued)

$$H(\omega) = \frac{1/j\omega C}{R + \left(1/j\omega C\right)} = \frac{1}{1 + j\omega RC} = \frac{a}{a + j\omega}$$

$$a = \frac{1}{RC}$$

and

$$\left|H(\omega)\right| = \frac{a}{\sqrt{a^2 + \omega^2}} \Rightarrow \left|H(0)\right| = 1, \lim_{w \to \infty} \left|H(\omega)\right| = 0.$$

$$\theta_h\left(\omega\right) = -\tan^{-1}\frac{\omega}{a}$$

Therefore, this circuit behaves as a low-pass filter.

Summary

- Linear time invariant systems
- Unit impulse response function
- Convolution formula: $y(t) = h(t) * g(t) = \int_{-\infty}^{\infty} h(\tau) g(t-\tau) d\tau$
- Low-pass, high-pass and band-pass filters

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Lecture eight

Lecture Aims

- To introduce Energy spectral density (ESD)
- Input and Output Energy spectral densities
- To introduce Power spectral density (PSD)
- Input and Output Power spectral densities

Signal Energy, Parseval's Theorem

Consider an energy signal $g(t)$, Parseval's Theorem states that

$$E_g = \int_{t=-\infty}^{\infty} |g(t)|^2 \, dt = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |G(\omega)|^2 \, d\omega$$

Proof:

$$E_g = \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega$$

Example

Consider the signal $g(t) = e^{-at}u(t)$ $(a > 0)$

Its energy is

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$$

We now determine E_g using the signal spectrum $G(\omega)$ given by

$$G(\omega) = \frac{1}{j\omega + a}$$

It follows

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{2\pi a}\left[\tan^{-1}\frac{\omega}{a}\right]_{-\infty}^{\infty} = \frac{1}{2a}$$

which verifies Parseval's theorem.

Energy Spectral Density

- Parseval's theorem can be interpreted to mean that the energy of a signal $g(t)$ is the result of energies contributed by all spectral components of a signal $g(t)$

- The contribution of a spectral component of frequency ω is proportional to $|G(\omega)|^2$

- Therefore, we can interpret $|G(\omega)|^2$ as the energy per unit bandwidth of the spectral components of $g(t)$ centered at frequency ω

- In other words, $|G(\omega)|^2$ is the energy spectral density of $g(t)$

Energy Spectral Density (continued)

The **energy spectral density (ESD)** $\Psi(\omega)$ is thus defined as

$$\Psi(\omega) = |G(\omega)|^2$$

and

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)d\omega$$

Thus, the ESD of the signal $g(t) = e^{-at}u(t)$ of the previous example is

$$\Psi(\omega) = |G(\omega)|^2 = \frac{1}{\omega^2 + a^2}$$

Energy of modulated signals (important)

Let $g(t)$ be a baseband energy signal with energy E_g.

The energy of the modulated signal $\varphi(t)$ = $g(t)\cos\omega_0 t$ is half the energy of $g(t)$.
That is,

$$E_\varphi = \frac{1}{2}E_g.$$

Proof: Go from the definition of energy being the integration of the magnitude squared of the signal over the whole time horizon. (ω_0 is assumed to be equal to or larger than 2π times the bandwidth of *g(t)*.)

The same applies to power signals. That is, if $g(t)$ is a power signal then

$$P_\varphi = \frac{1}{2}P_g.$$

(You will use this result when computing the efficiency of a Full AM system).

Time Autocorrelation Function and ESD

For a real signal the autocorrelation function $\psi_g(\tau)$ is defined as

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)dt$$

Do you remember the correlation of two signals (lecture three)? The autocorrelation function measure the correlation between $g(t)$ and all its translated versions.

Notice

$$\psi_g(\tau) = \psi_g(-\tau).$$

and

$$g(\tau) * g(-\tau) = \psi_g(\tau).$$

But, most important...

Time Autocorrelation Function and ESD

...the Fourier transform of the autocorrelation function is the Energy Spectral Density! That is

$$\psi_g(\tau) \Leftrightarrow \Psi(\omega) = \left|G(\omega)\right|^2$$

Proof:

$$F\left[\psi_g(\tau)\right] = \int_{-\infty}^{\infty} e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} g(t)g(t+\tau)dt\right]d\tau$$
$$= \int_{-\infty}^{\infty} g(t)\left[\int_{-\infty}^{\infty} g(\tau+t)e^{-j\omega\tau}d\tau\right]dt$$

The Fourier transform of $g(\tau+t)$ is $G(\omega)$ e$^{j\omega t}$. Therefore,

$$F\left[\psi_g(\tau)\right] = G(\omega)\int_{-\infty}^{\infty} g(t)e^{j\omega t}dt = G(\omega)G(-\omega) = \left|G(\omega)\right|^2$$

ESD of the Input and the Output

If $g(t)$ and $y(t)$ are the input and the corresponding output of a LTI system, then

$$Y(\omega) = H(\omega)G(\omega).$$

Therefore,

$$|Y(\omega)|^2 = |H(\omega)|^2 \, |G(\omega)|^2 .$$

This shows that

$$\Psi_y(\omega) = |H(\omega)|^2 \, \Psi_g(\omega).$$

Thus, the output signal ESD is $|H(\omega)|^2$ times the input signal ESD.

Signal Power and Power Spectral Density

The power P_g of a real signal $g(t)$ is given by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)dt.$$

All the results for energy signals can be extended to power signals.
Call $S_g(\omega)$ the Power Spectral Density (PSD) of $g(t)$. Thus,

$$P_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega)d\omega.$$

$S_g(\omega)$ can be found using the autocorrelation function.

Time autocorrelation Function of Power Signals

The (time) autocorrelation function $R_g(\tau)$ of a real deterministic power signal $g(t)$ is defined as

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t+\tau)dt$$

We have that

$$R_g(\tau) \Leftrightarrow S_g(\omega)$$

If $g(t)$ and $y(t)$ are the input and the corresponding output of a LTI system, then

$$S_y(\omega) = \left|H(\omega)\right|^2 S_g(\omega).$$

Thus, the output signal PSD is $|H(\omega)|^2$ times the input signal PSD.

Relationships among these signals and functions

Input

$$g(t) \Leftrightarrow G(\omega)$$

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t+\tau)dt$$

$$R_g(\tau) \Leftrightarrow S_g(\omega) = \lim_{T\to\infty} \frac{1}{T} |G(\omega)|^2$$

LTI system

$$h(t) \Leftrightarrow H(\omega)$$

Output

$$y(t) = h(t) * g(t) \Leftrightarrow H(\omega)G(\omega)$$

$$R_y(\tau) \Leftrightarrow S_y(\omega)$$

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{T} |H(\omega)|^2 |G(\omega)|^2$$
$$= |H(\omega)|^2 S_g(\omega)$$

Output PSD is $|H(\omega)|^2$ times

the input signal PSD.

Conclusions

We learned about

- Energy and Power Spectral Densities
- Time autocorrelation functions
- Input and output energies and powers

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Lecture nine

Lecture Aims

- To examine modulation process
- Baseband and bandpass signals
- Double Sideband Suppressed Carrier (DSB-SC)
  - Modulation
  - Demodulation
- Modulators
  - Nonlinear modulators
  - Switching modulators
  - Diode modulators

Modulation

- Modulation is a process that causes a shift in the range of frequencies in a signal.

- Two types of communication systems
  - Baseband communication: communication that does not use modulation
  - Carrier modulation: communication that uses modulation

- The baseband is used to designate the band of frequencies of the source signal. (e.g., audio signal 4kHz, video 4.3MHz)

Modulation (continued)

In analog modulation the basic parameter such as amplitude, frequency or phase of a sinusoidal carrier is varied in proportion to the baseband signal $m(t)$. This results in amplitude modulation (AM) or frequency modulation (FM) or phase modulation (PM).

The baseband signal $m(t)$ is the modulating signal.

The sinusoid is the carrier or modulator.

Why modulation?

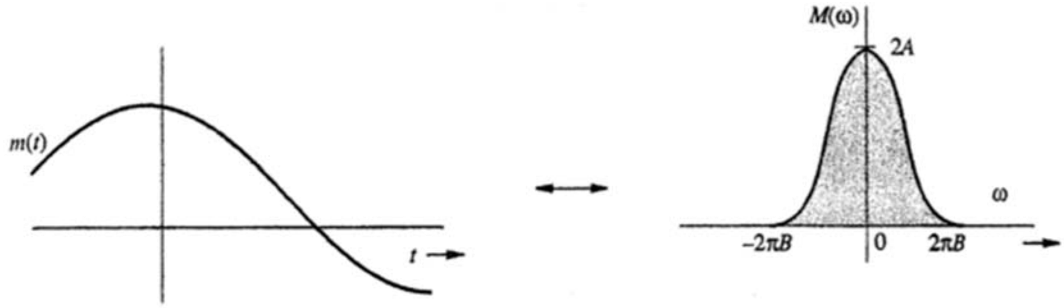
- To use a range of frequencies more suited to the medium

- To allow a number of signals to be transmitted simultaneously (frequency division multiplexing)

- To reduce the size of antennas in wireless links

Amplitude Modulation

- Carrier $A\cos(\omega_c t + \theta_c)$
  - Phase is constant $\theta_c = 0$
  - Frequency is constant
- Modulating signal $m(t)$

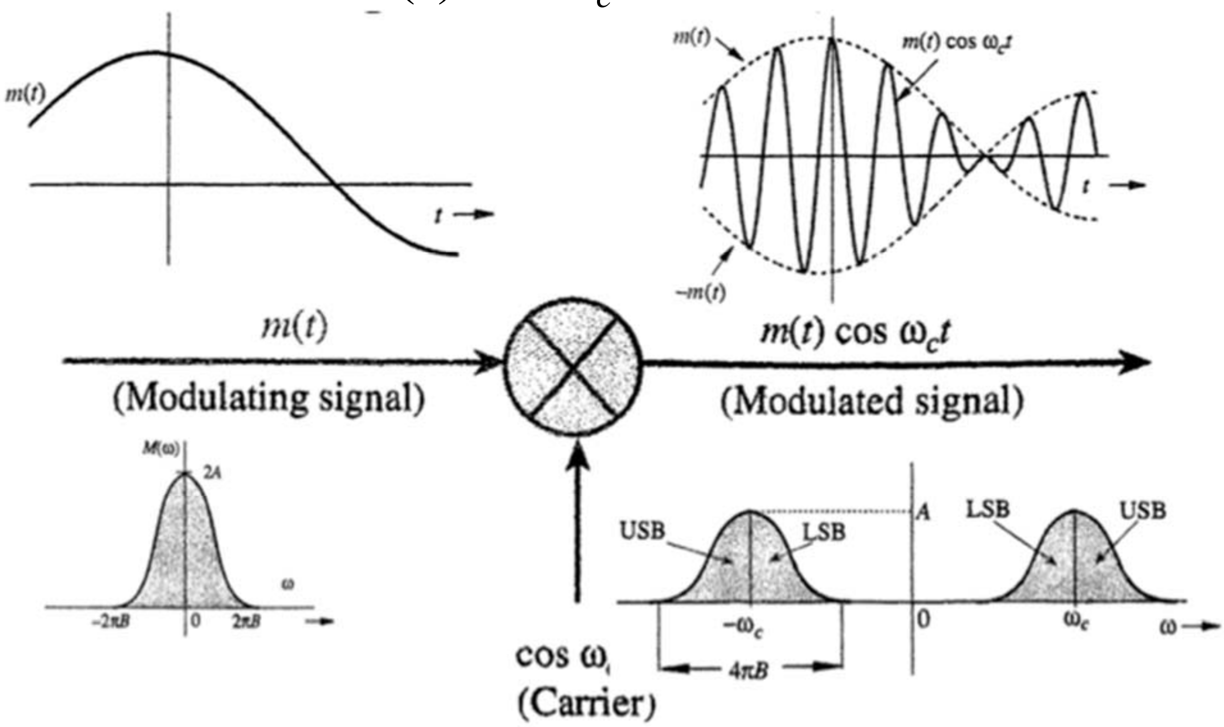


- With amplitude spectrum

  $m(t) \Leftrightarrow M(\omega)$

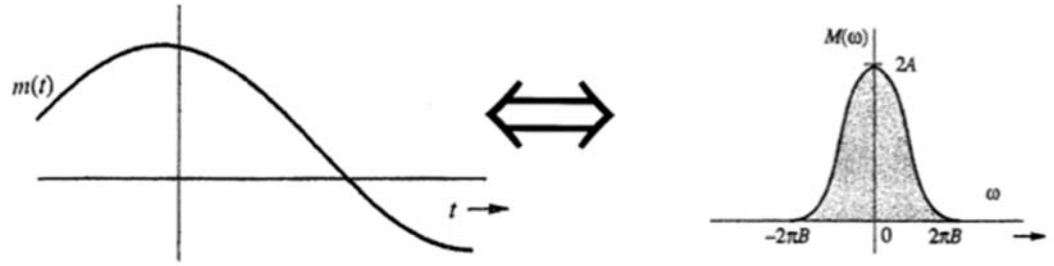
Modulated signal

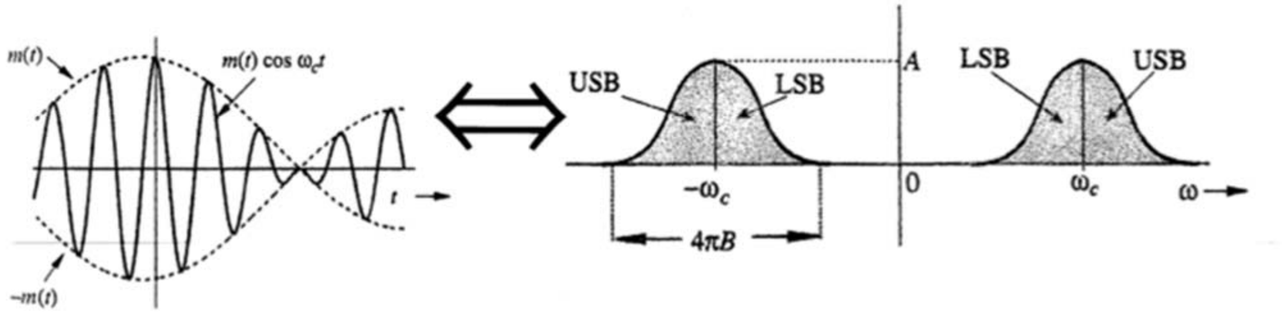
- Modulated signal: $m(t)\cos\omega_c t$



Modulated signal

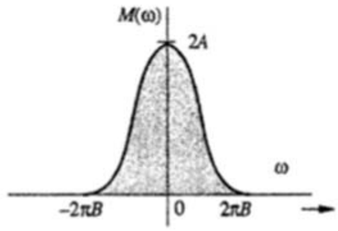
- Modulated signal: $m(t)\cos\omega_c t$



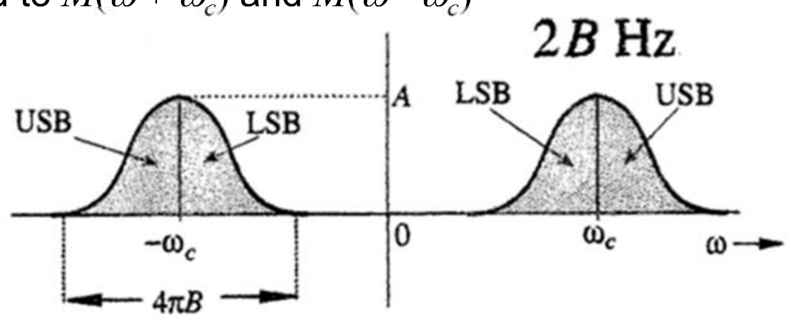


Modulated signal

- Baseband spectrum: $B\,Hz$



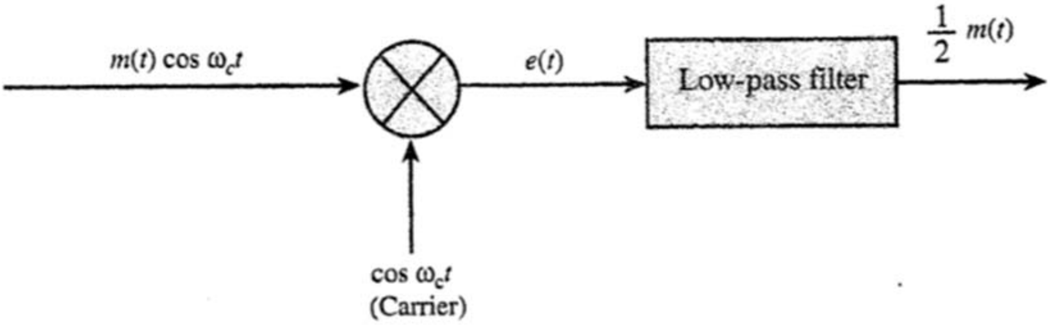
- $M(\omega)$ is shifted to $M(\omega + \omega_c)$ and $M(\omega - \omega_c)$



$$m(t)\cos\omega_c t \Leftrightarrow \frac{1}{2}\left[M(\omega+\omega_c)+M(\omega-\omega_c)\right]$$

Demodulation of DSB signal

- Process modulated signal $m(t)\cos\omega_c t$



- Multiply modulated signal with $\cos\omega_c t$

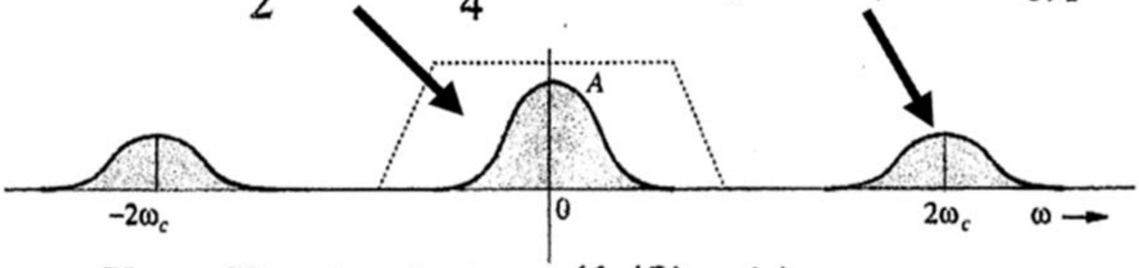
$$e(t) = m(t)\cos^2 \omega_c t = \frac{1}{2}\left[m(t) + m(t)\cos 2\omega_c t\right]$$

$$E(\omega) = \frac{1}{2}M(\omega) + \frac{1}{4}\left[M(\omega + 2\omega_c) + M(\omega - 2\omega_c)\right]$$

Demodulation of DSB signal

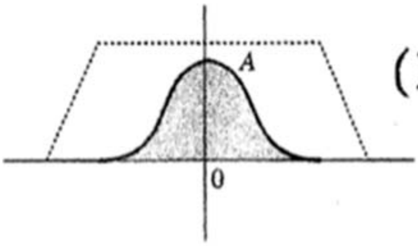
- Process modulated signal $m(t)\cos\omega_c t$

$$E(\omega) = \frac{1}{2}M(\omega) + \frac{1}{4}[M(\omega + 2\omega_c) + M(\omega - 2\omega_c)]$$



– Use a filter to select $(1/2)m(t)$

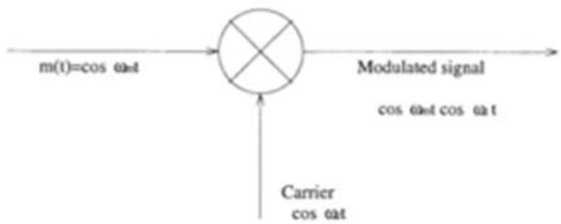
remove



$(1/2)m(t)\cos 2\omega_c t$

Example

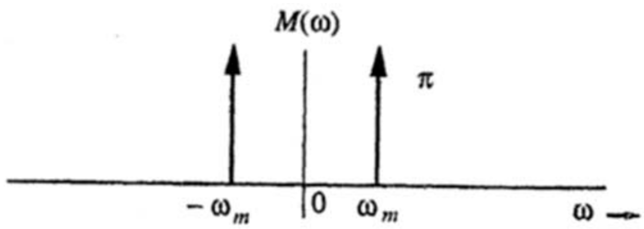
- Modulating signal $m(t)=\cos\omega_m t$
- Carrier $\cos\omega_c t$
- Modulated signal $\phi(t) = m(t)\cos\omega_c t = \cos\omega_m t\cos\omega_c t$



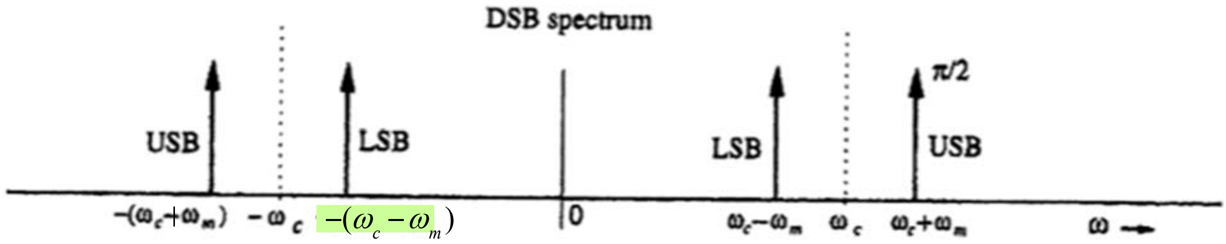
Amplitude spectrum

- Baseband signal

$$M(\omega) = \pi\left[\delta(\omega-\omega_m)+\delta(\omega+\omega_m)\right]$$

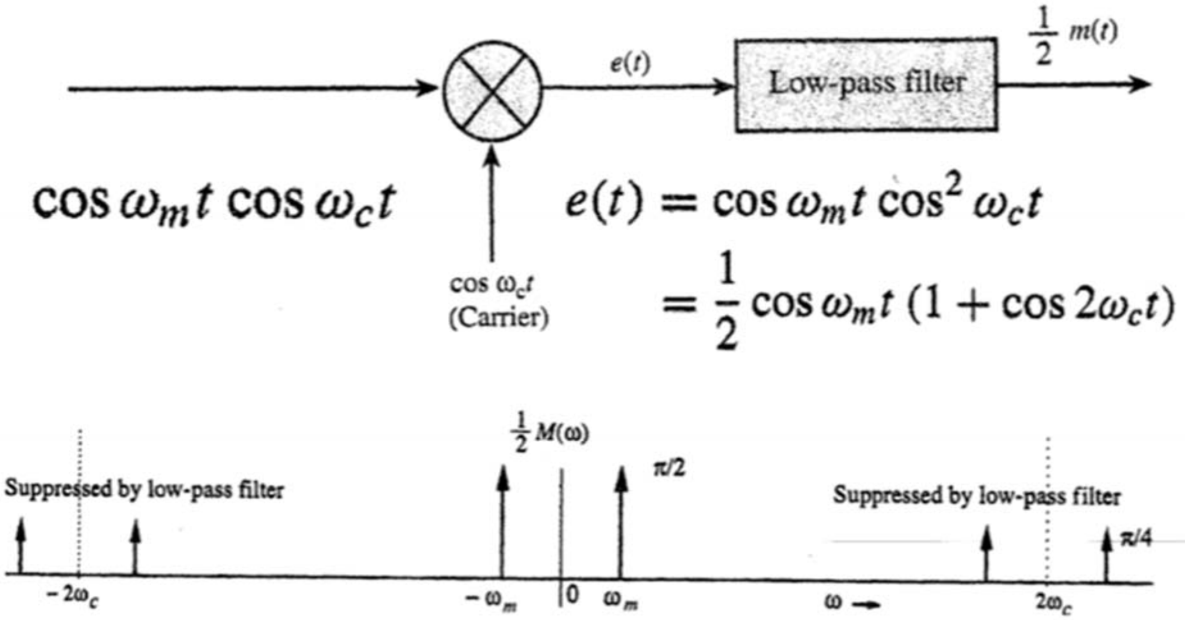


$$\varphi_{DSB-SC}(t) = \frac{1}{2}\left[\cos(\omega_c+\omega_m)t+\cos(\omega_c-\omega_m)t\right]$$



Demodulation of DSB signal

- Process modulated signal $m(t)\cos\omega_c t$



Modulators

- We need to implement multiplication $m(t)\cos\omega_c t$
- We can use
  - Nonlinear modulators
  - Switching modulators
- Switching modulators can be implemented using diode ring modulators

Nonlinear modulator

- Input-output characteristics of a nonlinear element

$$y(t) = ax(t) + bx^2(t)$$

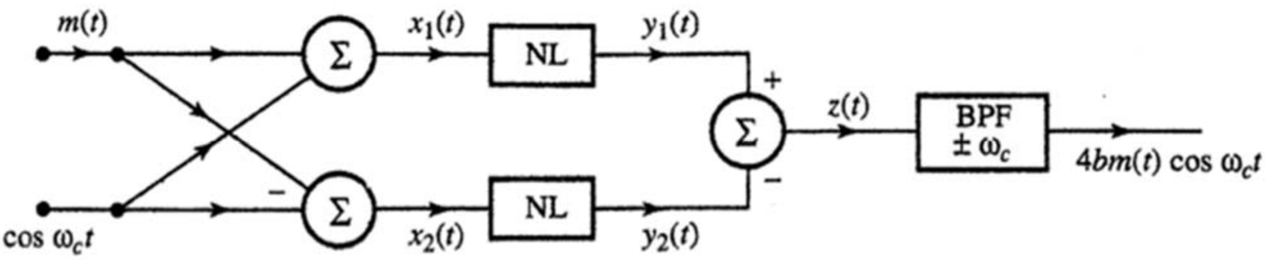
- Where $x(t)$ is the input signal and $y(t)$ is the output signal
- Consider to input signals

$$x_1(t) = \cos\omega_c t + m(t)$$
$$x_2(t) = \cos\omega_c t - m(t)$$

Nonlinear modulator

- Let us implement

$$z(t) = y_1(t) - y_2(t)$$
$$= \left[ax_1(t) + bx_1^{\ 2}(t)\right] - \left[ax_2(t) + bx_2^{\ 2}(t)\right]$$



$$z(t) = 2am(t) + 4bm(t)\cos\omega_c t$$

Switching modulator

- Consider a periodic signal of fundamental frequency ω_c

$$\phi(t) = \sum_{n=0}^{\infty} C_n \cos(n\omega_c t + \theta_n)$$

- Multiplication of modulating signal with this periodic signal gives

$$m(t)\phi(t) = \sum_{n=0}^{\infty} C_n m(t) \cos(n\omega_c t + \theta_n)$$

- The spectrum of the product $m(t)\phi(t)$ is the spectrum $M(\omega)$ shifted to

$$\pm\omega_c, \pm 2\omega_c, \ldots, \pm n\omega_c, \ldots$$

Square Pulse train as a modulator

- Consider a square pulse train

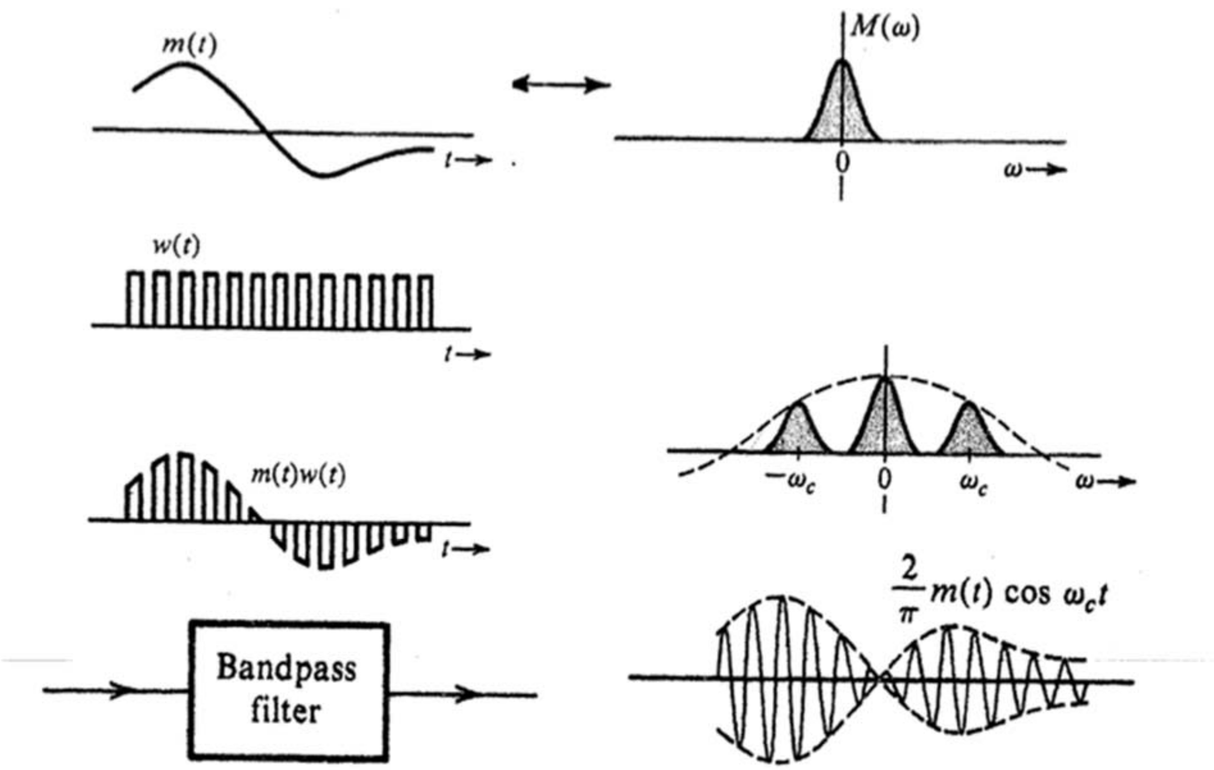
- The Fourier series for this periodic waveform is

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \cdots \right)$$

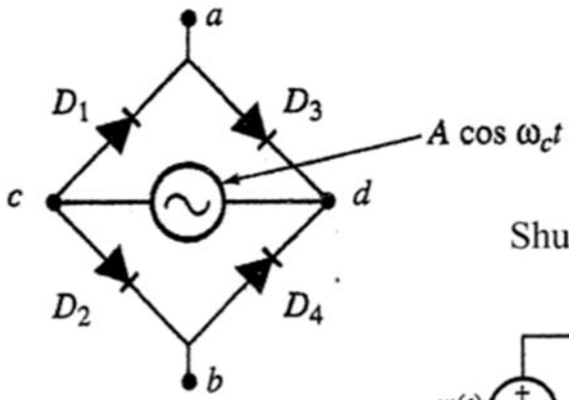
- The signal $m(t)w(t)$ is

$$m(t)w(t) = \frac{1}{2}m(t) + \frac{2}{\pi}\left[m(t)\cos\omega_c t - \frac{1}{3}m(t)\cos 3\omega_c t + \cdots \right.$$

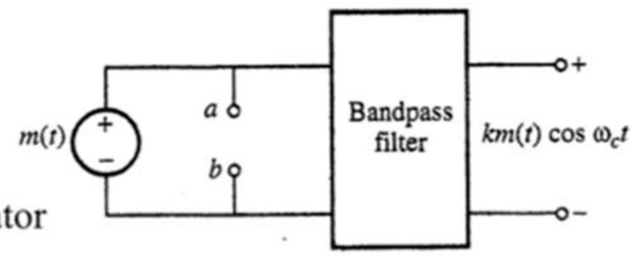
Switching modulator



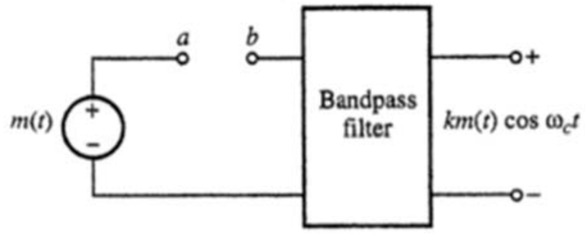
Diode Switches



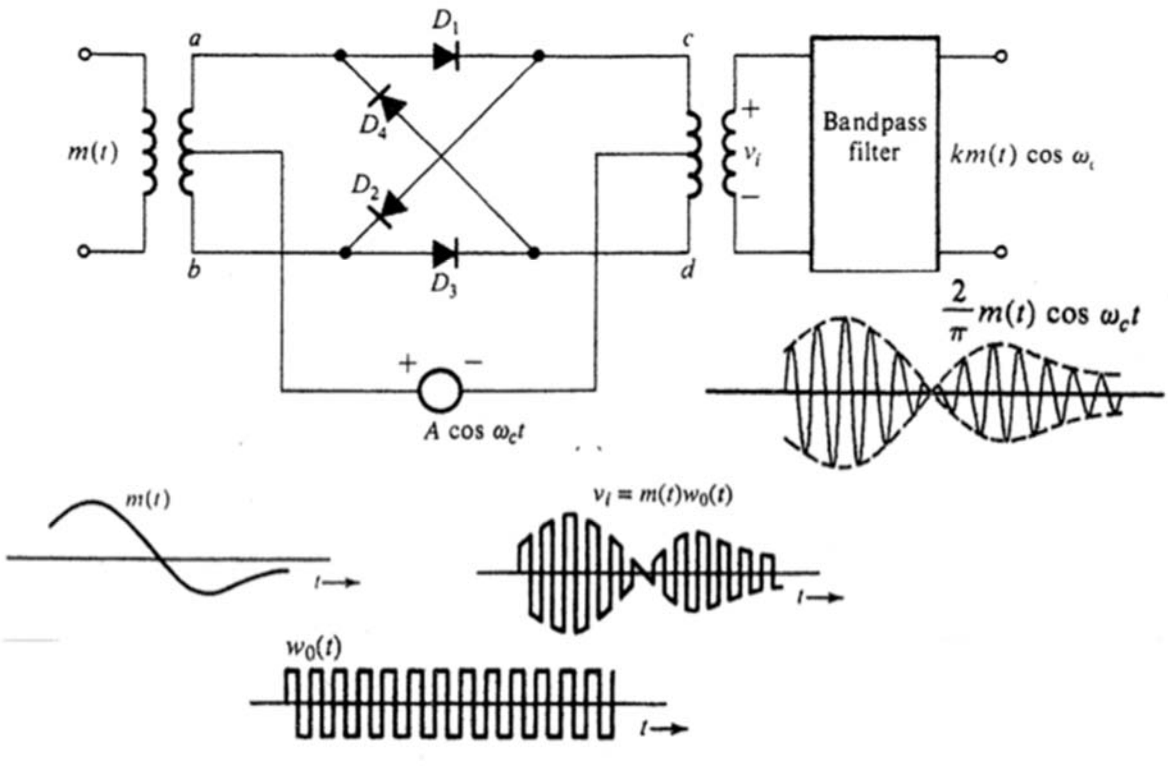
Shunt-bridge diode modulator



Series-bridge diode modulator



Ring modulator



Ring modulator

$$w_0(t) = \frac{4}{\pi}\left(\cos \omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \cdots \right)$$

$$v_i(t) = m(t)w_0(t) = \frac{4}{\pi}\left[m(t)\cos \omega_c t - \frac{1}{3}m(t)\cos 3\omega_c t + \frac{1}{5}m(t)\cos 5\omega_c t - \cdots \right]$$

Conclusions

We learned about

- Baseband and Carrier transmission
- Amplitude modulation (DSB-SC)
- Non-linear modulator
- Switching modulator
- Diode switches

EE1 and ISE1: Introduction to Signals and Communications

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Lecture ten

Lecture Aims

- To examine full AM process
- AM signal and its envelope
- Sideband carrier power
- Generation of AM signals
- Demodulation of AM signals

Double Sideband Suppressed Carrier

- A receiver must generate a carrier in frequency and phase synchronism with the carrier at the transmitter
- This calls for sophisticated receiver and could be quite costly
- An alternative is for the transmitter to transmit the carrier along with the modulated signal
- In this case the transmitter needs to transmit much larger power

Amplitude Modulation

- Carrier $A \cos(\omega_c t + \theta_c)$
  - Phase is constant $\theta_c = 0$
  - Frequency is constant.
- Modulation signal $m(t)$
- With amplitude spectrum $m(t) \Leftrightarrow M(\omega)$
- Full AM signal is

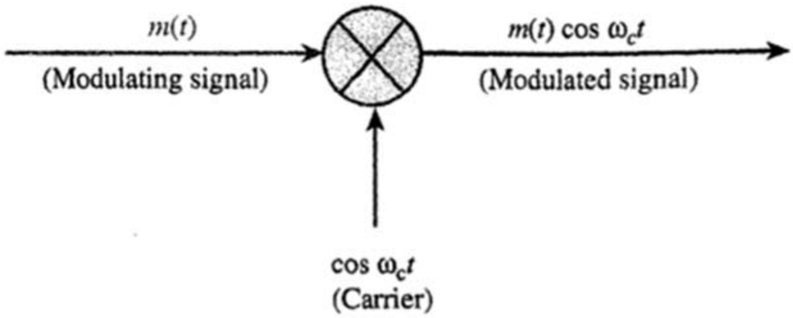
$$\varphi_{AM}(t) = A\cos\omega_c t + m(t)\cos\omega_c t$$
$$= \left[A + m(t)\right]\cos\omega_c t$$

- Spectrum of full AM signal

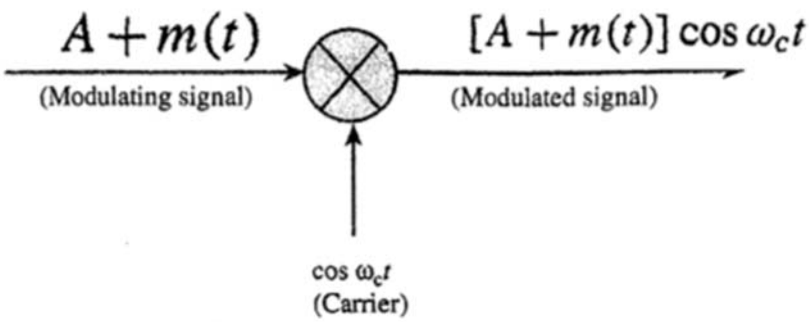
$$\varphi_{AM}(t) \Leftrightarrow \frac{1}{2}\left[M(\omega+\omega_c) + M(\omega-\omega_c)\right] + \pi A\left[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)\right]$$

Full AM Modulated signal

- DSB Modulated signal:

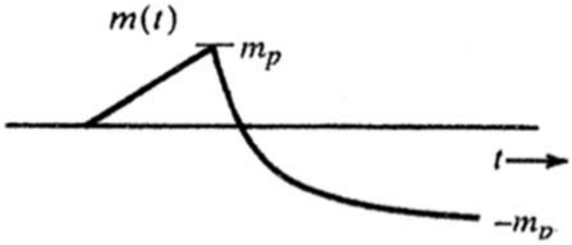


- Full AM signal



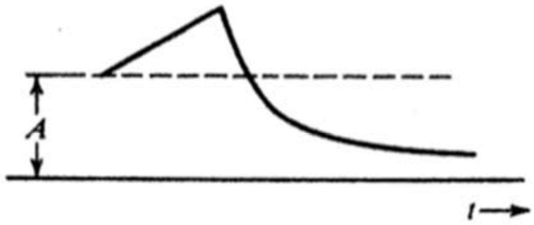
Full AM Modulated signal

- Signal



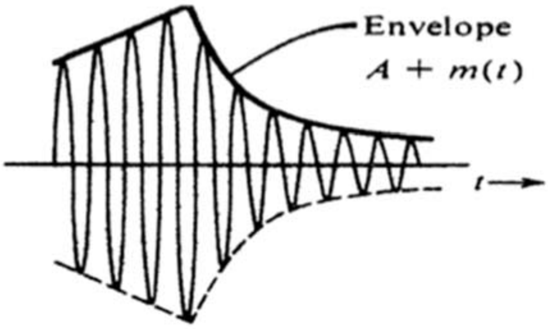
$A + m(t) > 0$ for all t

- Modulating signal



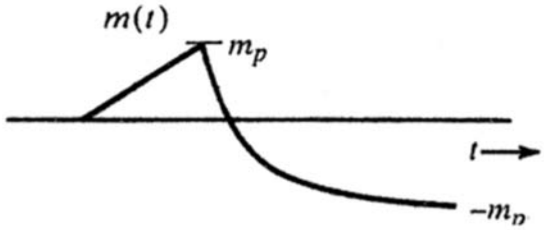
- Modulated signal:

$\left[A + m(t)\right]\cos\omega_c t$



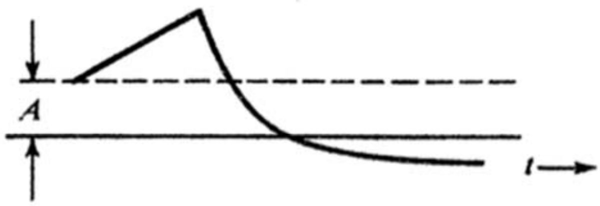
Envelope detection is not possible when

- Signal



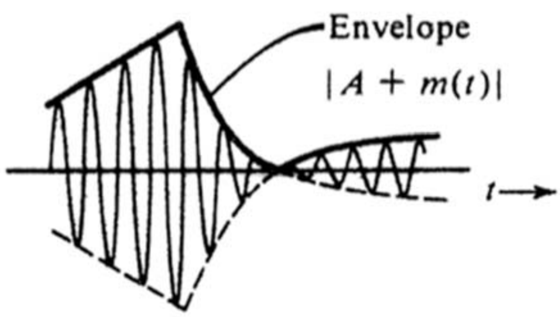
$A + m(t) \not> 0$ for all t

- Modulating signal



- Modulated signal:

$\left[A + m(t)\right]\cos\omega_c t$



Envelope detection condition

- Detection condition $A + m(t) \geq 0$
- Let m_p be the maximum negative value of $m(t)$. This means that $m(t) \geq -m_p$
- When we have $A \geq m_p$, we can use envelope detector
- The parameter $\mu = \frac{m_p}{A}$ is called the modulation index
- When $0 \leq \mu \leq 1$, we can use an envelope detector

Envelope detection example

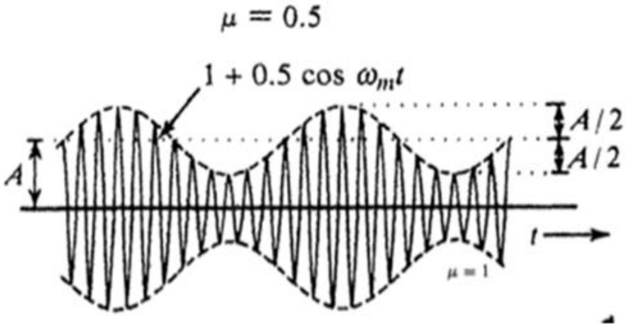
- Modulating signal $m(t) = B\cos\omega_m t$
- Modulating signal amplitude is $m_p = B$
- Hence $\mu = \frac{B}{A}$ and $B = \mu A$
- Modulating and modulated signals are

$$m(t) = B\cos\omega_m t = \mu A\cos\omega_m t$$

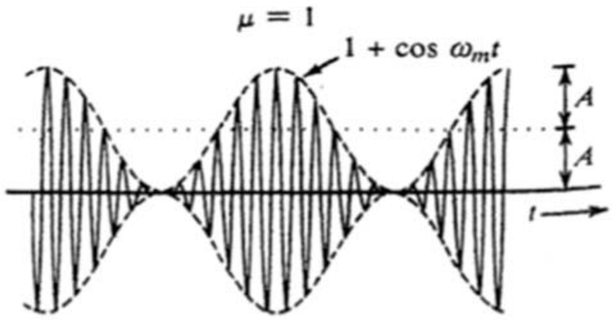
$$\varphi_{AM}(t) = [A + m(t)]\cos\omega_c t = A[1 + \mu\cos\omega_m t]\cos\omega_c t$$

Demodulation of DSB signal

- Consider modulation index to be $\mu = 0.5$



- For modulation index $\mu = 1$



Sideband and Carrier power

- Consider full AM signal

$$\varphi_{AM}(t) = \underbrace{A\cos\omega_c t}_{\text{carrier}} + \underbrace{m(t)\cos\omega_c t}_{\text{sidebands}}$$

- Power P_c of the carrier $A \cos \omega_c t$

$A^2/2$

- Power P_s of the sideband signals

$0.5\,\widetilde{m^2(t)}$

- Power efficiency

$$\eta = \frac{\text{useful power}}{\text{total power}} = \frac{P_s}{P_c + P_s} = \frac{\widetilde{m^2(t)}}{A^2 + \widetilde{m^2(t)}}100\%$$

Maximum power efficiency of Full AM

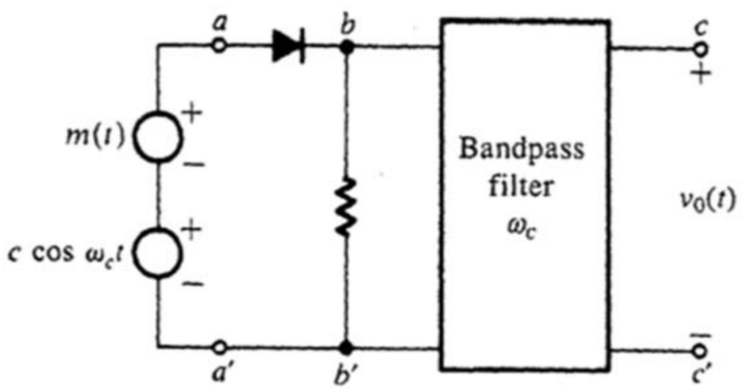
- When we have $m(t) = \mu A \cos \omega_m t$
- Signal power is $\widetilde{m^2(t)} = \dfrac{(\mu A)^2}{2}$
- When $0 \le \mu \le 1$
- When modulation index is unity, the efficiency is

  $\eta_{\max} = 33\%$
- When μ=0.3 the efficiency is

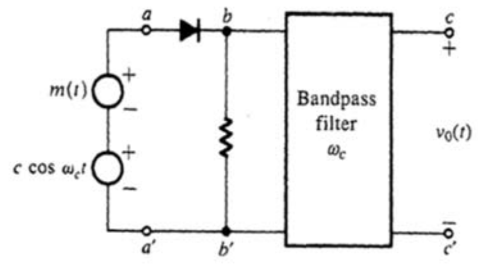
$$\eta = \frac{(0.3)^2}{2 + (0.3)^2} 100\% = 4.3\%$$

Generation of AM signals

- Full AM signals can be generated using DSB-SC modulators
- But we do not need to suppress the carrier at the output of the modulator, hence we do not need a balanced modulators
- Use a simple diode



Simple diode modulator design



- Input signal $c\cos\omega_c t + m(t)$
- Consider the case $c >> m(t)$
- Switching action of the diode is controlled by

  $c\cos\omega_c t$

- A switching waveform

  is generated. The diode open and shorts periodically with $w(t)$

- The signal is generated

$$v_{bb'}(t) = \left[c\cos\omega_c t + m(t)\right]w(t)$$

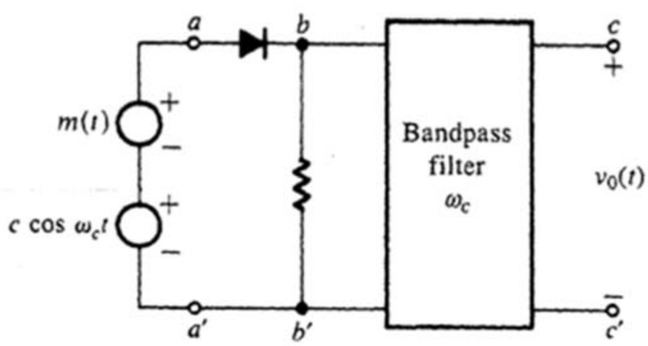
Diode Modulator

- Diode acts as a multiplier

$$v_{bb\mathrm{n}}(t) = \left[c\cos\omega_c t + m(t)\right]w(t)$$

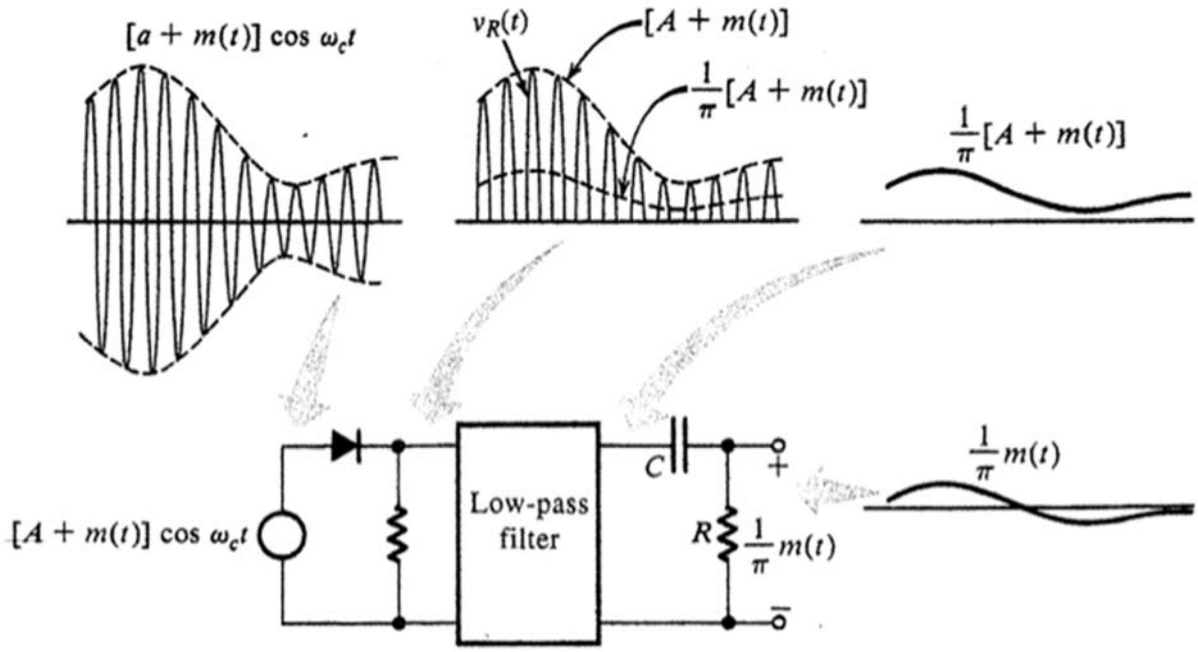
$$= \left[c\cos\omega_c t + m(t)\right]\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \cdots\right)\right]$$

$$= \underbrace{\frac{c}{2}\cos\omega_c t + \frac{2}{\pi}m(t)\cos\omega_c t}_{\text{AM}} + \underbrace{\text{other terms}}_{\text{suppressed by bandpass filter}}$$



Demodulation of AM signals

- Rectifier detector



Demodulation of AM signals

- Half-wave rectified signal υ_R is given by

$$\upsilon_R = \{[A + m(t)]\cos\omega_c t\}\, w(t)$$

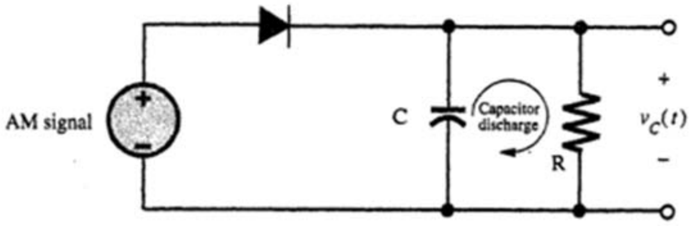
where $w(t)$

$$v_R = [A + m(t)]\cos\omega_c t\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \cdots\right)\right]$$

$$= \frac{1}{\pi}[A + m(t)] + \text{other terms of higher frequencies}$$

Demodulation of AM signals using an envelope detector

- Simple detector



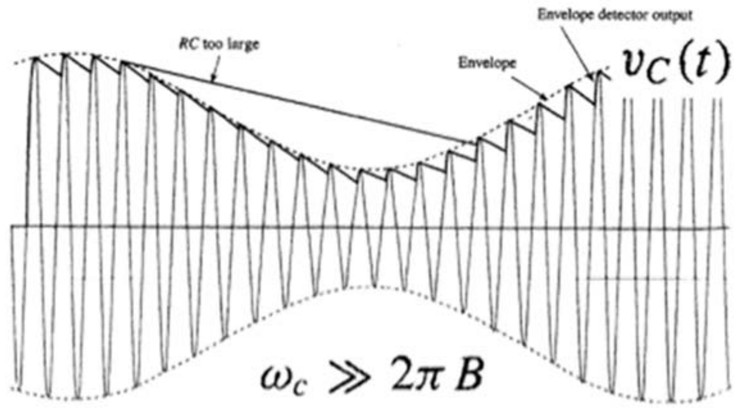
Large

$RC \gg 1/\omega_c$

But smaller than

$1/2\pi B$

- Detector operation

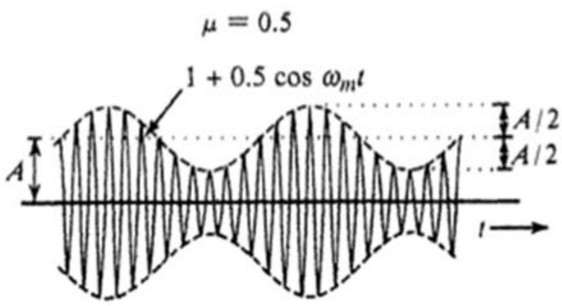


$$v_C(t) = A + m(t)$$

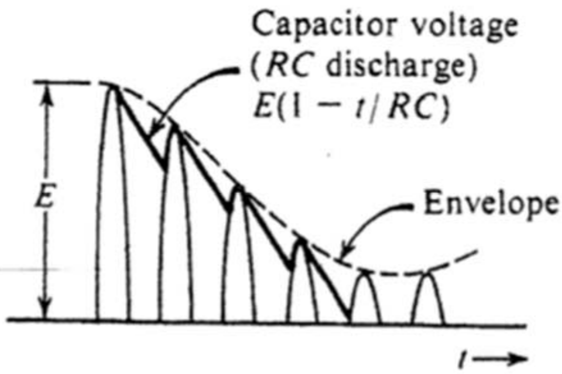
$$\omega_c \gg 2\pi B$$

Envelope detector example

- For the single tone



- Design envelope detector



$$v_C = Ee^{-t/RC}$$

$$v_C \simeq E\left(1 - \frac{t}{RC}\right)$$

Conclusions

- Examined full AM
- Sideband and carrier powers
- AM modulators
- AM demodulators

EE1 and ISE1: Introduction to Signals and Communications

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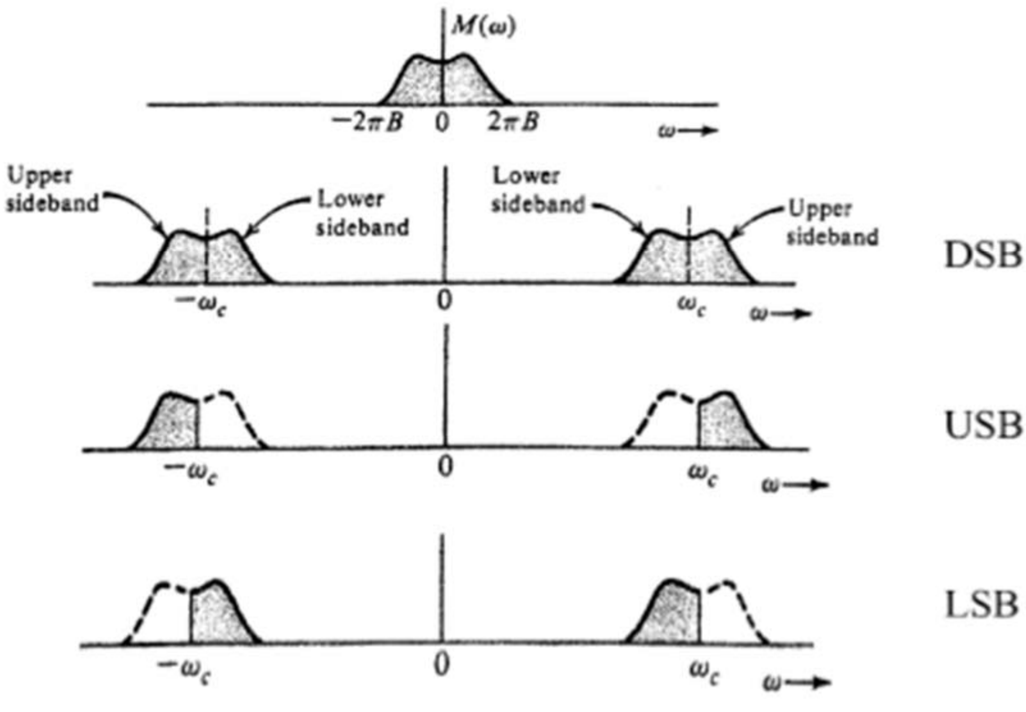
Lecture eleven

Lecture Aims

- To examine Single Sideband Modulation (SSB)
  - Time domain representation
  - Tone modulation
  - Generation of SSB signals
  - Demodulation of SSB signals

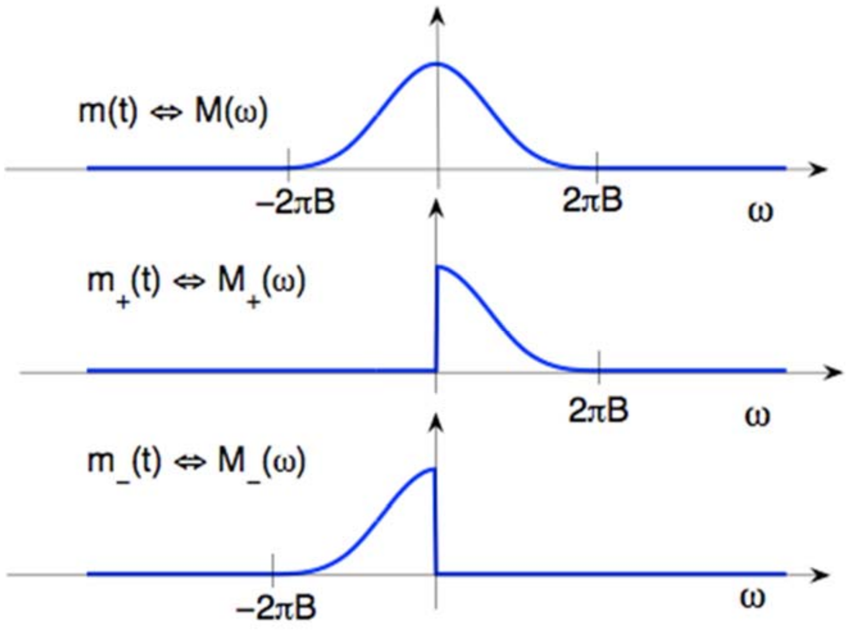
Modulation of Baseband Signals

Modulated Signal



Modulation of Baseband Signals

Splitting the baseband spectrum into USB and LSB



Single Sideband Generation

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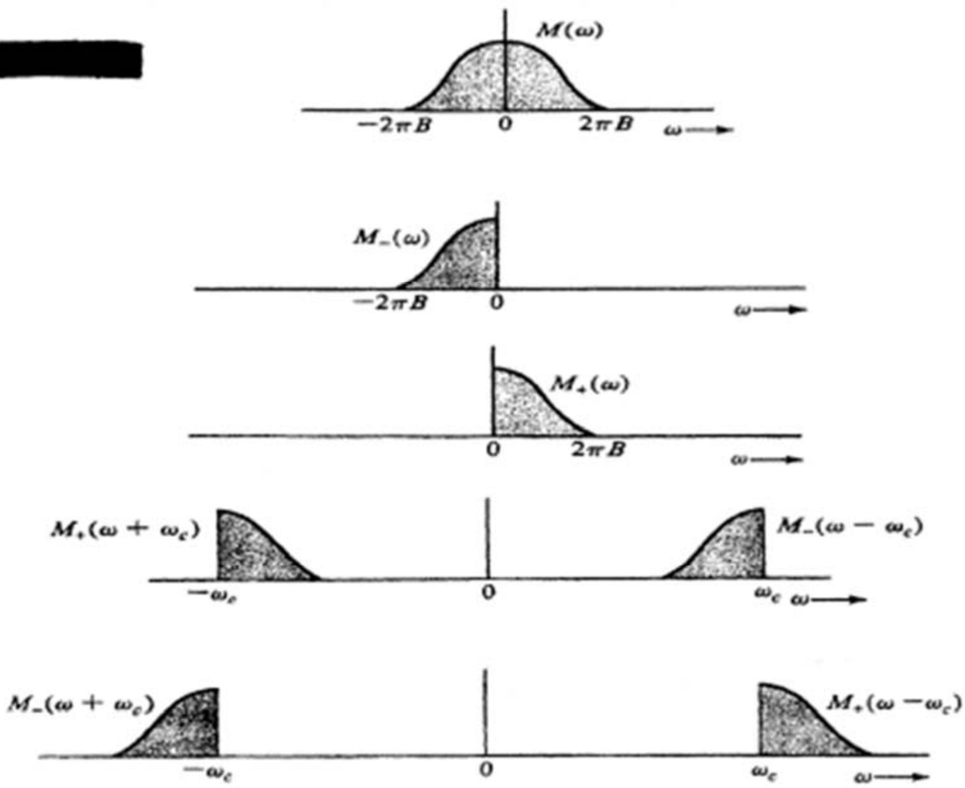


Time-Domain Representation of SSB signals

- Let $m_+(t)$ and $m_-(t)$ be the inverse Fourier transforms of $M_+(\omega)$ and $M_-(\omega)$.

- Because the amplitude spectra $|M_+(\omega)|$ and $|M_-(\omega)|$ are not even functions of ω, the signals $m_+(t)$ and $m_-(t)$ cannot be real. They are complex.

- It can be proven that $m_+(t)$ and $m_-(t)$ are conjugates. Moreover, $m_+(t) + m_-(t) = m(t)$. Hence,

$$m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right]$$
$$m_-(t) = \frac{1}{2}\left[m(t) - jm_h(t)\right]$$

Time-Domain Representation of SSB signals

To determine $m_h(t)$ note that

$$\begin{aligned} M_+(\omega) &= M(\omega)u(\omega) \\ &= \frac{1}{2}M(\omega)\left[1 + \operatorname{sgn}(\omega)\right] \\ &= \frac{1}{2}M(\omega) + \frac{1}{2}M(\omega)\operatorname{sgn}(\omega) \end{aligned}$$

Since $m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right]$, it follows $jm_h(t) \Leftrightarrow M(\omega)\operatorname{sgn}(\omega)$. Hence $M_h(\omega) = -jM(\omega)\operatorname{sgn}(\omega)$. But $1/\pi t \Leftrightarrow -j\operatorname{sgn}(\omega)$. Therefore

$$m_h(t) = m(t) * 1/\pi t = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\alpha)}{t-\alpha} d\alpha.$$

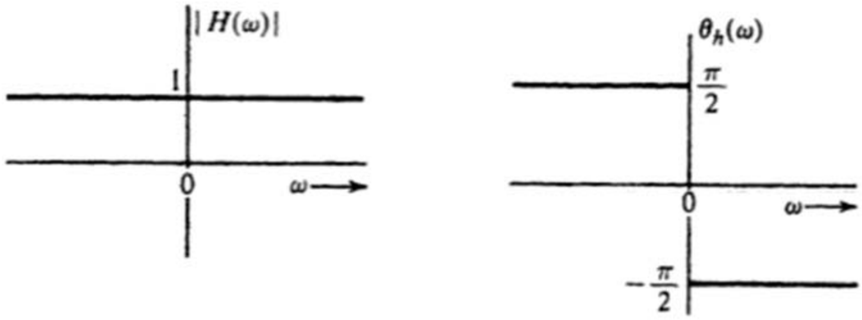
The right-hand side of this last equation defines the **Hilbert transform** of $m(t)$.

Hilbert Transform

The Hilbert Transform $m_h(t)$ is generated by passing $m(t)$ through a filter $h(t)$ with the following transfer function:

$$H(\omega) = -j\,\mathrm{sgn}(\omega) = \begin{cases} -j = e^{-j\pi/2} & \omega \ge 0 \\ j = e^{j\pi/2} & \omega < 0 \end{cases}$$

That is, $|H(\omega)| = 1$ and $\theta_h(\omega) = -\pi/2$, for $\omega \ge 0$



Time-Domain Representation of SSB Signals

We can now express the SSB signal in terms of $m(t)$ and $m_h(t)$.

$$\Phi_{USB}(\omega) = M_+(\omega - \omega_c) + M_-(\omega + \omega_c)$$

Inverse transform gives

$$\phi_{USB}(t) = m_+(t)e^{j\omega_c t} + m_-(t)e^{-j\omega_c t}$$

Using

$$m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right]$$

$$m_-(t) = \frac{1}{2}\left[m(t) - jm_h(t)\right]$$

$$\phi_{USB}(t) = m(t)\cos\omega_c t - m_h(t)\sin\omega_c t$$

Time-Domain Representation of SSB Signals

In a similar way we can show that

$$\phi_{LSB}(t) = m(t)\cos\omega_c t + m_h(t)\sin\omega_c t.$$

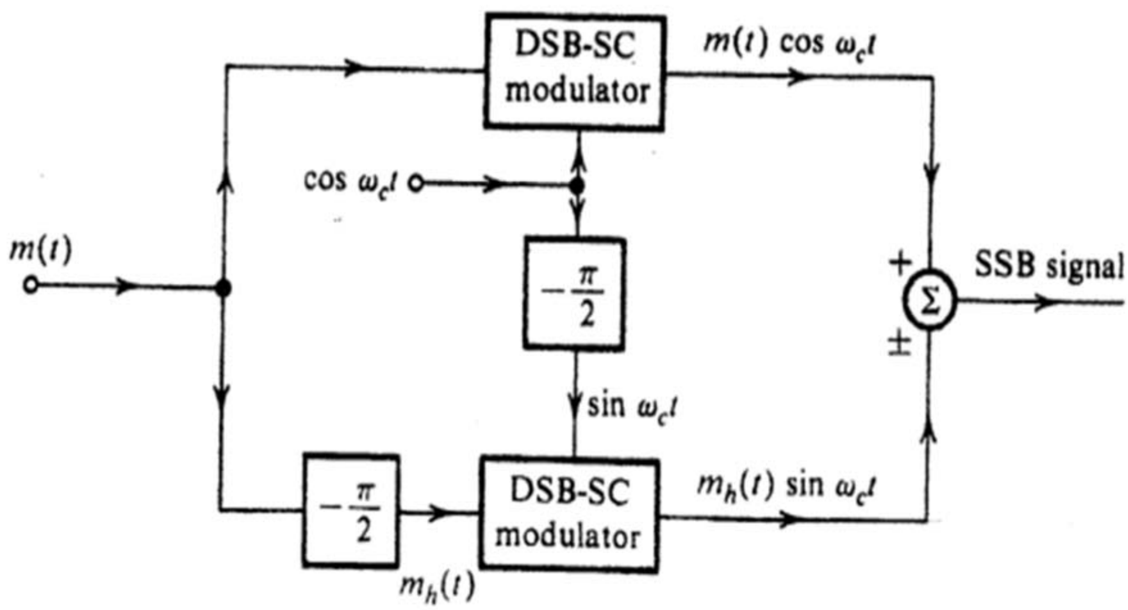
Hence a general SSB signal $\phi_{SSB}(t)$ can be expressed as

$$\phi_{SSB}(t) = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t,$$

where the minus sign applies to USB and the plus sign applies to LSB.

Generation of SSB Signals

Phase-Shift Method: $\phi_{SSB}(t) = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t$



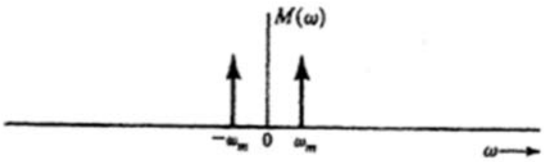
SSB Tone Modulation Example

- Consider single tone modulating signal $m(t) = \cos \omega_m t$.
- Hilbert transform requires phase shift by $\pi/2$.
- Delay in phase by $\pi/2$ yields $m_h(t) = \cos(\omega_m t - \pi/2) = \sin \omega_m t$.
- Using $\phi_{SSB}(t) = m(t)\cos \omega_c t \mp m_h(t)\sin \omega_c t$, we get

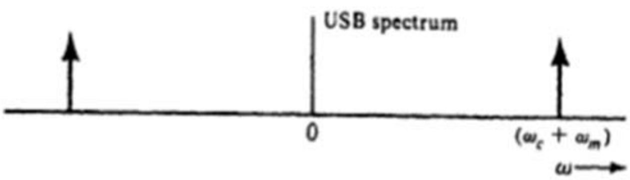
$$\phi_{SSB}(t) = \cos \omega_m t \cos \omega_c t \mp \sin \omega_m t \sin \omega_c t = \cos(\omega_c \pm \omega_m)t.$$

SSB Tone Modulation Example

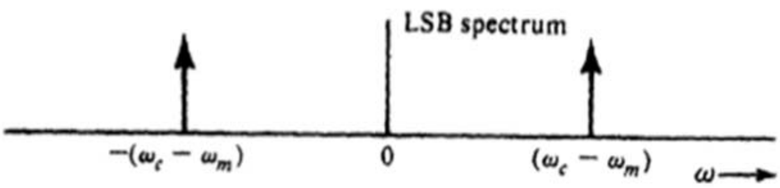
- Baseband spectrum



- USB spectrum

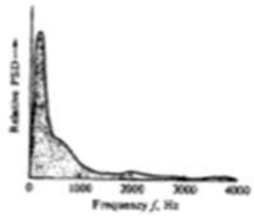


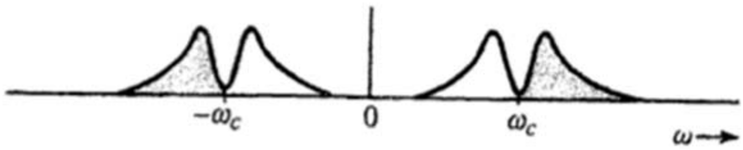
- LSB spectrum

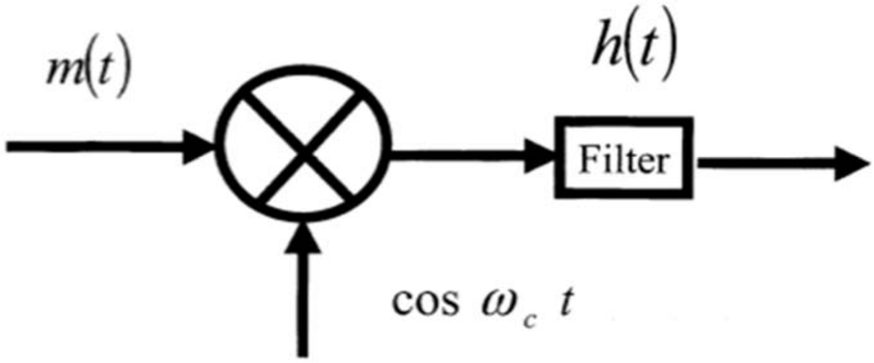


Generation of SSB Signals

Selective-filtering method:







Coherent demodulation of SSB-SC signals

The SSB demodulator is identical to the synchronous demodulator used for DSB-SC.

$$\phi_{SSB}(t) = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t$$

Hence

$$\begin{aligned}\phi_{SSB}(t)\cos(\omega_c t) &= \frac{1}{2}m(t)\left[1+\cos 2\omega_c t\right] \mp m_h(t)\sin 2\omega_c t \\ &= \frac{1}{2}m(t) + \frac{1}{2}\left[m(t)\cos 2\omega_c t \mp m_h(t)\sin 2\omega_c t\right]\end{aligned}$$

Thus, the product $\phi_{SSB}(\omega)\cos(\omega_c t)$ yields the baseband signal and another SSB signal with carrier $2\omega_c$. A low-pass filter will suppress the unwanted SSB terms.

Conclusions

- Hilbert Transform
- Single Side Band (SSB) signals
- Modulation and demodulation of SSB signals

EE1 and ISE1: Introduction to Signals and Communications

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Lecture twelve

Lecture Aims

- Angle Modulation
  - Phase and Frequency modulation
  - Concept of instantaneous frequency
  - Examples of phase and frequency modulation
  - Power of angle-modulated signals

Angle modulation

Consider a modulating signal $m(t)$ and a carrier $v_c(t) = A\cos(\omega_c t + \theta_c)$.

The carrier has three parameters that could be modulated: the amplitude A (AM) the frequency ω_c (FM) and the phase θ_c (PM).

The latter two methods are closely related since both modulate the argument of the cosine.

Instantaneous Frequency

- By definition a sinusoidal signal has a constant frequency and phase: $A\cos(\omega_c t + \theta_c)$
- Consider a generalized sinusoid with phase $\theta(t)$: $\phi(t) = A\cos\theta(t)$
- We define the instantaneous frequency ω_i as:

$$\omega_i(t) = \frac{d\theta}{dt}$$

- Hence, the phase is

$$\theta(t) = \int_{-\infty}^{t} \omega_i(\alpha)\,d\alpha.$$

Phase modulation

We can transmit the information of $m(t)$ by varying the angle θ of the carrier. In **phase modulation** (PM) the angle $\theta(t)$ is varied linearly with $m(t)$:

$$\theta(t) = \omega_c t + k_p m(t)$$

where k_p is a constant and ω_c is the carrier frequency. Therefore, the resulting PM wave is

$$\phi_{PM}(t) = A\cos\left[\omega_c t + k_p m(t)\right]$$

The instantaneous frequency in this case is given by

$$\omega_i(t) = \frac{d\theta}{dt} = \omega_c + k_p \dot{m}(t)$$

Frequency modulation

In PM the instantaneous frequency ω_i varies linearly with the **derivative** of $m(t)$. In **frequency modulation (FM)**, ω_i is varied linearly with $m(t)$. Thus

$$\omega_i(t) = \omega_c + k_f m(t).$$

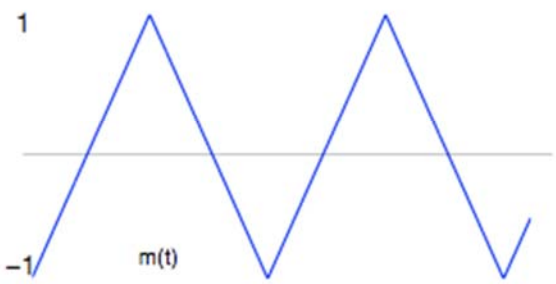
where k_f is a constant. The angle $\theta(t)$ is now

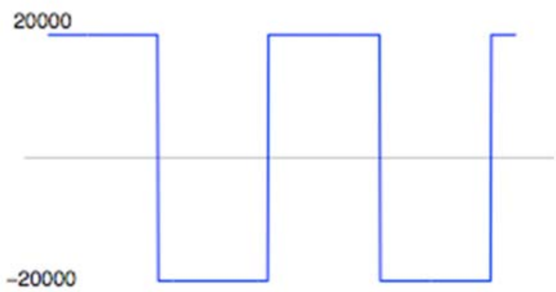
$$\theta(t) = \int_{-\infty}^{t} \left[\omega_c + k_f m(\alpha)\right] d\alpha = \omega_c t + k_f \int_{-\infty}^{t} m(\alpha) d\alpha.$$

The resulting FM wave is

$$\phi_{FM}(t) = A\cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha) d\alpha\right]$$

Example



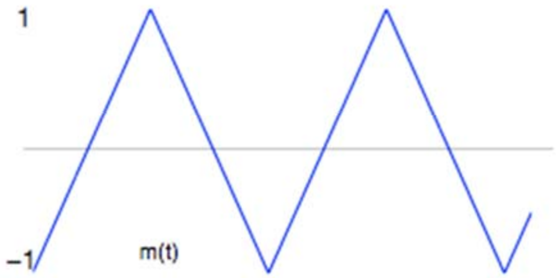


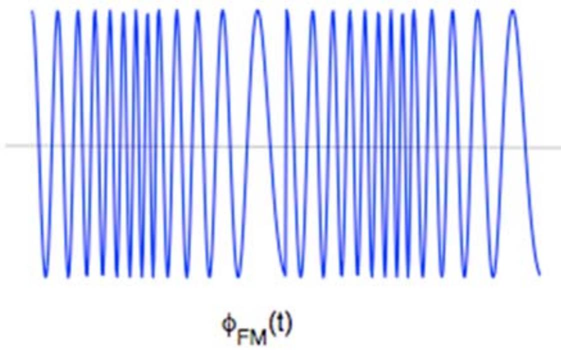
Sketch FM and PM signals if the modulating signal is the one above (on the left). The constants k_f and k_p are $2\pi \times 10^5$ and 10π, respectively, and the carrier frequency f_c =100$M\mathrm{H}_z$.

FM example

- Instantaneous angular frequency $\omega_i = \omega_c + k_f m(t)$
- Instantaneous frequency $f_i = f_c + \frac{k_f}{2\pi} m(t) = 10^8 + 10^5 m(t)$

$$(f_i)_{\min} = 10^8 + 10^5 \left[m(t)\right]_{\min} = 99.9MHz$$
$$(f_i)_{\max} = 10^8 + 10^5 \left[m(t)\right]_{\max} = 100.1MHz$$

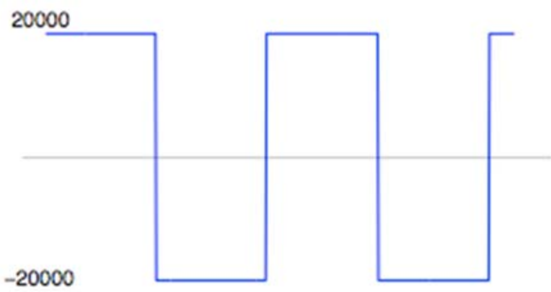


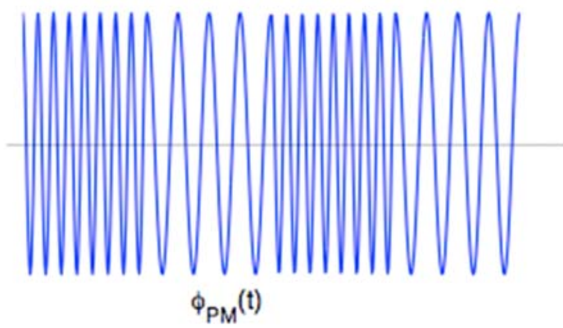


PM example

- Instantaneous frequency $f_i = f_c + \frac{k_p}{2\pi} \dot{m}(t) = 10^8 + 5\dot{m}(t)$

$$(f_i)_{\min} = 10^8 + 5\left[\dot{m}(t)\right]_{\min} = 10^8 - 10^5 = 99.9MHz$$
$$(f_i)_{\max} = 10^8 + 5\left[\dot{m}(t)\right]_{\max} = 10^8 + 10^5 = 100.1MHz$$





Power of an Angle-Modulated wave

- General angle modulated waveform

$$\phi(t) = A\cos\theta(t)$$

- Instantaneous phase and frequency vary with the time, but amplitude A remains constant.

- Thus, the power of angle–modulated waves is always $\frac{A^2}{2}$.

Conclusions

Examined

- Instantaneous frequency

- PM and FM modulations

- Examples of PM and FM signals

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Lecture thirteen

Lecture Aims

- To Study the bandwidth of angle modulated waves
  - Narrow-Band Angle Modulation
  - Carson's rule

Bandwidth of Angle Modulated waves

In order to study bandwidth of FM waves, define

$$a(t) = \int_{-\infty}^{t} m(\alpha)\, d\alpha$$

and

$$\hat{\phi}_{FM}(t) = Ae^{j\left[\omega_c t + k_f a(t)\right]} = Ae^{jk_f a(t)} e^{j\omega_c t}$$

The frequency modulated signal is

$$\phi_{FM}(t) = \mathrm{Re}\left\{\hat{\phi}_{FM}(t)\right\}$$

Bandwidth of Angle Modulated waves

Expanding the exponential $e^{jk_f a(t)}$ in power series yields

$$\hat{\phi}_{FM}(t) = A\left[1 + jk_f a(t) - \frac{k_f^2}{2!} a^2(t) + \cdots + j^n \frac{k_f^n}{n!} a^n(t) + \cdots\right] e^{j\omega_c t}$$

and

$$\phi_{FM}(t) = \mathrm{Re}\left\{\hat{\phi}_{FM}(t)\right\}$$

$$= A\left[\cos\omega_c t - k_f a(t)\sin\omega_c t - \frac{k_f^2}{2!} a^2(t)\cos\omega_c t + \frac{k_f^3}{3!} a^3(t)\sin\omega_c t + \cdots\right]$$

Narrow-Band Angle Modulation

The signal $a(t)$ is the integral of $m(t)$. It can be shown that if $M(\omega)$ is band limited to B, $A(\omega)$ is also band limited to B.

If $|k_f\, a(t)| \ll 1$ then all but the first term are negligible and

$$\phi_{FM}(t) \sim A\left[\cos\omega_c t - k_f a(t)\sin\omega_c t\right]$$

This case is called **narrow-band FM.**

Similarly, the **narrow-band PM** is given by

$$\phi_{PM}(t) \sim A\left[\cos\omega_c t - k_p m(t)\sin\omega_c t\right]$$

Narrow-Band Angle Modulation

Comparison of narrow band FM with Full AM.

Narrow band FM

$$\phi_{FM}(t) \sim A\left[\cos\omega_c t - k_f a(t)\sin\omega_c t\right]$$

Full AM

$$\left[A + m(t)\right]\cos\omega_c t = A\cos\omega_c t + m(t)\cos\omega_c t$$

Narrow band FM and full AM require a transmission bandwidth equal to $2B$ Hz. Moreover, the above equations suggest a way to generate narrowband FM or PM signals by using DSB-SC modulator

Wide-Band FM

● Assume that $|k_f a(t)| \ll 1$ is not satisfied.

● Cannot ignore higher order terms, but power series expansion analysis becomes complicated.

● The precise characterization of the FM bandwidth is mathematically intractable.

● Use an empirical rule (Carson's rule) which applies to most signals of interests.

Bandwidth equation

● Take the angular frequency deviation as $\Delta\omega = k_f m_p$ where $m_p = \max_t |m(t)|$ and frequency deviation as $\Delta f = \dfrac{k_f m_p}{2\pi}$.

● The transmission bandwidth of an FM signal is, with good approximation, given by

$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

Carson's rule

- The formula

$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

goes under the name of **Carson's rule**.

- If we define frequency deviation ratio as $\beta = \frac{\Delta f}{B}$
- Bandwidth equation becomes

$$B_{FM} = 2B\left(\beta + 1\right)$$

Wide-Band PM

- All results derived for FM can be applied to PM.
- Angular frequency deviation $\Delta\omega = k_p \dot{m}_p$ and frequency deviation $\Delta f = \frac{k_p \dot{m}_p}{2\pi}$ where we assume $\dot{m}_p = \max_t |\dot{m}(t)|$
- The bandwidth for the PM signal will be

$$B_{PM} = 2\left(\frac{k_p \dot{m}_p}{2\pi} + B\right) = 2\left(\Delta f + B\right)$$

Conclusions

Examined

- Narrowband FM
- Wideband FM and PM
- Carson's rule

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Lecture fourteen

Lecture Aims

- To verify bandwidth calculations for FM using single tone modulating signals

Verification of FM bandwidth

- To verify Carson's rule

$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

- Consider a single tone modulating sinusoid

$$m(t) = \alpha \cos \omega_m t \qquad a(t) = \int_{-\infty}^{t} m(\tau) d\tau = \frac{\alpha}{\omega_m} \sin \omega_m t$$

- We can express the FM signal as

$$\hat{\varphi}_{FM}(t) = Ae^{j(\omega_c t + \frac{k_f \alpha}{\omega_m} \sin \omega_m t)}$$

Verification of FM bandwidth

- The angular frequency deviation is $\Delta\omega = k_f m_p = \alpha k_f$
- Since the bandwidth of $m(t)$ is $B = f_m Hz$, the frequency deviation ratio (or modulation index) is

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m} = \frac{\alpha k_f}{\omega_m}$$

- Hence the FM signal become

$$\hat{\varphi}_{FM}(t) = Ae^{(j\omega_c t + j\beta \sin \omega_m t)} = Ae^{j\omega_c t}\left(e^{j\beta \sin \omega_m t}\right)$$

Verification of FM bandwidth

The exponential term $e^{j\beta \sin \omega_m t}$ is a periodic signal with period $2\pi/\omega_m$ and can be expanded by the exponential Fourier series:

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$$

where

$$C_n = \frac{\omega_m}{2\pi}\int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt$$

Bessel functions

By changing variables $\omega_m t = x$, we get

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(j\beta\sin x - jnx)}dx$$

This integral is denoted as the Bessel function $J_{\mathrm{n}}(\beta)$ of the first kind and order n. It cannot be evaluated in closed form but it has been tabulated.

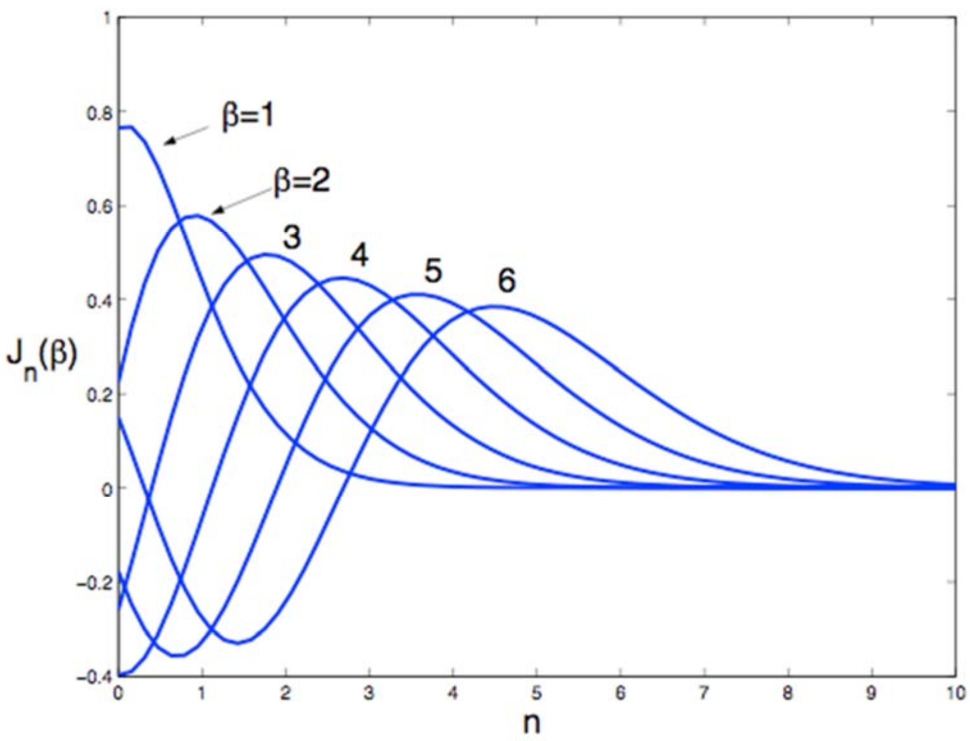
Hence the FM waveform can be expressed as

$$\hat{\varphi}_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)e^{(j\omega_c t + jn\omega_m t)}$$

and

$$\varphi_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t$$

Bessel functions of the first kind



Bandwidth calculation for FM

The FM signal for single tone modulation is

$$\varphi_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t.$$

The modulated signal has ‘theoretically’ an infinite bandwidth made of one carrier at frequency ω_c and an infinite number of sidebands at frequencies $\omega_c \pm \omega_m$, $\omega_c \pm 2\omega_m$, ..., $\omega_c \pm n\omega_m$, ... However

- for a fixed β, the amplitude of the Bessel function $J_n(\beta)$ decreases as n increases. This means that for any fixed β there is only a finite number of significant sidebands.
- As $n > \beta + 1$ the amplitude of the Bessel function becomes negligible. Hence, the number of significant sidebands is $\beta + 1$.

This means that with good approximation the bandwidth of the FM signal is

$$B_{FM} = 2nf_m = 2\left(\beta+1\right)f_m = 2\left(\Delta f + B\right).$$

Example

Estimate the bandwidth of the FM signal when the modulating signal is the one shown in Fig. 1 with period $T = 2 \times 10^{-4}$ sec, the carrier frequency is $f_c = 100\text{MHz}$ and $k_f = 2\pi \times 10^5$.

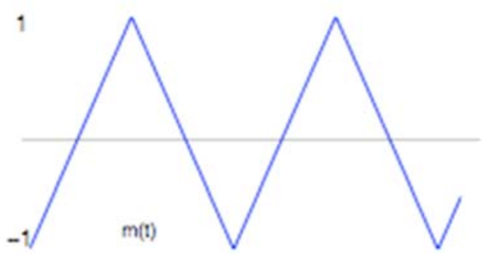


Figure 1: The modulating signal $m(t)$

Repeat the problem when the amplitude of $m(t)$ is doubled.

Example

- Peak amplitude of $m(t)$ is $m_p = 1$.
- Signal period is $T = 2 \times 10^{-4}$, hence fundamental frequency is $f_0 = 5\text{kHz}$.
- We assume that the essential bandwidth of $m(t)$ is the third harmonic. Hence the modulating signal bandwidth is $B = 15\text{kHz}$.
- The frequency deviation is:

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}\left(2\pi \times 10^5\right)(1) = 100kHz.$$

- Bandwidth of the FM signal:

$$B_{FM} = 2\left(\Delta f + B\right) = 230kHz.$$

Example

- Doubling amplitude means that $m_p = 2$.
- The modulating signal bandwidth remains the same, i.e., $B = 15\text{kHz}$.
- The new frequency deviation is :

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}\left(2\pi \times 10^5\right)(2) = 200kHz.$$

- The new bandwidth of the FM signal is :

$$B_{FM} = 2\left(\Delta f + B\right) = 430kHz.$$

Example

Now estimate the bandwidth of the FM signal if the modulating signal is time expanded by a factor 2.

- The time expansion by a factor 2 reduces the signal bandwidth by a factor 2. Hence the fundamental frequency is now f_0 = 2.5kHz and B = 7.5kHz.
- The peak value stays the same, i.e., $m_p = 1$ and

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}\left(2\pi \times 10^5\right)\left(1\right) = 100kHz.$$

- The new bandwidth of the FM signal is:

$$B_{FM} = 2\left(\Delta f + B\right) = 2\left(100 + 7.5\right) = 215kHz.$$

Second Example

An angle modulated signal with carrier frequency $\omega_c = 2\pi \times 10^5$ rad/s is given by:

$$\varphi_{FM}(t) = 10\cos\left(\omega_c t + 5\sin 3000t + 10\sin 2000\pi t\right).$$

- Find the power of the modulated signal
- Find the frequency deviation Δf
- Find the deviation ratio $\beta = \frac{\Delta f}{B}$
- Estimate the bandwidth of the FM signal

Second Example

- The carrier amplitude is 10 therefore the power is $P = 10^2/2 = 50$.
- The signal bandwidth is $B = 2000\pi/2\pi = 1000\text{Hz}$.
- To find the frequency deviation we find the instantaneous frequency:

$$\omega_i = \frac{d}{dt}\theta(t) = \omega_c + 15{,}000\cos 3000t + 20{,}000\pi\cos 2000\pi t.$$

The angle deviation is the maximum of $15{,}000\cos 3000t + 20{,}000\pi\cos 2000\pi t$. The maximum is: $\Delta\omega = 15{,}000 + 20{,}000\pi$rad/s. Hence, the frequency deviation is

$$\Delta f = \frac{\Delta\omega}{2\pi} = 12{,}387.32Hz.$$

- The modulation index is

$$\beta = \frac{\Delta f}{B} = 12.387.$$

- The bandwidth of the FM signal is: $B_{FM} = 2(\Delta f + B) = 26{,}774.65Hz.$

Conclusions

- Verified bandwidth calculation for FM using single tone modulating signal
- Examined Bessel functions and their properties
- Examined two examples and calculated FM bandwidths

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Lecture fifteen

Lecture Aims

- To identify how resilient FM is to non-linear distortion
- To outline FM modulators and demodulators

Angle Modulation and non-linearities

- FM signals are constant envelope signals, therefore they are less susceptible to non-linearities
- Example: a non-linear device whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = a_1 x(t) + a_2 x^2(t)$$

- if $x(t) = \cos\left[\omega_c t + \psi(t)\right]$
- Then

$$y(t) = a_1 \cos\left[\omega_c t + \psi(t)\right] + a_2 \cos^2\left[\omega_c t + \psi(t)\right]$$

$$= \frac{a_2}{2} + a_1 \cos\left[\omega_c t + \psi(t)\right] + \frac{a_2}{2}\cos\left[2\omega_c t + 2\psi(t)\right]$$

Angle Modulation and non-linearities

- For FM wave

$$\psi(t) = k_f \int m(\alpha) d\alpha$$

- The output waveform is

$$y(t) = \frac{a_2}{2} + a_1 \cos\left[\omega_c t + k_f \int m(\alpha) d\alpha\right] + \frac{a_2}{2}\cos\left[2\omega_c t + 2k_f \int m(\alpha) d\alpha\right]$$

- Unwanted signals can be removed by means of a bandpass filter

Higher order non-linearities

- Consider higher order non-linearities

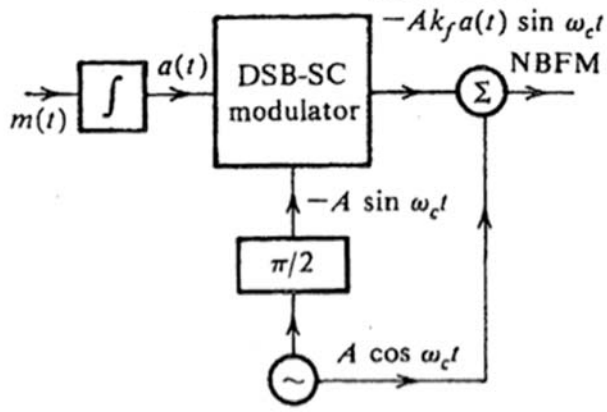
$$y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + \cdots + a_n x^n(t)$$

- If the input signal is an FM wave, $y(t)$ will have the form

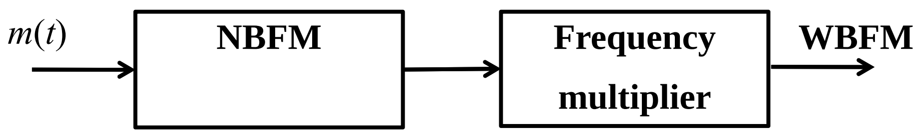
$$y(t) = c_0 + c_1 \cos\left[\omega_c t + k_f \int m(\alpha)d\alpha\right] + c_2 \cos\left[2\omega_c t + 2k_f \int m(\alpha)d\alpha\right]$$
$$+ \cdots + c_n \cos\left[n\omega_c t + nk_f \int m(\alpha)d\alpha\right]$$

- The deviations are Δf, $2\Delta f$, ..., $n\Delta f$

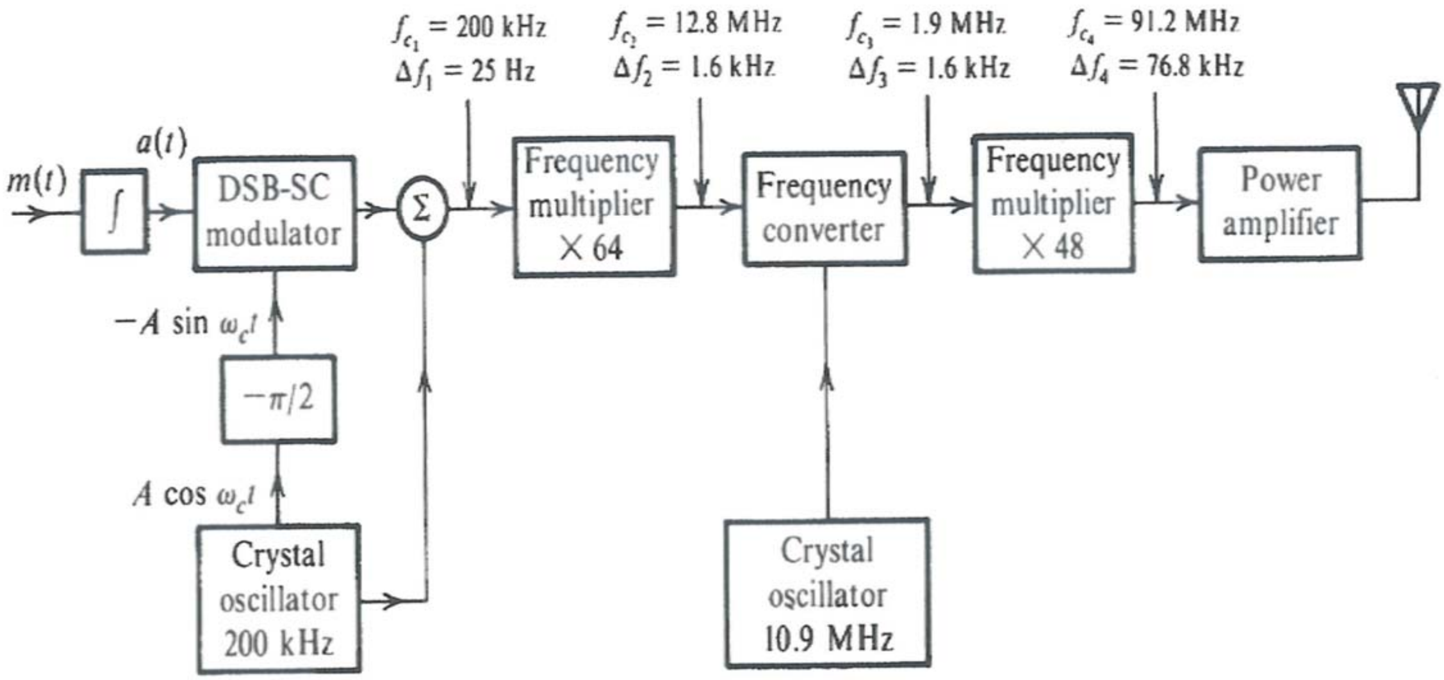
- Narrowband signal is generated using



- NBFM signal is then converted to WBFM using



Armstrong indirect FM transmitter



Direct Method of FM Generation

- The modulating signal $m(t)$ can control a voltage controlled oscillator to produce instantaneous frequency

$$\omega_i(t) = \omega_c + k_f m(t)$$

- A voltage controlled oscillator can be implemented using an LC parallel resonant circuit with centre frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

- If the capacitance is varied by $m(t)$

$$C = C_0 - km(t)$$

Direct Method of FM Generation

- The oscillator frequency is given by

$$\omega_i(t) = \frac{1}{\sqrt{LC_0\left[1-\frac{km(t)}{C_0}\right]}} = \frac{1}{\sqrt{LC_0}\left[1-\frac{km(t)}{C_0}\right]^{1/2}}$$

- If $\frac{km(t)}{C_0} \ll 1$, the binomial series expansion gives

$$\omega_i(t) \sim \frac{1}{\sqrt{LC_0}}\left[1+\frac{km(t)}{2C_0}\right]$$

- This gives the instantaneous frequency as a function of the modulating signal.

Demodulation of FM signals

- The FM demodulator is given by a differentiator followed by an envelope detector

- Output of the ideal differentiator

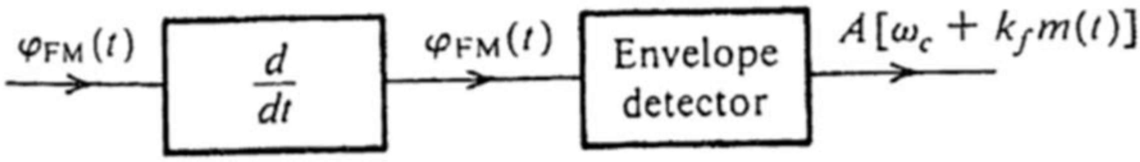
$$\varphi_{FM}(t) = \frac{d}{dt}\left\{A\cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha)\,d\alpha\right]\right\}$$

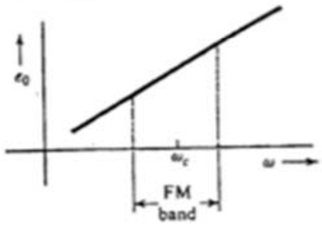
$$= A\left[\omega_c + k_f m(t)\right]\sin\left[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha)\,d\alpha\right]$$

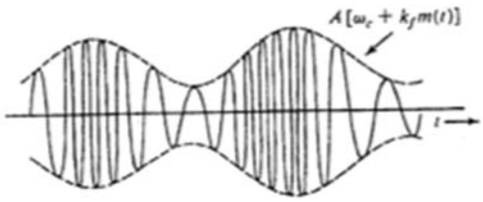
- The above signal is both amplitude and frequency modulated. Hence, an envelope detector with input $\varphi_{FM}(t)$ yields an output proportional to

$$A\left[\omega_c + k_f m(t)\right]$$

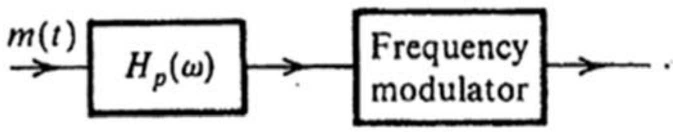
● As $\Delta\omega = k_f m_p < \omega_c$ and $\omega_c + k_f m(t) > 0$ for all t. The modulating signal $m(t)$ can be obtained using and envelope detector



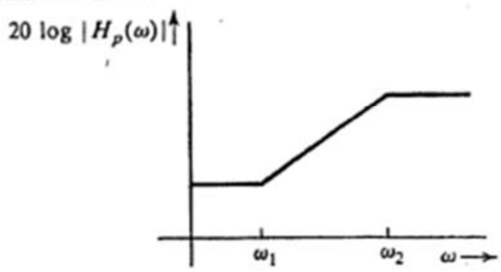


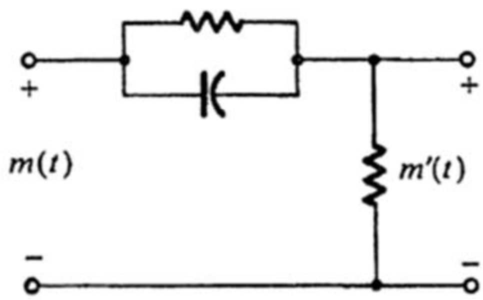


● To improve noise immunity of FM signals we use a pre-emphasis circuit ant transmitter

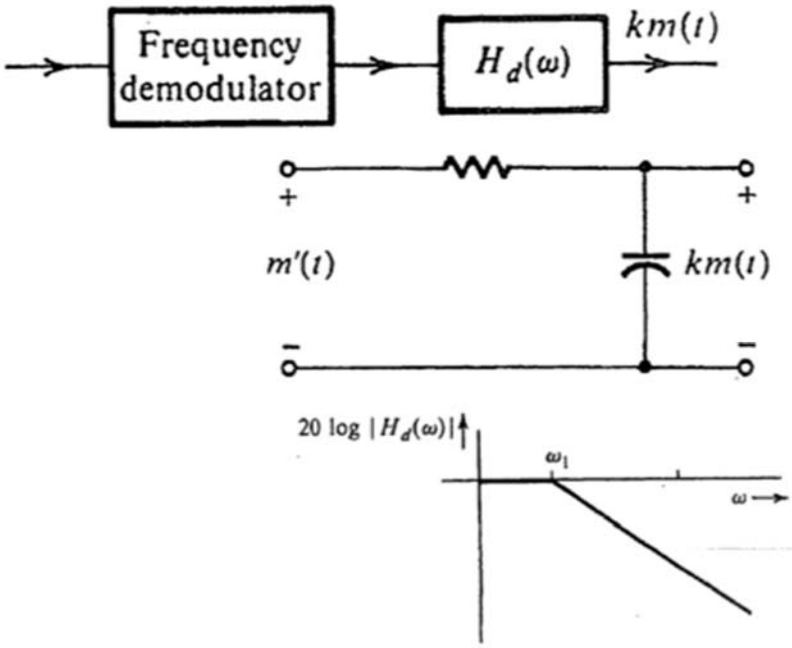


$$H_p(\omega) = K\frac{j\omega + \omega_1}{j\omega + \omega_2}$$

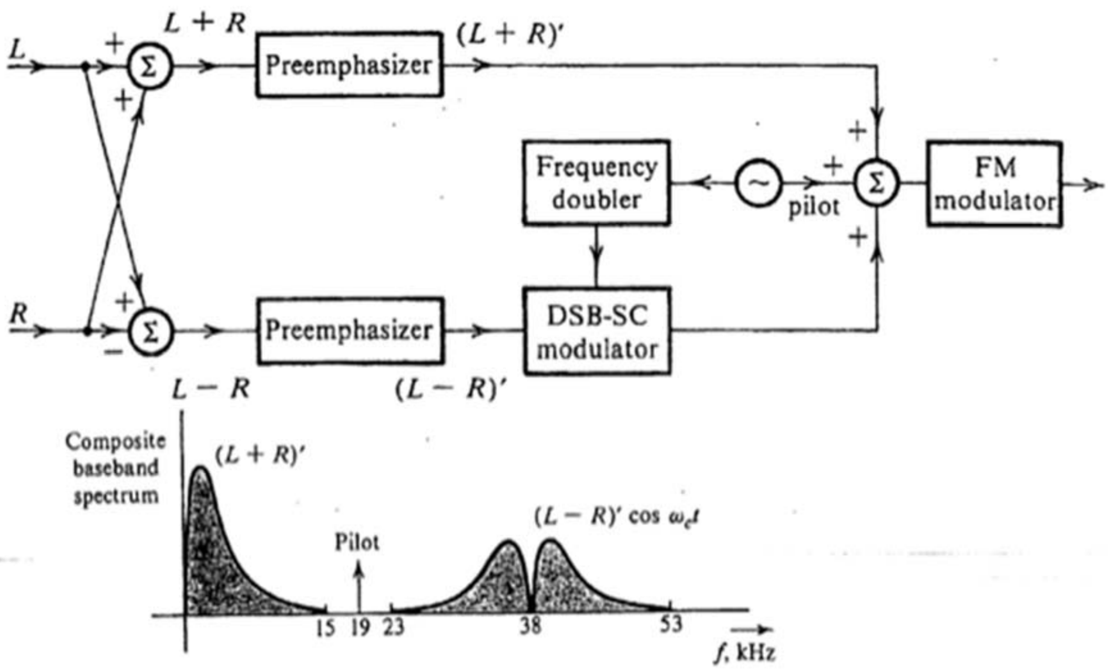




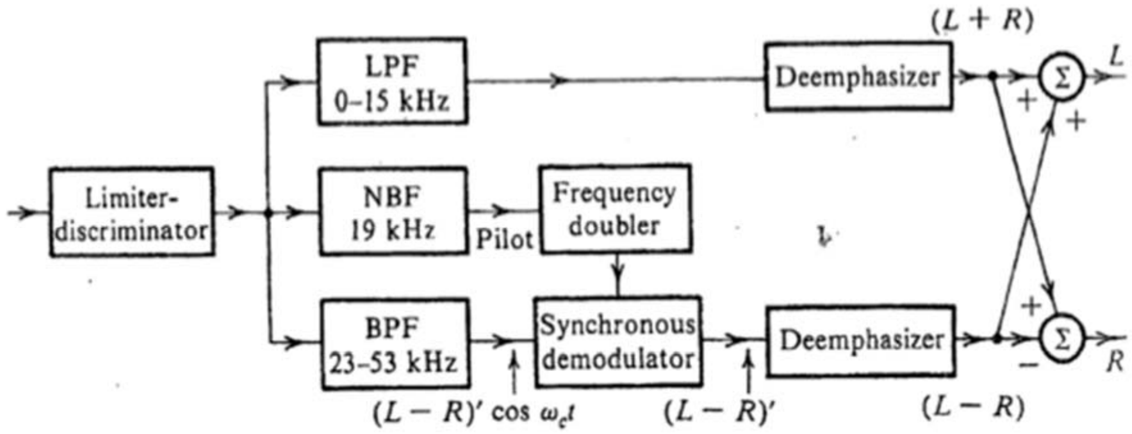
- Receiver de-emphasis circuit



- Transmitter



● Receiver



Conclusions

● FM modulators: direct and indirect methods

● FM demodulator

● Pre-emphasis and de-emphasis circuits to improve noise immunity of signals

● FM stereo transmitter and receiver

EE1 and ISE1: Introduction to Signals and Communications

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Lecture sixteen

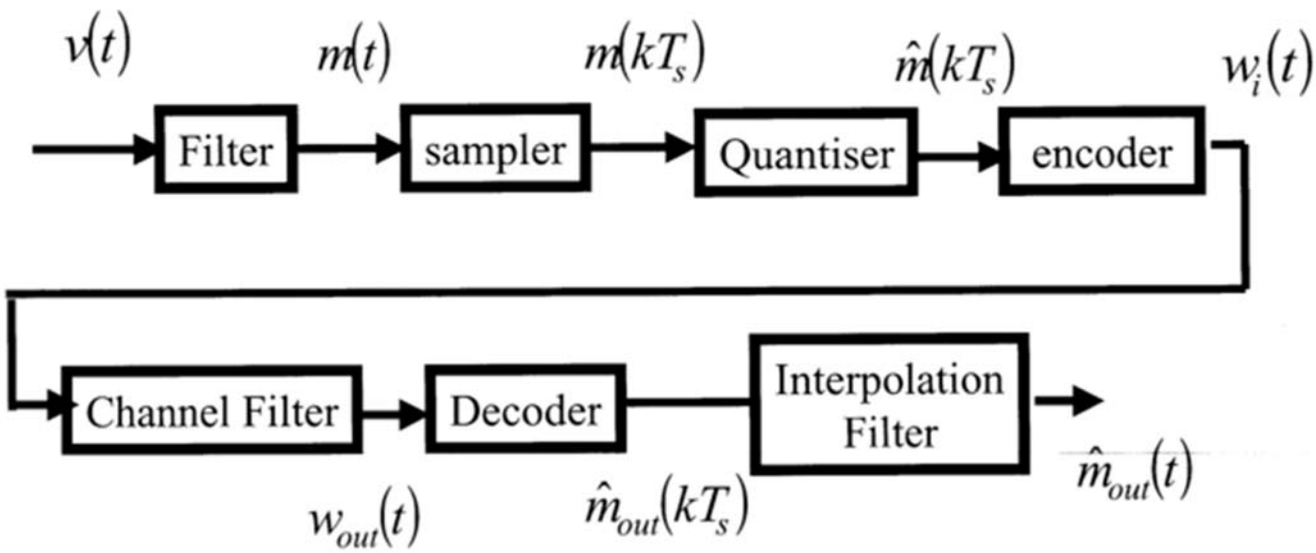
Lecture Aims

- Outline digital communication systems

Why digital modulation

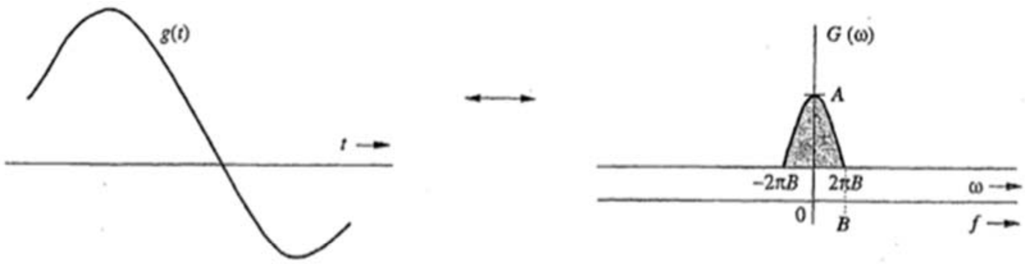
- More resilient to noise
- Viability of regenerative repeaters
- Digital hardware more flexible
- It is easier to multiplex digital signals

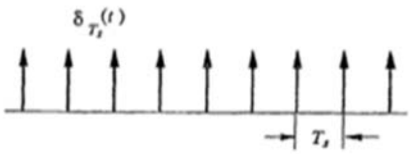
Digital Transmission System



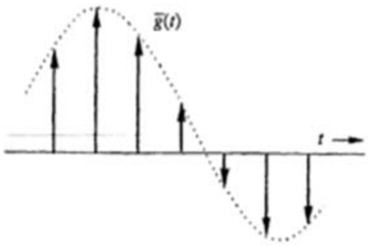
Analogue waveform

- Analogue waveform and its spectrum

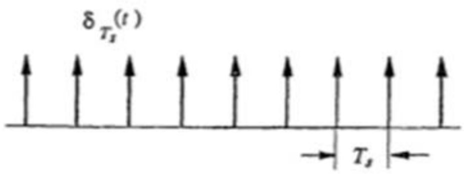




Sampling waveform

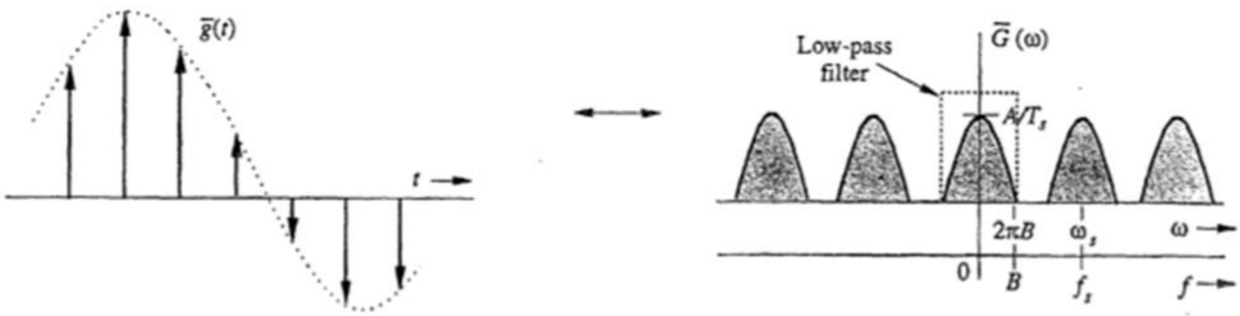


Sampled waveform



$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

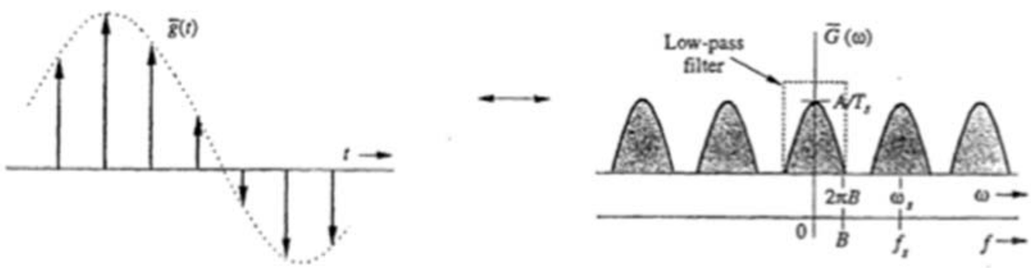
$$\delta_{T_s}(t) = \frac{1}{T_s}\left[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \cdots\right]$$



$$\bar{g}(t) = g(t)\delta_{T_s}(t)$$

$$= \frac{1}{T_s}\left[g(t) + 2g(t)\cos\omega_s t + 2g(t)\cos 2\omega_s t + 2g(t)\cos 3\omega_s t + \cdots\right]$$

Sampled signal spectrum



Sampling frequency

$$f_s = 1/T_s \; Hz$$

Sampling time

$$T_s = 1/f_s$$

Also have

Sampled signal spectrum

$$\overline{G}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(\omega - n\omega_s)$$

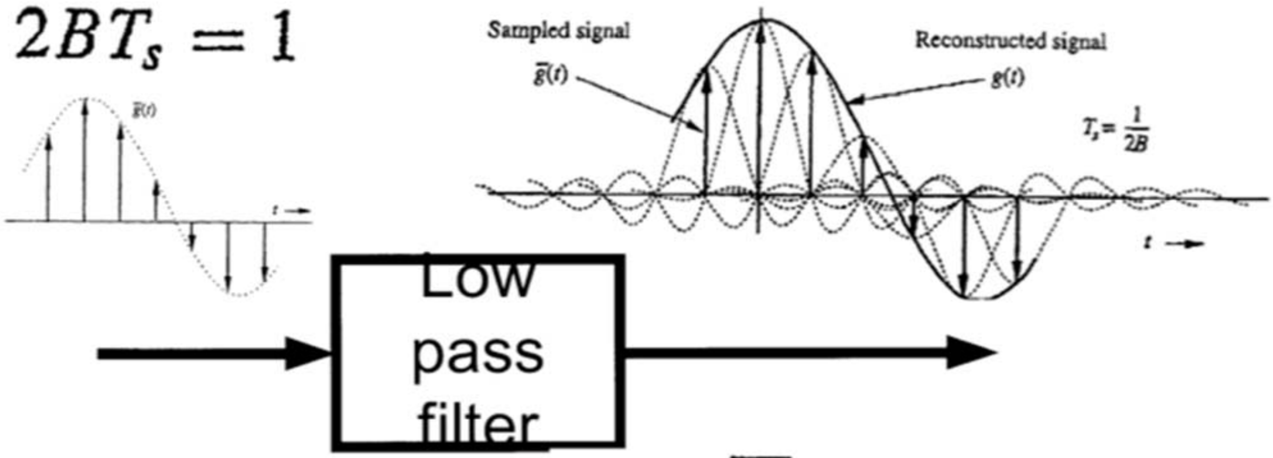
Sampling frequency must satisfy

$$f_s > 2B$$

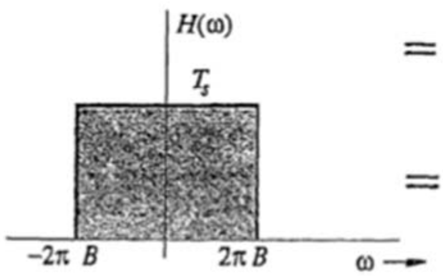
$$T_s < \frac{1}{2B}$$

Signal construction using better filter

$$2BT_s = 1$$



$$h(t) = \text{sinc}\,(2\pi Bt)$$

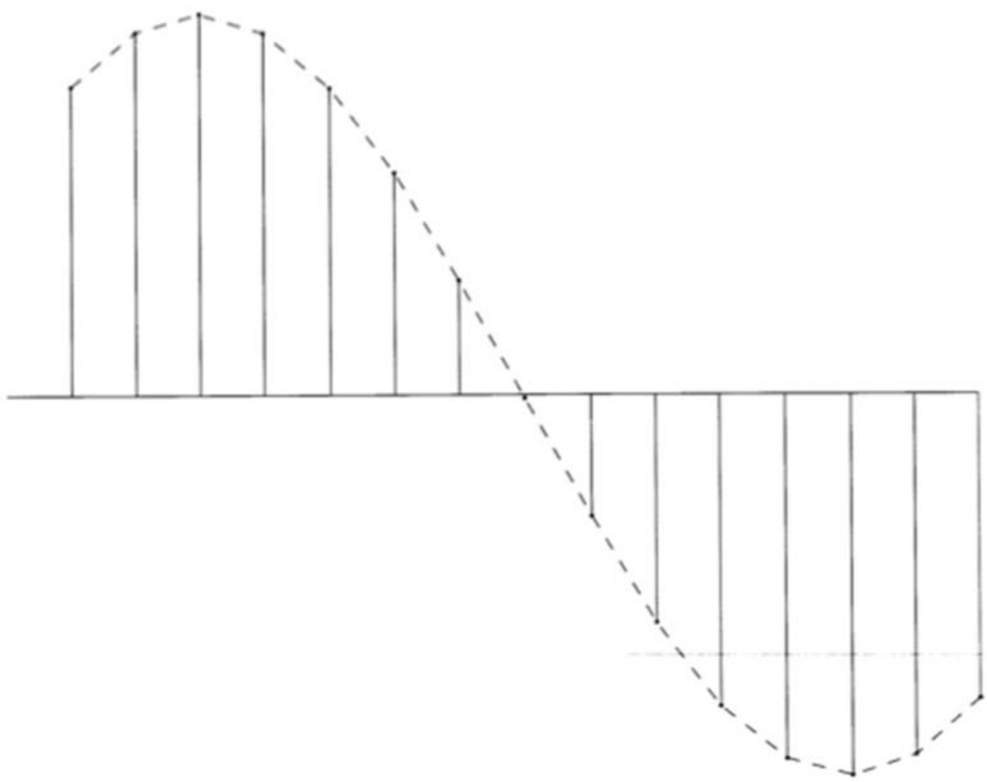


$$g(t) = \sum_k g(kT_s)h(t - kT_s)$$

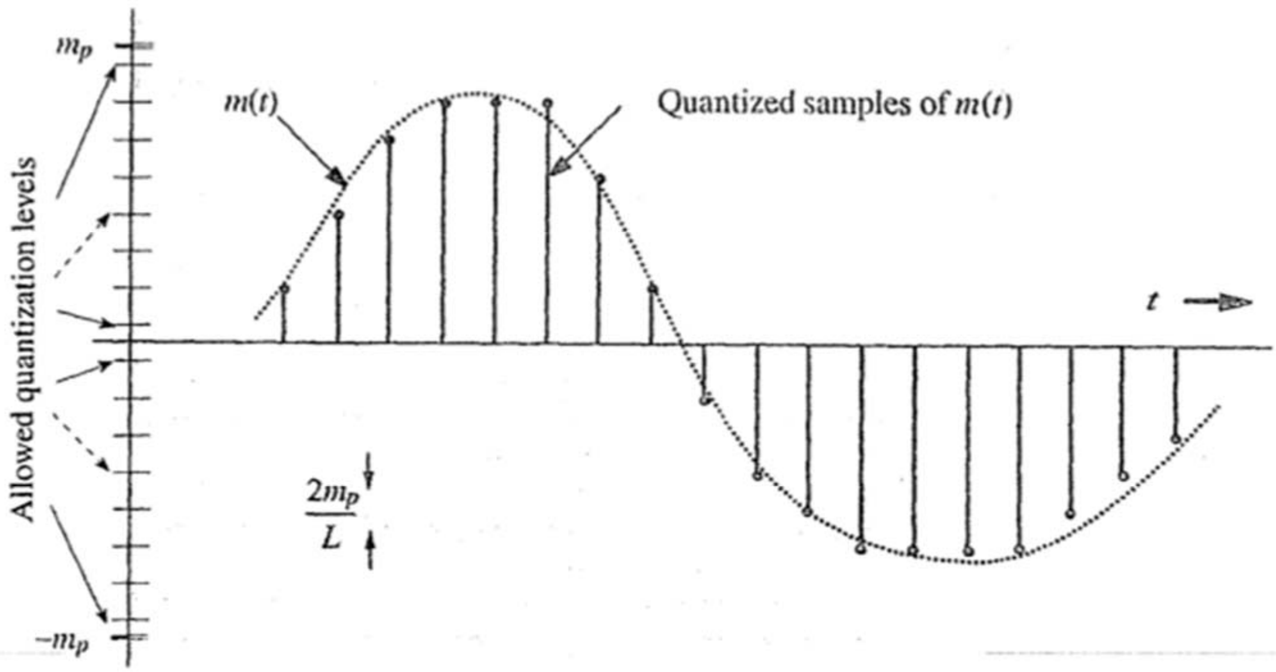
$$= \sum_k g(kT_s)\,\text{sinc}\,[2\pi B(t - kT_s)]$$

$$= \sum_k g(kT_s)\,\text{sinc}\,(2\pi Bt - k\pi)$$

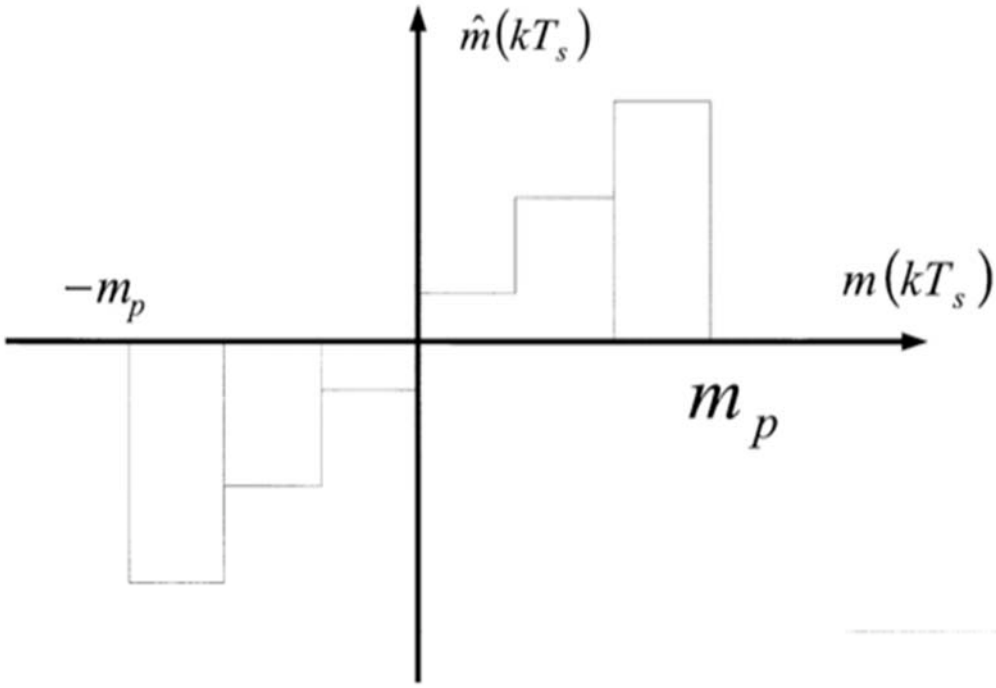
Sampled Waveform



Quantized waveform



Uniform quantizer



Minimum and maximum voltage

$$\max\left(m(t)\right) = m_p$$

$$\min\left(m(t)\right) = -m_p$$

$$n = \text{number of bits}$$

$$L = 2^n = \text{number of levels}$$

$$m_i = \text{voltage boundaries}$$

$$i = 0, 1, 2 \cdots L$$

$$m_0 = -m_p \qquad\qquad m_L = m_p$$

Voltage range values

$$\Delta = \text{step size} = \frac{\max\left(m(t)\right) - \min\left(m(t)\right)}{L}$$

$$m_0 = \min\left(m(t)\right) = -m_p$$

$$m_i = \min\left(m(t)\right) + i\Delta$$

$$m_L = \max\left(m(t)\right) = m_p$$

$$\Delta = \frac{m_p - \left(-m_p\right)}{L} = \frac{2m_p}{L}$$

$$m_i > m\left(kT_s\right) \geq m_{m-1}$$

$$\hat{m}\left(kT_s\right) = \frac{m_i + m_{i-1}}{2}$$

Quantization and binary representation

- Assume the amplitude of the analog signal $m(t)$ lie in the range $(-m_p, m_p)$.

- with quantization, this interval is partitioned into L sub-intervals, each of magnitude $\delta u = 2m_p / L$.

- Each sample amplitude is approximated by the midpoint value of the subinterval in which the sample falls.

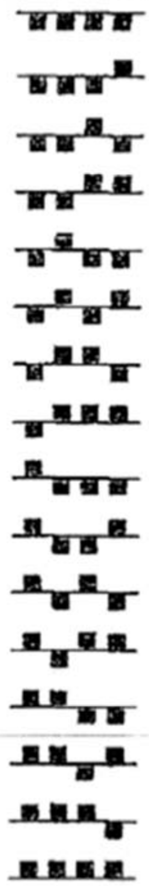
- Thus, each sample of the original signal can take on only one of the L different values.

- Such a signal is known as an L-ary digital signals

- In practice, it is better to have binary signals

Alternatively we can use A sequence of four binary pulses to get 16 distinct patterns

Digit	Binary equivalent	Pulse code waveform
0	0000	
1	0001	
2	0010	
3	0011	
4	0100	
5	0101	
6	0110	
7	0111	
8	1000	
9	1001	
10	1010	
11	1011	
12	1100	
13	1101	
14	1110	
15	1111	



Examples

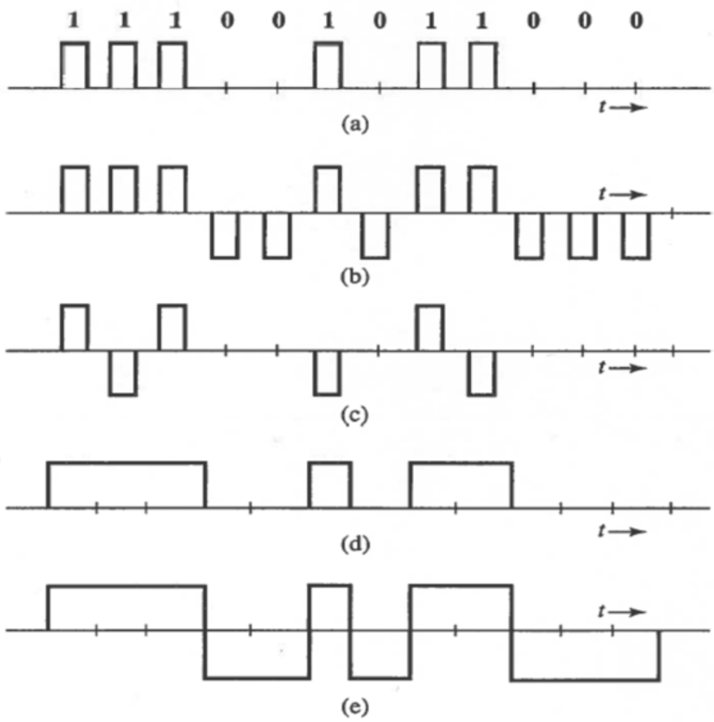
1. Audio Signal (Low Fidelity, used in telephone lines).

- Audio signal frequency from 0 to 15 kHz. Subjective tests show signal articulation (intelligibility) is not affected by components above 3.4 kHz. So, assume bandwidth B = 4 kHz.
- Sampling frequency $f_s = 2B = 8$ kHz that means 8,000 samples per second.
- Each sample is quantized with L = 256 levels, that is a group of 8 bits to encode each sample $2^8 = 256$
- Thus a telephone line requires 8 x 8,000 = 64,000 bits per second (64 kbps).

2. Audio Signal (High Fidelity, used in CD)

- Bandwidth 20 kHz, we assume a bandwidth of B = 22.05kHz.
- Sampling frequency $f_s = 2B = 44.1$ kHz, this means 44,100 samples per seconds.
- Each sample is quantized with L = 65,536 levels, 16 bits per sample.
- Thus, a Hi-Fi audio signal requires 16 x 44,100 $\simeq$706 kbps.

Transmission or Line Coding



On-off return-to-zero

Polar return-to-zero

Bi-polar return-to-zero

On-off non-return-to-zero

Polar non-return-to-zero

Desirable Properties of Line Coding

- Transmission bandwidth as small as possible
- Power efficiency
- Error detection and correction capability
- Favorable power spectral density (e.g., avoid dc component for use of ac coupling and transformers)
- Adequate timing content
- Transparency (independent of info bits, to avoid timing problem)

Digital Modulation

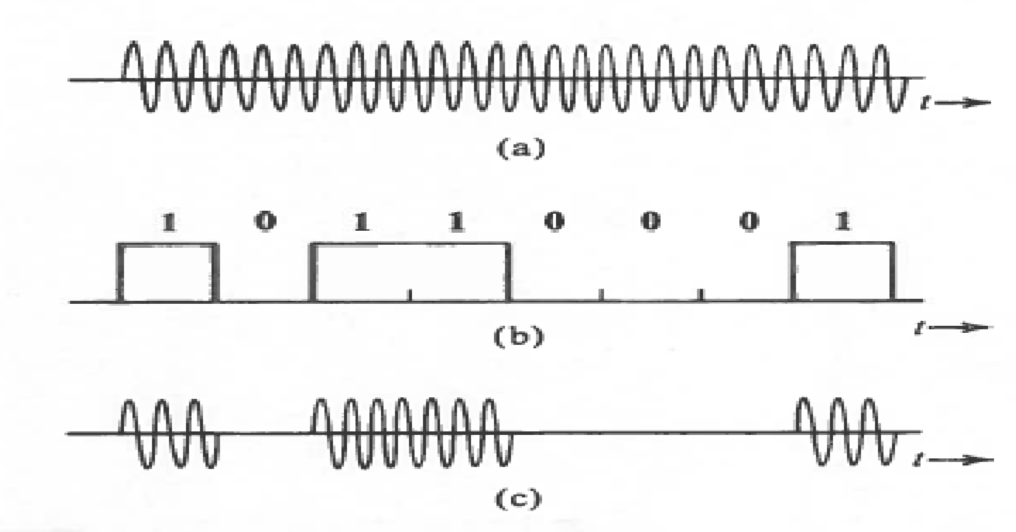
- The process of modulating a digital signal is called keying

- As for the analogue case, we can choose one of the three parameters of a sine wave to modulate

  1. Amplitude modulation, called *Amplitude Shift Keying* (ASK)
  2. Phase modulation*, Phase Shift Keying* (PSK)
  3. Frequency modulation *Frequency Shift Keying* (FSK)

- In some cases, the data can be sent by simultaneously modulating phase and amplitude, this is called *Quadrature Amplitude Phase Shift Keying* (QASK)

Amplitude Shift Keying (ASK)



On-off non-return-to-zero

Amplitude shift keying (ASK) = on-off keying (OOK)

$$s_0(t) = 0$$
$$s_1(t) = A\cos(2\pi f_c t)$$

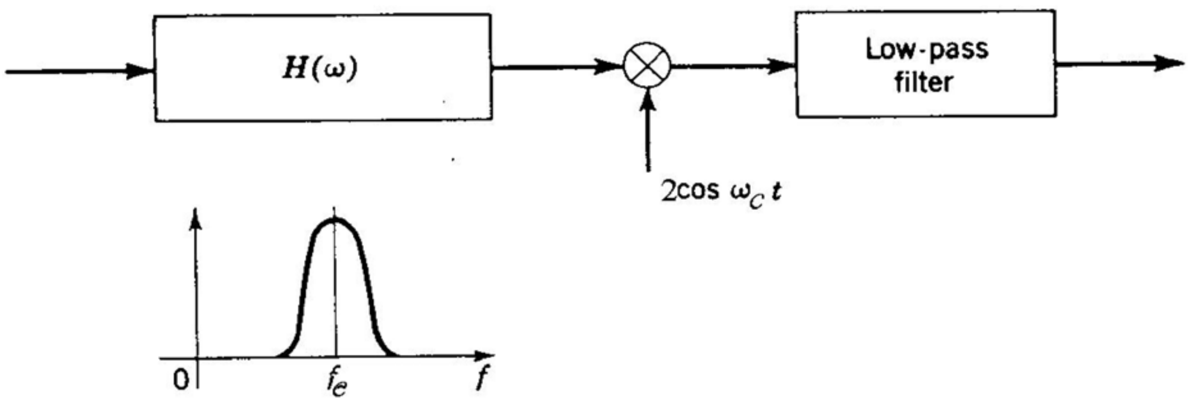
or $\quad s(t) = A(t)\cos(2\pi f_c t), \quad A(t) \in \{0, A\}$

How to recover ASK transmitted symbol?

- Coherent (synchronous) detection
  - Use a BPF to reject out-of-band noise
  - Multiply the incoming waveform with a cosine of the carrier frequency
  - Use a LPF
  - Requires carrier regeneration (both frequency and phase synchronization by using a phase-lock loop)

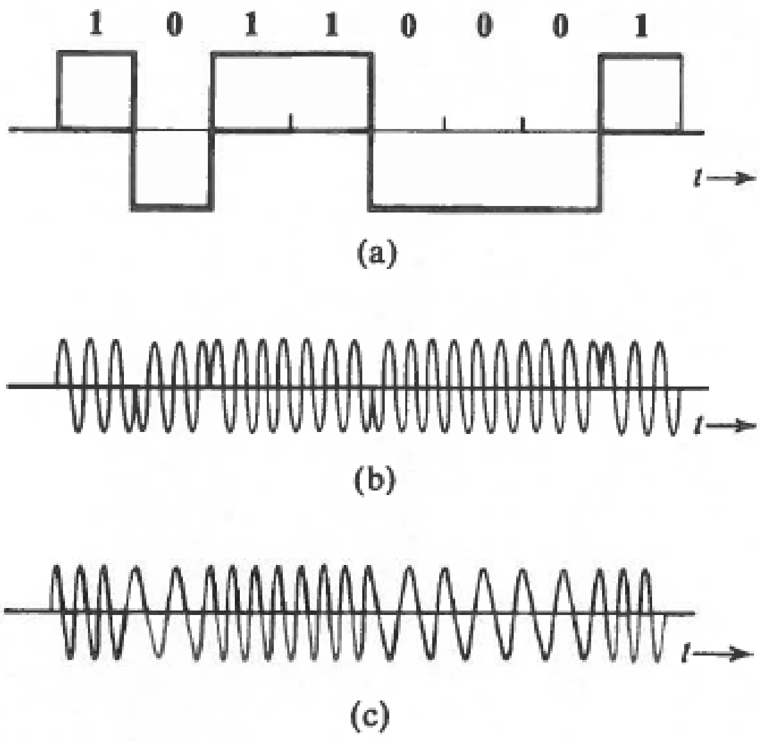
- Noncoherent detection (envelope detection etc.)
  - Makes no explicit efforts to estimate the phase

Coherent Detection of ASK



Assume an ideal band-pass filter with unit gain on $[f_c - W, f_c + W]$. For a practical band-pass filter, $2W$ should be interpreted as the equivalent bandwidth.

Phase and Frequency Shift Keying (PSK, FSK)



$m(t)$: Polar non-return-to-zero

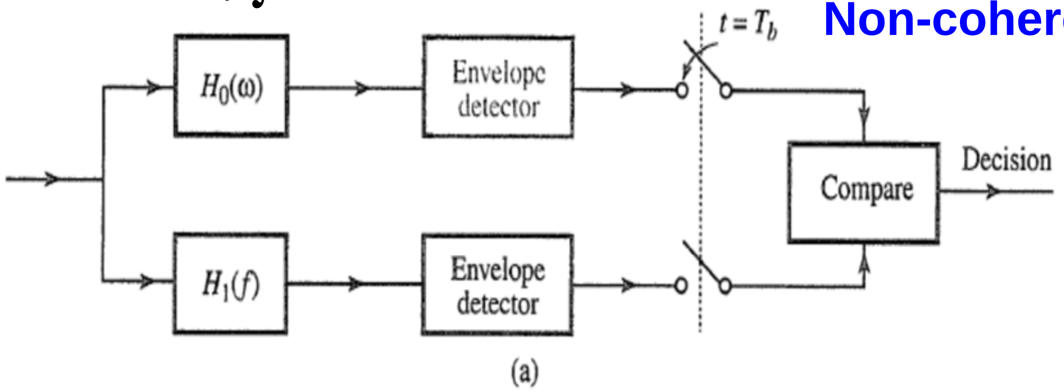
$$\varphi_{PSK} = m(t)\cos(\omega_c t)$$

$$\varphi_{FSK} = \cos[\omega_c t + k_f \int m(t)dt]$$

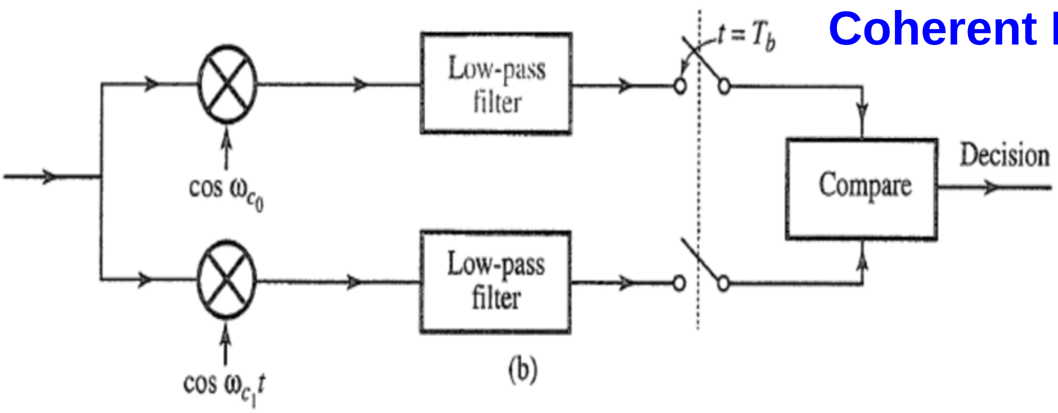
FSK Non-coherent and Coherent Detection

$$\varphi_{FSK} = \cos[\omega_c t + k_f \int m(t)dt]$$

Non-coherent Detection

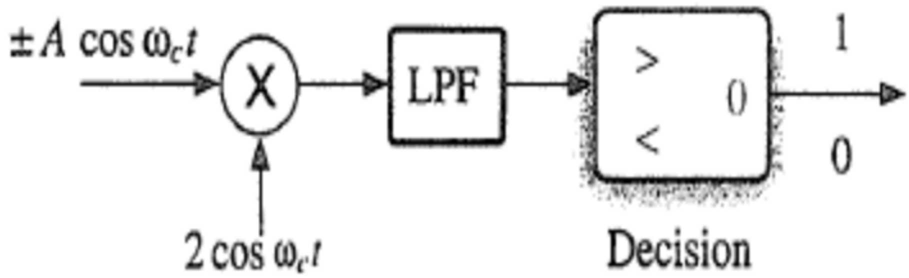


Coherent Detection



PSK Coherent Detection

$$\varphi_{PSK} = m(t)\cos(\omega_c t)$$



Envelop detection is not applicable to PSK

Signal Bandwidth, Channel Bandwidth & Channel Capacity (Maximum Data Rate)

- Signal bandwidth: the range of frequencies present in the signal
- Channel bandwidth: the range of signal bandwidths allowed (or carried) by a communication channel without significant loss of energy or distortion
- Channel capacity (maximum data rate): the maximum rate (in bits/second) at which data can be transmitted over a given communication channel

Two Views of Nyquist Rate

- Nyquist rate: 2 times of the bandwidth

- Sampling rate: For a given signal of bandwidth B Hz, the sampling rate must be at least 2B Hz to enable full signal recovery (i.e., avoid aliasing)

- Signaling rate: A noiseless communication channel with bandwidth B Hz can support the maximum rate of 2B symbols (signals, pulses or codewords) per second – so called the "baud rate"

Channel Capacity (Maximum Data Rate) with Channel Bandwidth B Hz

- Noiseless channel
  - Each symbol represents a signal of *M* levels (where *M*=2 and 4 for binary symbol and QPSK, respectively)
  - Channel capacity (maximum data rate): bits/second

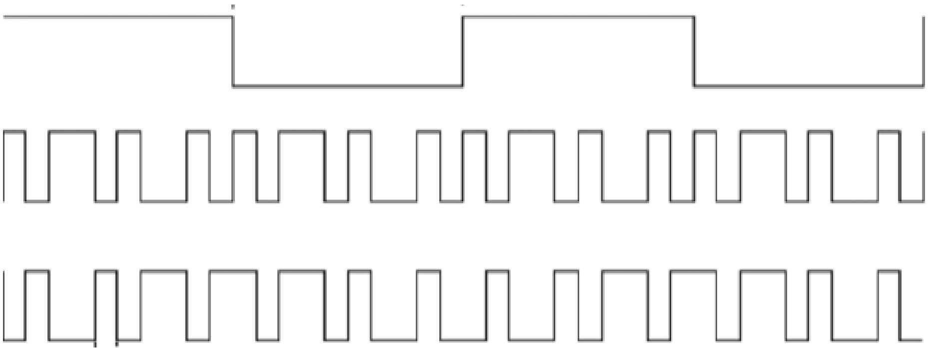
$$C = 2B \log_2 M$$

- Noisy channel
  - Shannon's channel capacity (maximum data rate): bits/second

$$C = B \log_2 (S / N)$$

where *S* and *N* denote the signal and noise power, respectively

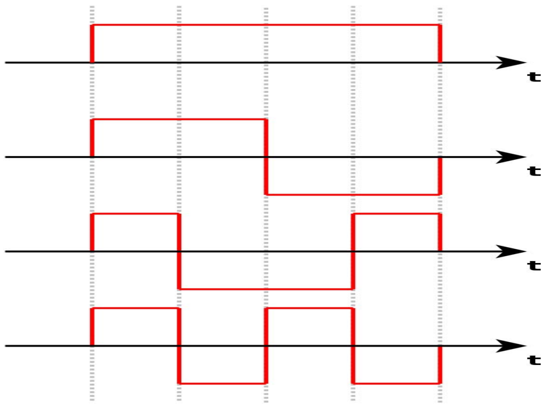
Introduction to CDMA (Code Division Multiple Access)



Courtesy by Marcos Vicente on Wikipedia

- Each user data (bit) is represented by a number of "chips" – pseudo random code – forming a spread-spectrum technique
- Pseudo random codes
  - Appear random but can be generated easily
  - Have close to zero auto-correlation with non-zero time offset (lag)
  - Have very low cross-correlation (almost orthogonal) for simultaneous use by multiple users – thus the name, CDMA

Use of Orthogonal Codes for Multiple Access



- Transmission: Spread each information bit using a code
- Detection: Correlate the received signal with the correspond code
- Orthogonal spreading codes ensure low mutual interference among concurrent transmissions
- Use codes to support multiple concurrent transmissions – Code-division multiple access (CDMA), besides time-division multiple access (TDMA) and frequency-diversion multiplex (FDMA)
- FDMA – 1G, TDMA – 2G, CDMA – 3G, 4G… cellular networks

Conclusions

- Highlighted digital communication systems
- Importance of digital communication
- Sampling and quantization
- Modulation of digital signals
- Channel bandwidth and capacity
- Introduction to CDMA