EE1 and EIE1: Introduction to Signals and Communications

Professor Kin K. Leung

EEE and Computing Departments

Imperial College

kin.leung@imperial.ac.uk

Lecture one

Course Aims

To introduce:

1. How signals can be represented and interpreted in time and frequency domains
2. Basic principles of communication systems
3. Methods for modulating and demodulating signals to carry information from an source to a destination

Recommended text book

B.P Lathi and Z. Ding, *Modern Digital and Analog Communication Systems*, Oxford University Press

- Highly recommended
- Well balanced book
- It will be useful in the future
- Slides based on this book, most of the figures are taken from this book

Handouts

- Copies of the transparencies
- Problem sheets and solutions
- Everything is on the web
  http://www.commsp.ee.ic.ac.uk/~kkleung/Intro_Signals_Comm_2018

Syllabus

- Fundamentals of Signals and Systems
  - Energy and power
  - Trigonometric and Exponential Fourier Series
  - Fourier transform
  - Linear system and convolution integral
- Modulation
  - Amplitude modulation: DSB, Full AM, SSB
  - Angle modulation: PM, FM
- Advanced Topics: Digital communications, CDMA

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Three examples of communication systems

Another example of Communication Systems...

From the movie ’The Blues Brothers’

Communication Systems

A **source** originates a message, such as a human voice, a television picture, a teletype message.

The message is converted by an input **transducer** into an electrical waveform (**baseband signal**).

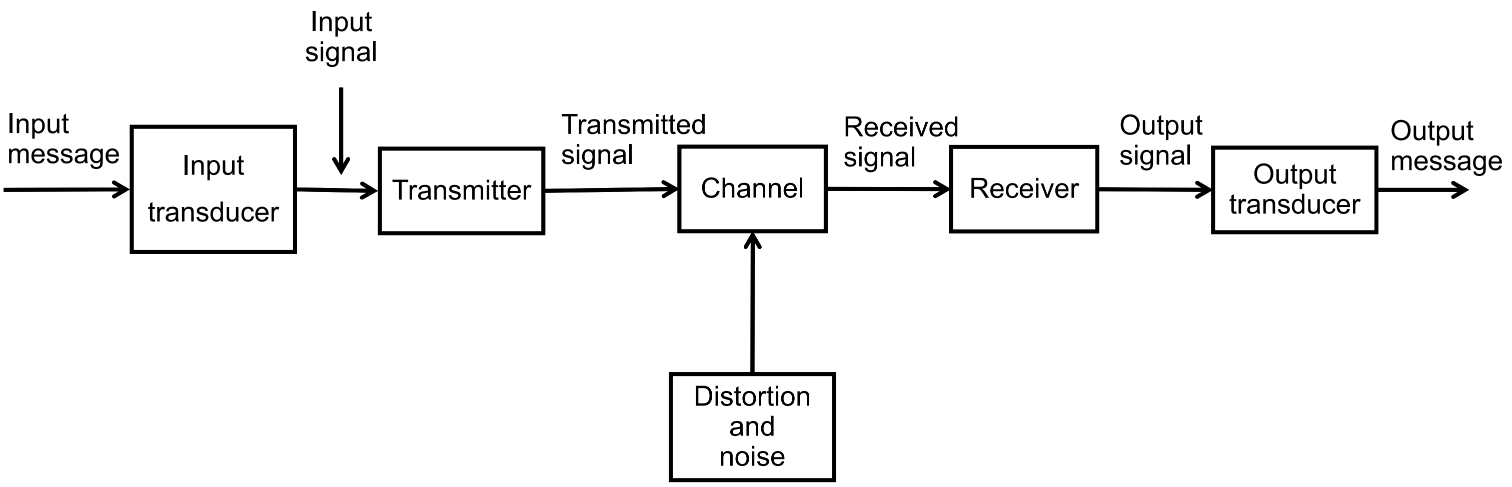
The **transmitter** modifies the baseband for efficient transmission.

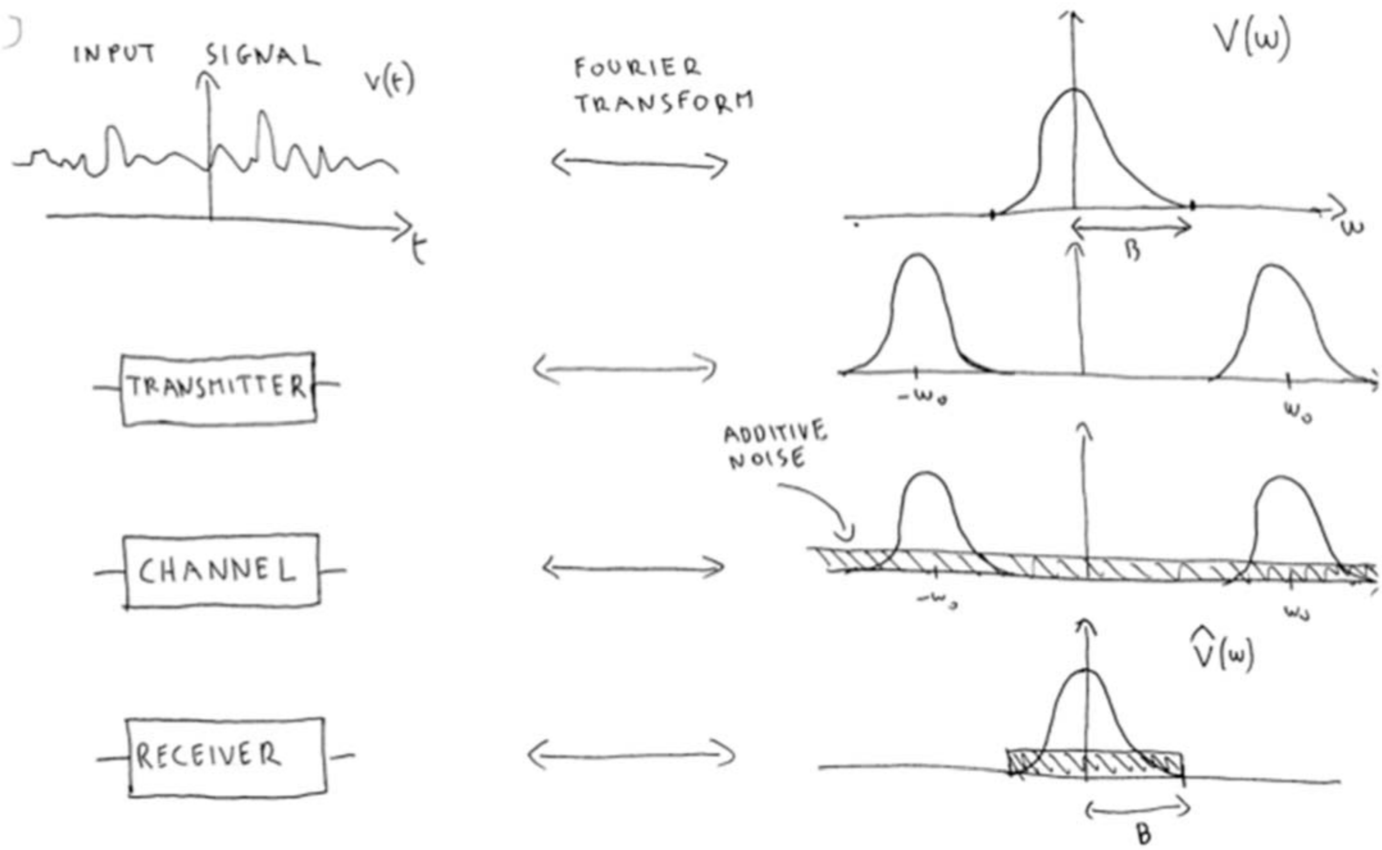
The **channel** is a medium such as a coaxial cable, an optical fiber, a radio link.

The **receiver** processes the signal received to undo modifications made at the transmitter and the channel.

The **output transducer** converts the signal into the original form.

Communications





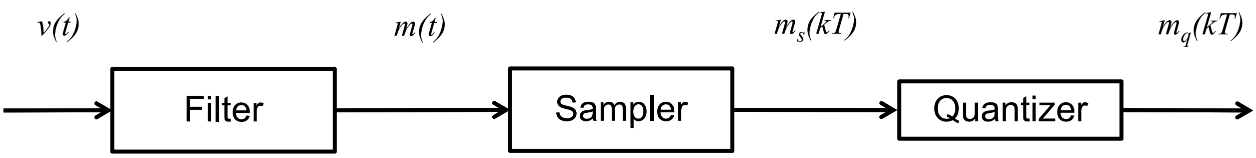
Analog and digital messages

- Message are digital or analog.
- Digital messages are constructed with a finite number of symbols. Example: a Morse-coded telegraph message.
- Analog messages are characterized by data whose values vary over a continuous range. For example, the temperature of a certain location.

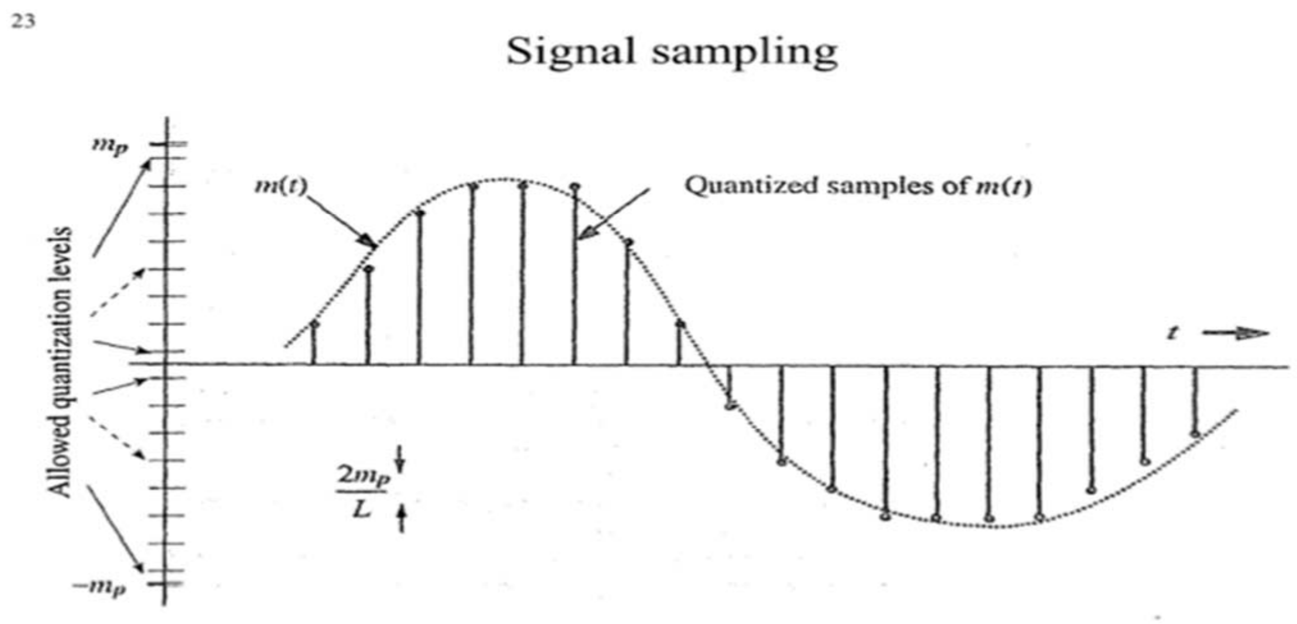
Digital Transmission

Digital signals are more robust to noise.

An analog signal is converted to a digital signal by means of an analog-to-digital (A/D) converter.

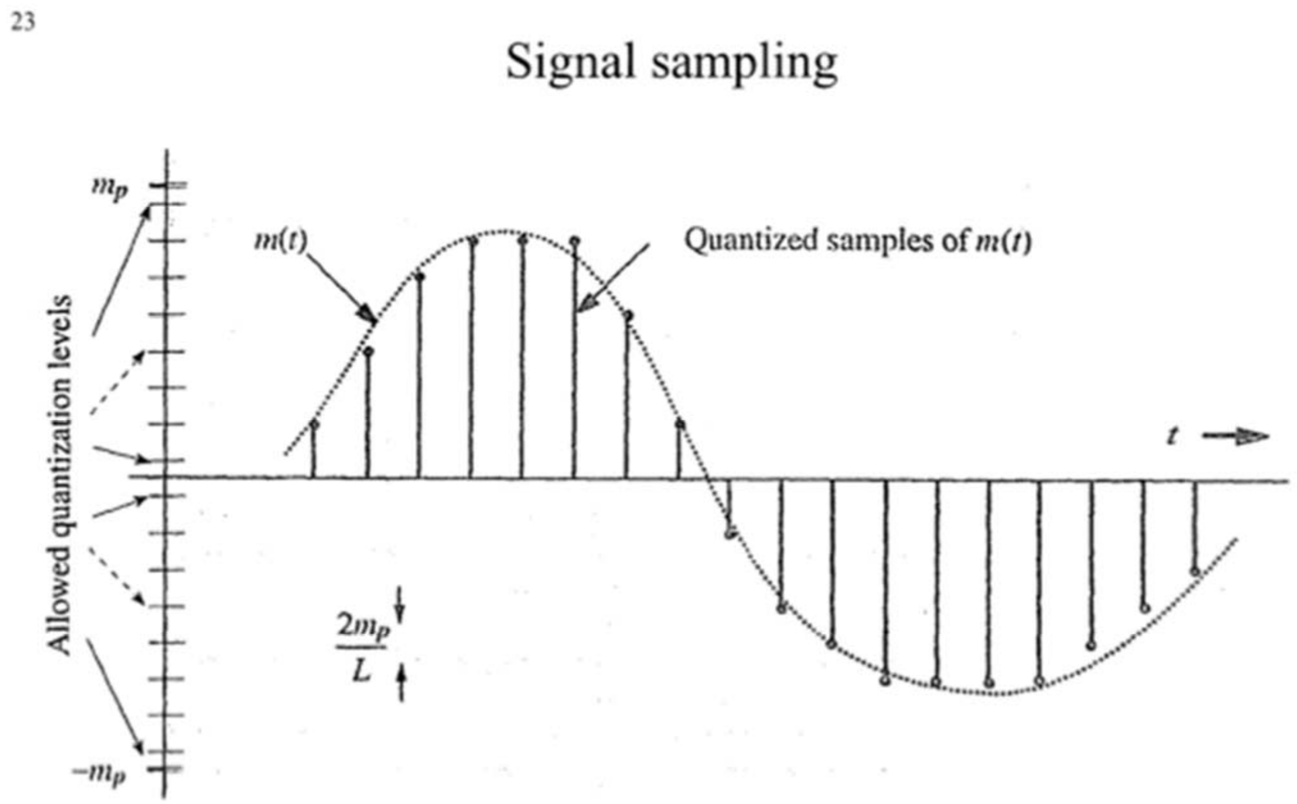


A/D conversion



The signal $m(t)$ is first sampled in the time domain.

The amplitude of the signal samples $m_s(kT)$ is partitioned into a finite number of intervals (quantization).



Sampling theorem

The sampling theorem states that

If the highest frequency in the signal spectrum is B, the signal can be reconstructed from its samples taken at a rate not less than $2B$ sample per second.

What did we learn today?

- The main elements of a communication systems
- The importance of the Fourier transforms
- Concept of signal bandwidth
- Analog and digital signals

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Lecture two

Lecture Aims

- To introduce signals
- Classifications of signals
- Some particular signals

Signals

- A signal is a set of information or data
- Examples
  - a telephone or television signal
  - monthly sales of a corporation
  - the daily closing prices of a stock market
- We deal exclusively with signals that are functions of time

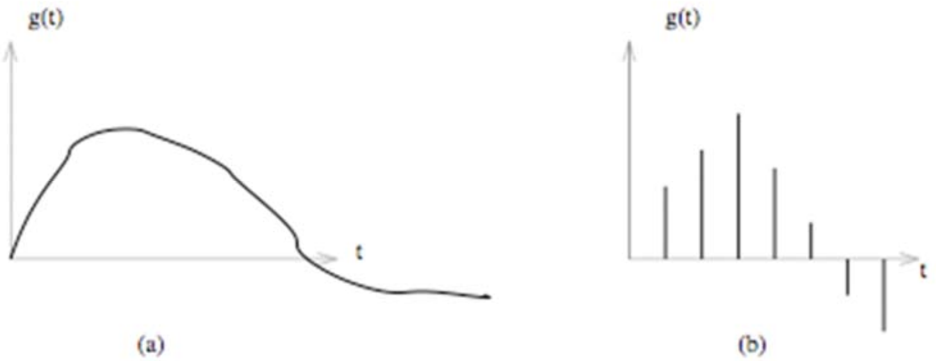
How can we measure a signal?
How can we distinguish two different signals?

Classifications of Signals

- Continuous-time and discrete-time signals
- Analog and digital signals
- Periodic and aperiodic signals
- Energy and power signals
- Deterministic and probabilistic signals

Continuous-time and discrete-time signals

- A signal that is specified for every value of time t is a continuous-time signal
- A signal that is specified only at discrete values of t is a discrete-time signals



Continuous-time and discrete-time signals (continued)

- A discrete-time signal can be obtained by **sampling** a continuous-time signal.

- In some cases, it is possible to ’undo’ the sampling operation. That is, it is possible to get back the continuous-time signal from the discrete-time signal.

Sampling Theorem

The sampling theorem states that if the highest frequency in the signal spectrum is B, the signal can be reconstructed from its samples taken at a rate not less than $2B$ samples per second.

Analog and digital signals

- A signal whose amplitude can take on any value in a continuous range is an analog signal

- The concept of analog and digital signals is different from the concept of continuous-time and discrete-time signals

- For example, we can have a digital and continuous-time signal, or a analog and discrete-time signal

Analog and digital signals (continued)

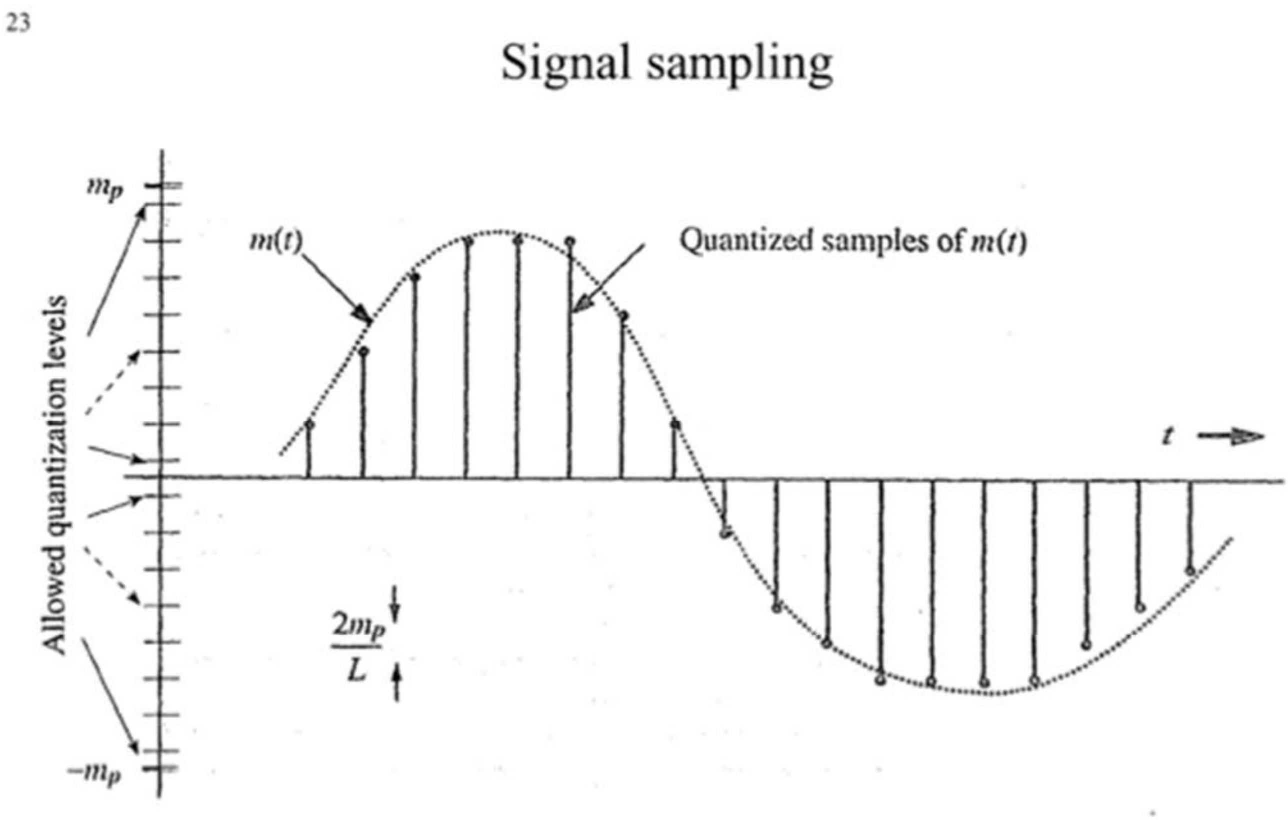
- One can obtain a digital signal from an analog one using a *quantizer*

- The amplitude of the analog signal is partitioned into L intervals. Each sample is approximated to the midpoint of the interval in which the original value falls

- Quantization is a **lossy** operation

Notice that:
One can obtain a digital discrete-time signal by sampling and quantizing an analog continuous-time signal

Conversion of continuous-time analog to discrete-time digital signal



Periodic and aperiodic signals

- A signal $g(t)$ is said to be periodic if for some positive constant T_0,

$$g(t) = g(t + T_0) \text{ for all } t$$

- A signal is aperiodic if it is **not** periodic

Same *famous* periodic signals:

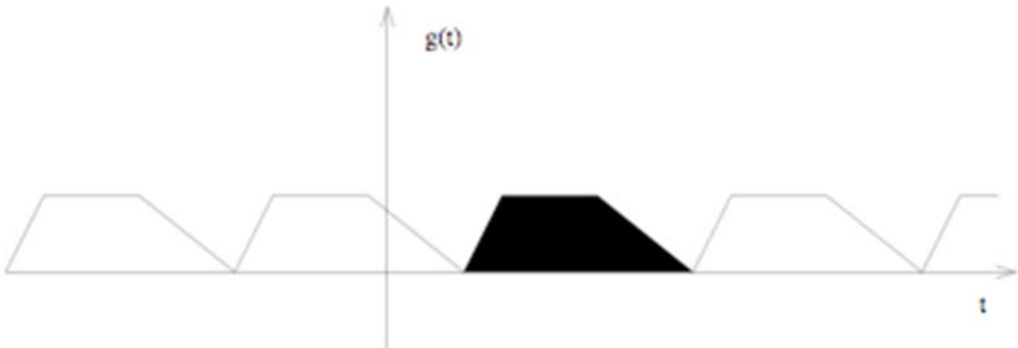
$$\sin \omega_0 t,\ \ \cos \omega_0 t,\ \ e^{j\omega_0 t},$$

where $\omega_0 = 2\pi/T_0$ and T_0 is the period of the function

(Recall that $e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$)

Periodic Signal

A periodic signal $g(t)$ can be generated by periodic extension of any segment of $g(t)$ of duration T_0



Energy and power signal

First, define energy

- The signal energy E_g of $g(t)$ is defined (for a real signal) as

$$E_g \;=\; \int_{-\infty}^{\infty} g^2(t)dt.$$

- In the case of a complex valued signal $g(t)$, the energy is given by

$$E_g = \int_{-\infty}^{\infty} g^*(t)g(t)dt = \int_{-\infty}^{\infty} |g(t)|^2\, dt.$$

- A signal $g(t)$ **is an energy signal** if $E_g < \infty$

Power

A necessary condition for the energy to be finite is that the signal amplitude goes to zero as time tends to infinity.
In case of signals with infinite energy (e.g., periodic signals), a more meaningful measure is the signal power.

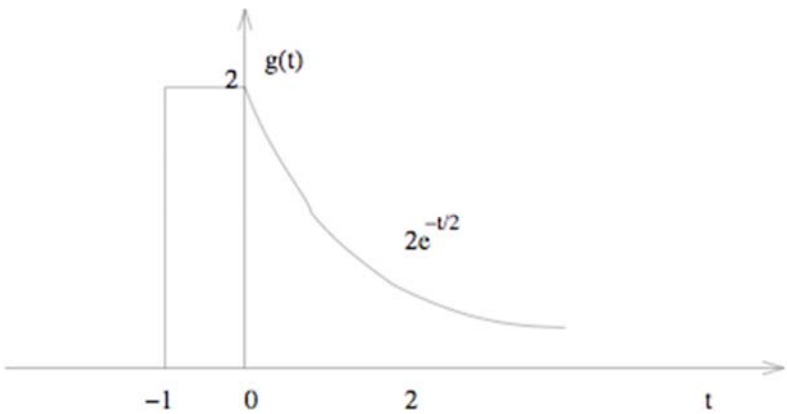
$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt$$

A signal is a power signal if

$$0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt < \infty$$

A signal cannot be an energy and a power signal at the same time

Energy signal example



Signal Energy calculation

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-1}^{0} (2)^2 dt + \int_{0}^{\infty} 4e^{-t} dt = 4 + 4 = 8.$$

Power signal example

Assume $g(t)$ = $A\cos(\omega_0 t + \theta)$, its power is given by

$$\begin{aligned}
P_g &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(w_0 t + \theta) dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} \left[1 + \cos(2w_0 t + 2\theta) \right] dt \\
&= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} dt + \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} \cos(2w_0 t + 2\theta) dt \\
&= A^2 / 2
\end{aligned}$$

Power of Periodic Signals

Show that the power of a periodic signal $g(t)$ with period T_0 is

$$P_g = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |g(t)|^2 \, dt$$

Another important parameter of a signal is the **time average**:

$$g_{average} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) dt.$$

Deterministic and probabilistic signals

- A signal whose physical description is known completely is a deterministic signal.
- A signal known only in terms of probabilistic descriptions is a random signal.

Summary

- Signal classification
- Power of a periodic signal of period T_0

$$P_g = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |g(t)|^2 \, dt$$

- Time average

$$g_{average} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)dt$$

- Power of a sinusoid $A\cos(2\pi f_0 t + \theta)$ is $\dfrac{A^2}{2}$

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Lecture three

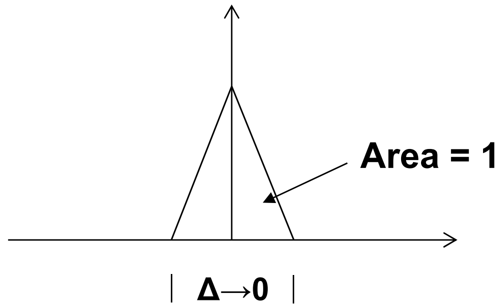
Lecture Aims

- To introduce some useful signals
- To present analogies between vectors and signals
  - Signal comparison: correlation
  - Energy of the sum of orthogonal signals
  - Signal representation by orthogonal signal set

Useful Signals: Unit impulse function

The unit impulse function or Dirac function is defined as

$$\delta(t) = 0 \quad t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



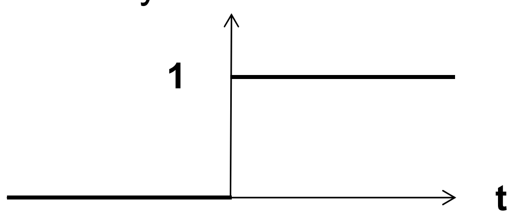
Multiplication of a function by an impulse:

$$g(t)\delta(t-T) = g(T)\delta(t-T)$$
$$\int_{-\infty}^{\infty} g(t)\delta(t-T)\,dt = g(T).$$

Useful Signals: Unit step function

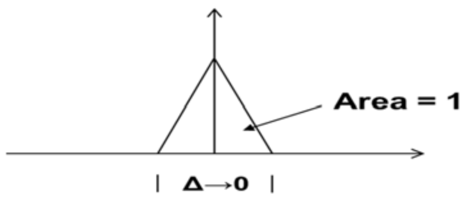
Another useful signal is the unit step function $u(t)$, defined by

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Observe that

$$\int_{-\infty}^{t} \delta(\alpha)\,d\alpha = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



Therefore

$$\frac{du(t)}{dt} = \delta(t).$$

Use intuition to understand this relationship: The derivative of a 'unit step jump' is an unit impulse function.

Useful Signals: Sinusoids

Consider the sinusoid

$$x(t) = C\cos(2\pi f_0 t + \theta)$$

f_0 (measured in Hertz) is the frequency of the sinusoid and $T_0 = 1/f_0$ is the period.

Sometimes we use ω_0 (radiant per second) to express $2\pi f_0$.

Important identities

$$e^{\pm jx} = \cos x \pm j\sin x, \cos x = \frac{1}{2}\left[e^{jx} + e^{-jx}\right], \sin x = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right],$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$a\cos x + b\sin x = C\cos(x+\theta)$$

with $C = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{-b}{a}$

Signals and Vectors

- Signals and vectors are closely related. For example,
  - A vector has components
  - A signal has also its components
- Begin with some basic vector concepts
- Apply those concepts to signals

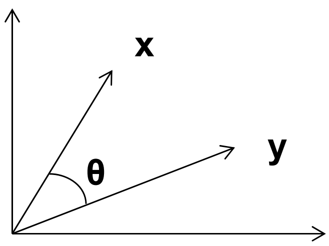
Inner product in vector spaces

x is a certain vector.

It is specified by its magnitude or length $|x|$ and direction.

Consider a second vector $\mathbf{y}$.

We define the inner or scalar product of two vectors as



$$<\mathbf{y}, \mathbf{x}> = |x||y| \cos \theta.$$

Therefore, $|x|^2 = <\mathbf{x}, \mathbf{x}>$.

When $<\mathbf{y}, \mathbf{x}> = 0$, we say that $\mathbf{y}$ and $\mathbf{x}$ are orthogonal (geometrically, $\theta = \pi/2$).

Signals as vectors

The same notion of inner product can be applied for signals.
What is the useful part of this analogy?
We can use some geometrical interpretation of vectors to understand signals!
Consider two (energy) signals $x(t)$ and $y(t)$.
The inner product is defined by

$$\left\langle x(t), y(t) \right\rangle = \int_{-\infty}^{\infty} x(t) y(t) dt$$

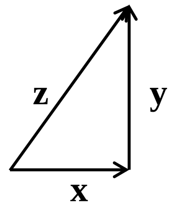
For complex signals

$$\left\langle x(t), y(t) \right\rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

where $y^*(t)$ denotes the complex conjugate of $y(t)$.

Two signals are orthogonal if $<x(t), y(t)> = 0$.

Energy of orthogonal signals



If vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, and if $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$|\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$ (Pythagorean Theorem).

If signals $x(t)$ and $y(t)$ are orthogonal and if $z(t) = x(t) + y(t)$ then

$$E_{\mathbf{z}} = E_x + E_y .$$

Proof for real $x(t)$ and $y(t)$:

$$\begin{aligned} E_z &= \int_{-\infty}^{\infty} (x(t) + y(t))^2\, dt \\ &= \int_{-\infty}^{\infty} x^2(t)dt + \int_{-\infty}^{\infty} y^2(t)dt + 2\int_{-\infty}^{\infty} x(t)y(t)dt \\ &= E_x + E_y + 2\int_{-\infty}^{\infty} x(t)y(t)dt \\ &= E_x + E_y \end{aligned}$$

since $\int_{-\infty}^{\infty} x(t)y(t)dt = 0.$

Power of orthogonal signals

The same concepts of orthogonality and inner product extend to power signals.

For example, $g(t) = x(t) + y(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)$ and $\omega_1 \neq \omega_2$.

$$P_x = \frac{C_1^2}{2}, \qquad P_y = \frac{C_2^2}{2}.$$

The signal $x(t)$ and $y(t)$ are orthogonal: $\langle x(t), y(t) \rangle = 0$. Therefore,

$$P_g = P_x + P_y = \frac{C_1^2}{2} + \frac{C_2^2}{2}.$$

Signal comparison: Correlation

If vectors $\mathbf{x}$ and $\mathbf{y}$ are given, we have the correlation measure as

$$c_n = \cos\theta = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{|\mathbf{x}||\mathbf{y}|}$$

Clearly, $-1 \leq c_\text{n} \leq 1$.

In the case of energy signals:

$$c_n = \frac{1}{\sqrt{E_y E_x}} \int_{-\infty}^{\infty} y(t)x(t)dt$$

again $-1 \leq c_\text{n} \leq 1$.

Best friends, worst enemies and complete strangers

- $c_\text{n} = 1$. **Best friends**. This happens when $g(t) = Kx(t)$ and K is positive. The signals are aligned, maximum similarity.

- $c_\text{n} = -1$. **Worst Enemies**. This happens when $g(t) = Kx(t)$ and K is negative. The signals are again aligned, but in opposite directions. The signals *understand* each others, but they do not like each others.

- $c_\text{n} = 0$. **Complete Strangers** The two signals are orthogonal. We may view orthogonal signals as unrelated signals.

Correlation

Why do we bother poor undergraduate students with correlation?
Correlation is widely used in engineering.
For instance

- To design receivers in many communication systems
- To identify signals in radar systems
- For classifications

Correlation examples

Find the correlation coefficients between:

- $x(t) = \mathrm{A}_0 \cos(\omega_0 t)$ and $y(t) = \mathrm{A}_1 \sin(\omega_1 t)$.
- $x(t) = \mathrm{A}_0 \cos(\omega_0 t)$ and $y(t) = \mathrm{A}_1 \cos(\omega_1 t)$ and $\omega_0 \neq \omega_1$.
- $x(t) = \mathrm{A}_0 \cos(\omega_0 t)$ and $y(t) = \mathrm{A}_1 \cos(\omega_0 t)$.
- $x(t) = \mathrm{A}_0 \sin(\omega_0 t)$ and $y(t) = \mathrm{A}_1 \sin(\omega_1 t)$ and $\omega_0 \neq \omega_1$.
- $x(t) = \mathrm{A}_0 \sin(\omega_0 t)$ and $y(t) = \mathrm{A}_1 \sin(\omega_0 t)$.
- $x(t) = \mathrm{A}_0 \sin(\omega_0 t)$ and $y(t) = -\mathrm{A}_1 \sin(\omega_0 t)$.

Correlation examples

Find the correlation coefficients between:

- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_1 t)$ and $\omega_0 \neq \omega_1$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \cos(\omega_0 t)$ and $y(t) = A_1 \cos(\omega_0 t)$ $\quad c_{x,y} = 1.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_1 t)$ and $\omega_0 \neq \omega_1$ $\quad c_{x,y} = 0.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = A_1 \sin(\omega_0 t)$ $\quad c_{x,y} = 1.$
- $x(t) = A_0 \sin(\omega_0 t)$ and $y(t) = -A_1 \sin(\omega_0 t)$ $\quad c_{x,y} = -1.$

Signal representation by orthogonal signal sets

- Examine a way of representing a signal as a sum of orthogonal signals
- We know that a vector can be represented as the sum of orthogonal vectors
- The results for signals are parallel to those for vectors
- Review the case of vectors and extend to signals

Orthogonal vector space

Consider a three-dimensional Cartesian vector space described by three mutually orthogonal vectors, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

$$<\mathbf{x}_{\mathrm{m}}, \mathbf{x}_{\mathrm{n}}> = \begin{cases} 0 & m \neq n \\ |\mathbf{x}_{\mathrm{m}}|^2 & m = n \end{cases}$$

Any three-dimensional vector can be expressed as a linear combination of those three vectors: $\mathbf{g} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3$

where $a_i = \dfrac{<\mathbf{g},\mathbf{x}_{\mathrm{i}}>}{|\mathbf{x}_{\mathrm{i}}|^2}$

In this case, we say that this set of vectors is *complete*.

Such vectors are known as a **basis** vector.

Orthogonal signal space

Same notions of completeness extend to signals.
A set of mutually orthogonal signals $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ is complete if it can represent any signal belonging to a certain space. For example:

$$g(t) \sim a_1 x_1(t) + a_2 x_2(t) + \ldots + a_N x_N(t)$$

If the approximation error is zero for any $g(t)$ then the set of signals $x_1(t)$, $x_2(t)$, ..., $x_N(t)$ is complete. In general, the set is complete when $N \rightarrow \infty$.
Infinite dimensional space (this will be more clear in the next lecture).

Summary

- Analogies between vectors and signals
- Inner product and correlation
- Energy and Power of orthogonal signals
- Signal representation by means of orthogonal signal

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Lecture four

Lecture Aims

- Trigonometric Fourier series
- Fourier spectrum
- Exponential Fourier series

Trigonometric Fourier series

- Consider a signal set

$$\{1, \cos\omega_0 t, \cos 2\omega_0 t, ..., \cos n\omega_0 t, ..., \sin\omega_0 t, \sin 2\omega_0 t, ..., \sin n\omega_0 t, ...\}$$

- A sinusoid of frequency $n\omega_0 t$ is called the n^{th} harmonic of the sinusoid, where n is an integer.
- The sinusoid of frequency ω_0 is called the fundamental harmonic.
- This set is orthogonal over an interval of duration $T_0 = 2\pi/\omega_0$, which is the period of the fundamental harmonic.

Trigonometric Fourier series

The components of the set $\{1, \cos\omega_0 t, \cos 2\omega_0 t, ..., \cos n\omega_0 t, ..., \sin\omega_0 t, \sin 2\omega_0 t, ..., \sin n\omega_0 t, ...\}$ are orthogonal as

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t dt = \begin{cases} 0 & m \neq n \\ \dfrac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t dt = 0$$ for all m and n

$\int_{T_0}$ means integral over an interval from $t = t_1$ to $t = t_1 + T_0$ for any value of t_1.

Trigonometric Fourier series

This set is also complete in T_0. That is, any signal in an interval $t_1 \leq t \leq t_1 + T_0$ can be written as the sum of sinusoids. Or

$$g(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + ... + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + ...$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Series coefficients

$$a_n = \frac{\langle g(t), \cos n\omega_0 t\rangle}{\langle \cos n\omega_0 t, \cos n\omega_0 t\rangle} \qquad b_n = \frac{\langle g(t), \sin n\omega_0 t\rangle}{\langle \sin n\omega_0 t, \sin n\omega_0 t\rangle}$$

Trigonometric Fourier Coefficients

Therefore

$$a_n = \frac{\int_{t_1}^{t_1+T_0} g(t)\cos n\omega_0 t dt}{\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t dt}$$

As

$$\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t dt = T_0/2, \qquad \int_{t_1}^{t_1+T_0} \sin^2 n\omega_0 t dt = T_0/2.$$

We get

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t) dt$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\cos n\omega_0 t dt \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\sin n\omega_0 t dt \qquad n = 1, 2, 3, \ldots$$

Compact Fourier series

Using the identity

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n)$$

where

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad \theta_n = \tan^{-1}(-b_n/a_n).$$

The trigonometric Fourier series can be expressed in compact form as

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad t_1 \le t \le t_1 + T_0.$$

For consistency, we have denoted a_0 by C_0.

Periodicity of the Trigonometric series

We have seen that an arbitrary signal $g(t)$ may be expressed as a trigonometric Fourier series over any interval of T_0 seconds.

What happens to the Trigonometric Fourier series outside this interval?

Answer: The Fourier series is periodic of period T_0 (the period of the fundamental harmonic).

Proof:

$$\phi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad \text{for all } t$$

and

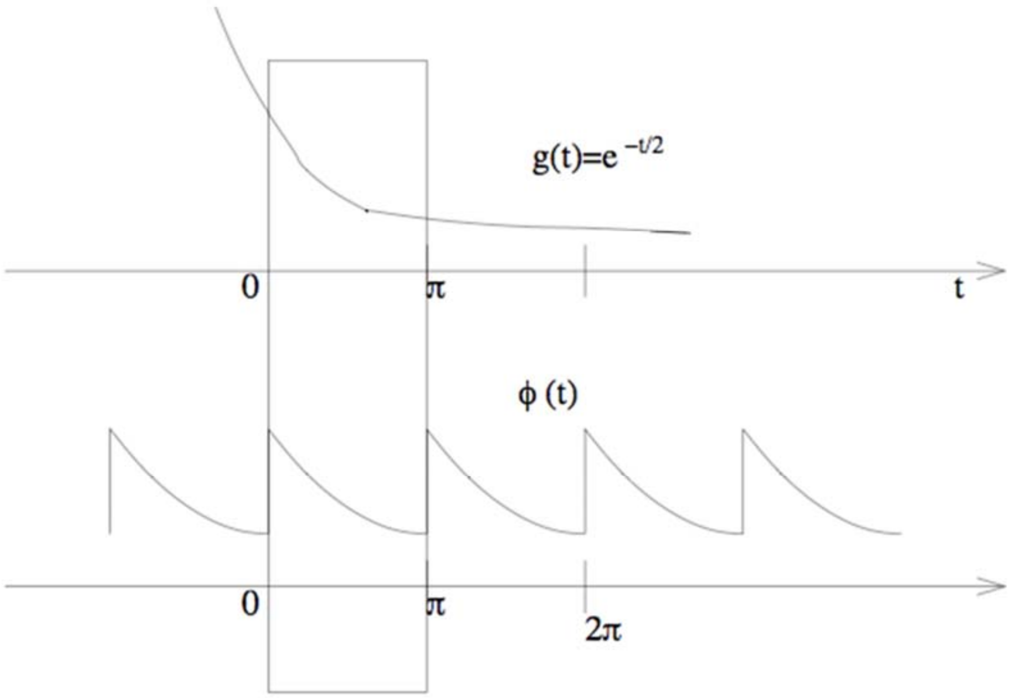
$$\begin{aligned}
\phi(t+T_0) &= C_0 + \sum\nolimits_{n=1}^{\infty} C_n \cos\left[n\omega_0\left(t+T_0\right)+\theta_n\right] \\
&= C_0 + \sum\nolimits_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + 2n\pi + \theta_n\right) \\
&= C_0 + \sum\nolimits_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) \\
&= \phi(t) \qquad \text{for all } t
\end{aligned}$$

Properties of trigonometric series

- The trigonometric Fourier series is a periodic function of period $T_0 = 2\pi/\omega_0$.

- If the function $g(t)$ is periodic with period T_0, then a Fourier series representing $g(t)$ over an interval T_0 will also represent $g(t)$ for all t.

Example



Example

$\omega_0 = 2\pi / T_0 = 2$ rad / s.

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2nt + \theta_n)$$

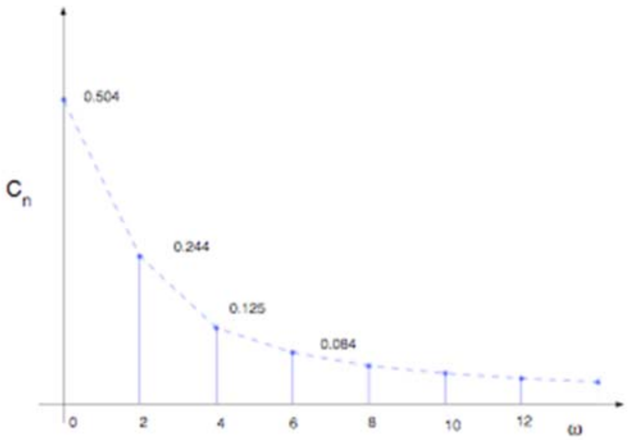
n	0	1	2	3	4
C_n	0.504	0.244	0.125	0.084	0.063
θ_n	0	-75.96	-82.87	-85.84	-86.42

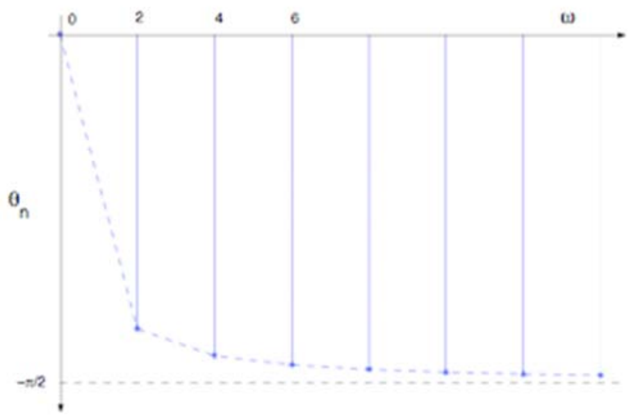
We can plot

- the amplitude C_n versus ω this gives us the **amplitude spectrum**
- the phase θ_n versus ω (**phase spectrum**).

This two plots together are the **frequency spectra** of $g(t)$.

Amplitude and phase spectra





Exponential Fourier Series

Consider a set of exponentials

$$e^{jn\omega_0 t} \qquad n = 0, \pm 1, \pm 2, \ldots$$

The components of this set are orthogonal.

A signal $g(t)$ can be expressed as an exponential series over an interval T_0:

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn\omega_0 t} dt$$

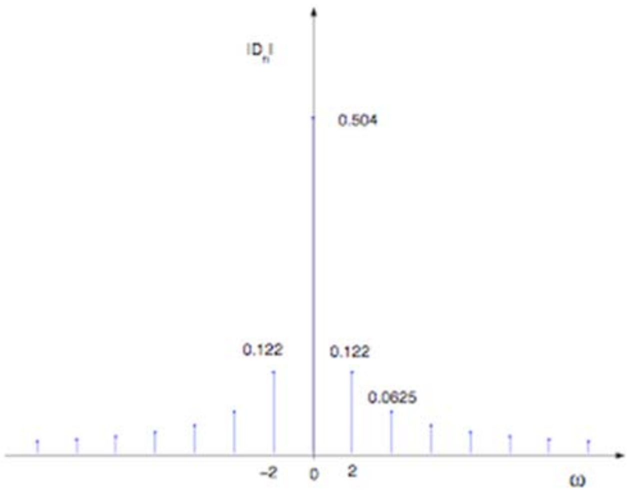
Trigonometric and exponential Fourier series

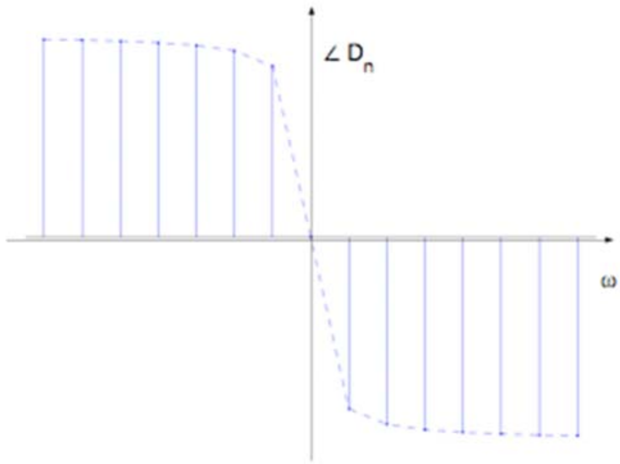
Trigonometric and exponential Fourier series are related. In fact, a sinusoid in the trigonometric series can be expressed as a sum of two exponentials using Euler's formula.

$$
\begin{aligned}
C_n \cos(n\omega_0 t + \theta_n) &= \frac{C_n}{2}\left[e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right] \\
&= \left(\frac{C_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \left(\frac{C_n}{2} e^{-j\theta_n}\right) e^{-jn\omega_0 t} \\
&= D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}
\end{aligned}
$$

$$
D_n = \frac{1}{2} C_n e^{j\theta_n} \qquad D_{-n} = \frac{1}{2} C_n e^{-j\theta_n}
$$

Amplitude and phase spectra. Exponential case





Parseval's Theorem

Trigonometric Fourier series representation $g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$.
The power is given by

$$P_g = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2.$$

Exponential Fourier series representation $g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$.
Power for the exponential representation

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Conclusions

- Trigonometric Fourier series
- Exponential Fourier series
- Amplitude and phase spectra
- Parseval's theorem

EE1 and EIE1: Introduction to Signals and Communications

Professor Kin K. Leung

EEE and Computing Departments

Imperial College

kin.leung@imperial.ac.uk

Lecture five

Lecture Aims

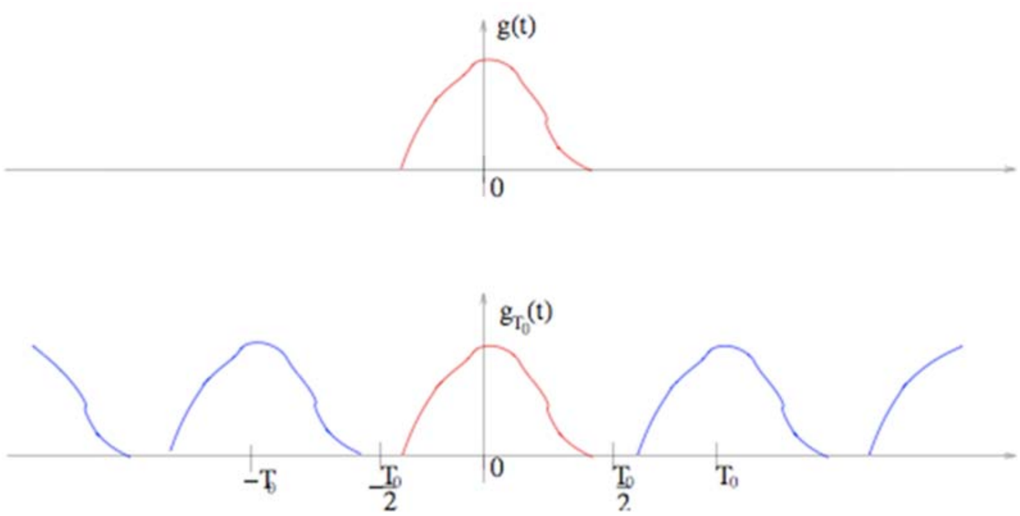
- To introduce Fourier integral, Fourier transformation
- To present transforms of some useful functions
- To discuss some properties of the Fourier transform

Introduction

- We electrical engineers think of signals in terms of their spectral content.
- We have studied the spectral representation of periodic signals.
- We now extend this spectral representation to the case of aperiodic signals.

Aperiodic signal representation

We have an aperiodic signal $g(t)$ and we consider a periodic version $g_{T0}(t)$ of such signal obtained by repeating $g(t)$ every T_0 seconds.



The periodic signal $g_{T0}(t)$

The periodic signal $g_{T0}(t)$ can be expressed in terms of $g(t)$ as follows:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0).$$

Notice that, if we let $T_0 \to \infty$, we have

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t).$$

The Fourier representation of $g_{T0}(t)$

The signal $g_{T0}(t)$ is periodic, so it can be represented in terms of its Fourier series.
The basic intuition here is that the Fourier series of $g_{T0}(t)$ will also represent $g(t)$ in the limit for $T_0 \to \infty$.
The exponential Fourier series of $g_{T0}(t)$ is

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

where

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t)\, e^{-jn\omega_0 t} dt$$

and

$$\omega_0 = \frac{2\pi}{T_0}.$$

The Fourier representation of $g_{T_0}(t)$

Integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. So we can write

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}dt.$$

If we define a function

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

then we can write the Fourier coefficients D_n as follows:

$$D_n = \frac{1}{T_0}G(n\omega_0).$$

Computing the $\lim_{T_0\to\infty} g_{T_0}(t)$

Thus $g_{T_0}(t)$ can be expressed as:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0}.$$

Assuming $\frac{1}{T_0} = \frac{\Delta\omega}{2\pi}$ (i.e., replace notation ω_0 by $\Delta\omega$), we get

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{j(n\Delta\omega)t}.$$

In the limit for $T_0 \to \infty$, $\Delta\omega \to 0$ and $g_{T_0}(t) \to g(t)$.

We thus get:

$$g(t) = \lim_{T_0\to\infty} g_{T_0}(t) = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{G(n\Delta\omega)\Delta\omega}{2\pi} e^{j(n\Delta\omega)t}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega.$$

Fourier Transform and Inverse Fourier Transform

What we have just learned is that, from the spectral representation $G(\omega)$ of $g(t)$, that is, from

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt,$$

we can obtain $g(t)$ back by computing

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega.$$

Fourier transform of $g(t)$:

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt.$$

Inverse Fourier transform:

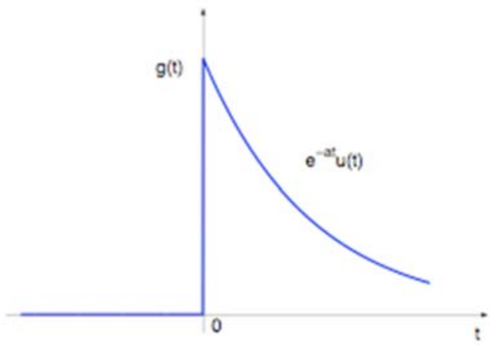
$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega.$$

Fourier transform relationship:

$$g(t) \Leftrightarrow G(\omega).$$

Example

Find the Fourier transform of $g(t) = e^{-at}u(t)$.



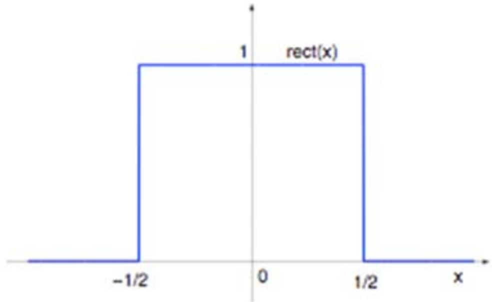
$$G(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}.$$

Since $\left|e^{-j\omega t}\right| = 1$, we have that $\lim_{t\to\infty} e^{-at}e^{-j\omega t}$. Therefore:

$$G(\omega) = \frac{1}{a+j\omega}, \quad \left|G(\omega)\right| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \theta_g(\omega) = -\tan^{-1}(\frac{\omega}{a}).$$

Some useful functions

The Unit Gate Function:

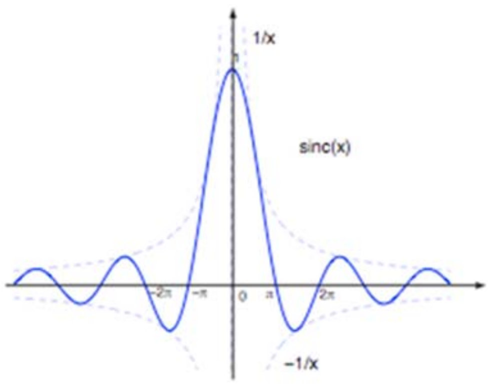


The unit gate function rect(x) is defined as:

$$rect(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$

Some useful functions

The function $\sin(x)/x$ '**sine over argument**' function is denoted by $\mathrm{sinc}(x)$:



- $\mathrm{sinc}(x)$ is an even function of x.
- $\mathrm{sinc}(x)$ = 0 when $\sin(x)$ = 0 and $x \neq 0$.
- Using L'Hospital's rule, we find that $\mathrm{sinc}(0)$ = 1
- $\mathrm{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ and a monotonically decreasing function $1/x$.

Example

Find the Fourier transform of $g(t) = \text{rect}(\, t/\tau\,)$.

$$G(\omega) = \int_{-\infty}^{\infty} rect\left(\frac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin(\omega\tau/2)}{\omega}$$

$$= \tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau \sin c(\omega\tau/2).$$

Therefore

$$rect(t/\tau) \;\Leftrightarrow\; \tau \sin c(\omega\tau/2)$$

Example

Find the Fourier transform of the unit impulse $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1.$$

Therefore

$$\delta(t) \quad \Leftrightarrow \quad 1$$

Find the inverse Fourier transform of $\delta(\omega)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\; e^{j\omega t} d\omega = \frac{1}{2\pi}.$$

Therefore

$$1 \quad \Leftrightarrow \quad 2\pi\delta(\omega)$$

Example

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\, e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}.$$

Therefore

$$e^{j\omega_0 t} \quad \Leftrightarrow \quad 2\pi\delta(\omega-\omega_0)$$

and

$$e^{-j\omega_0 t} \quad \Leftrightarrow \quad 2\pi\delta(\omega+\omega_0)$$

Example

Find the Fourier transform of the everlasting sinusoid $\cos(\omega_0 t)$.

Since

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and using the fact that $e^{j\omega_0 t} \Leftrightarrow 2\pi\delta\left(\omega-\omega_0\right)$ and $e^{-j\omega_0 t} \Leftrightarrow 2\pi\delta\left(\omega+\omega_0\right)$, we discover that

$$\cos(\omega_0 t) \Leftrightarrow \pi\left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right].$$

Summary

Fourier transform of $g(t)$:

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt,$$

Inverse Fourier transform:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega.$$

Fourier transform relationship:

$$g(t) \Leftrightarrow G(\omega).$$

Important Fourier transforms:

$$rect(t/\tau) \;\Leftrightarrow\; \tau \sin c(\omega\tau/2)$$

$$\delta(t) \quad \Leftrightarrow \quad 1$$

$$\cos(\omega_0 t) \;\Leftrightarrow\; \pi\left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right].$$

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Professor Kin K. Leung

EEE and Computing Departments

Imperial College

kin.leung@imperial.ac.uk

Lecture six

Lecture Aims

- To present some properties of the Fourier transform

Topics Covered

- Fourier transform table
- Symmetry of Fourier transformation
- Time and Frequency shifting property
- Convolution
- Time differentiation and time integration
- Please read Lathi & Ding

Some properties of Fourier transform

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$

Some properties of Fourier transform

6	$\delta(t)$	1
7	1	$2\pi\delta(\omega)$
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$

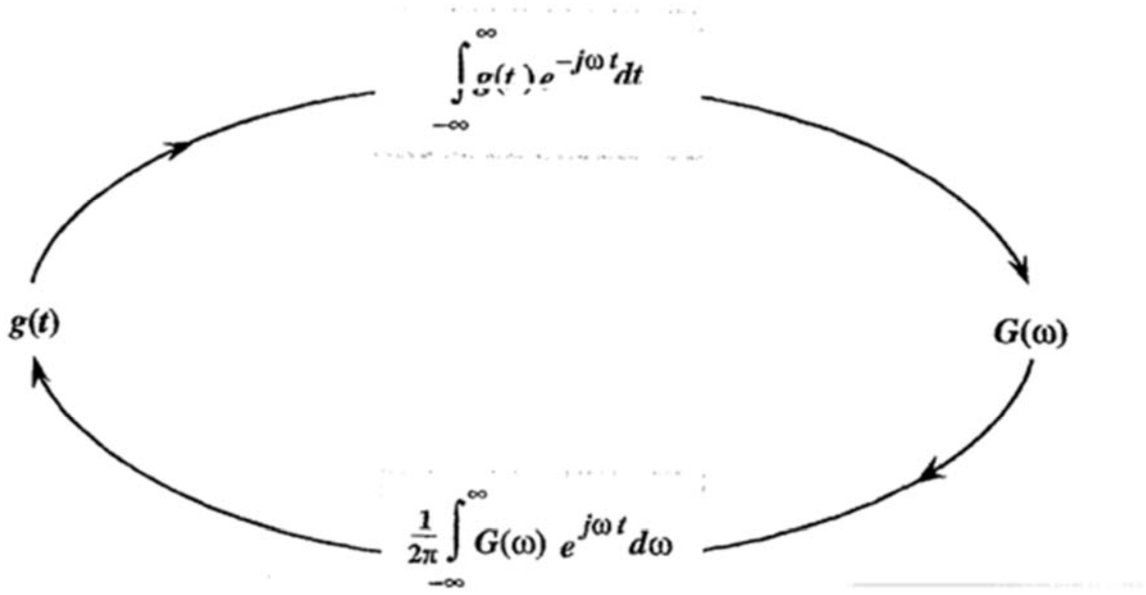
Some properties of Fourier transform

12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t \, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin \omega_0 t \, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin \omega_0 t \, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos \omega_0 t \, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau \operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	

Some properties of Fourier transform

19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$

Fourier transform pair



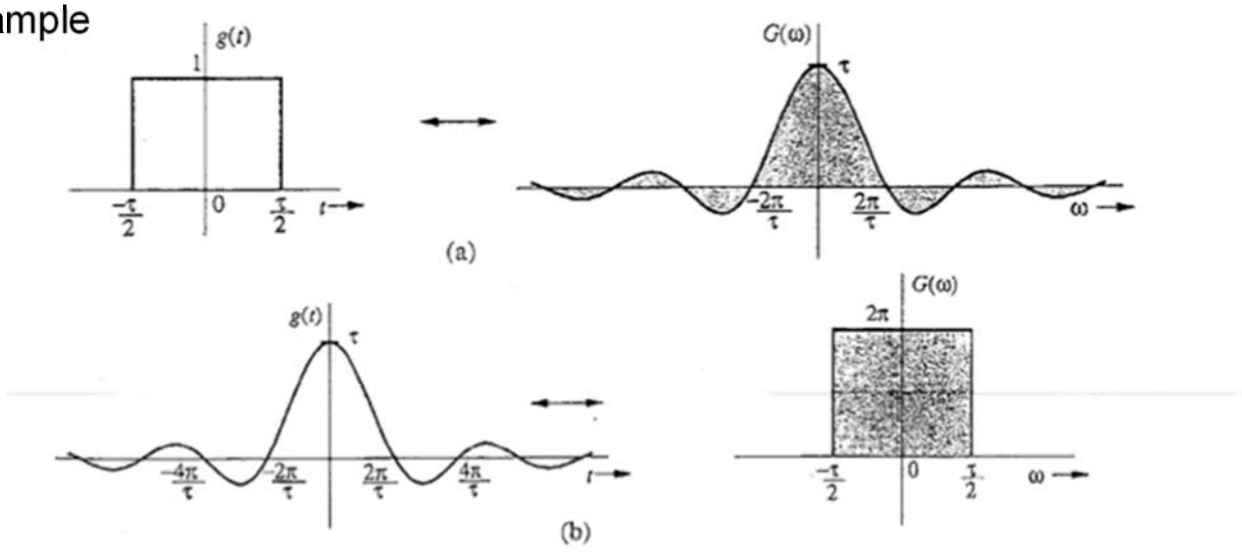
Symmetry Property

- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(\omega)$$

- Then $$G(t) \Leftrightarrow 2\pi g(-\omega)$$

- Example



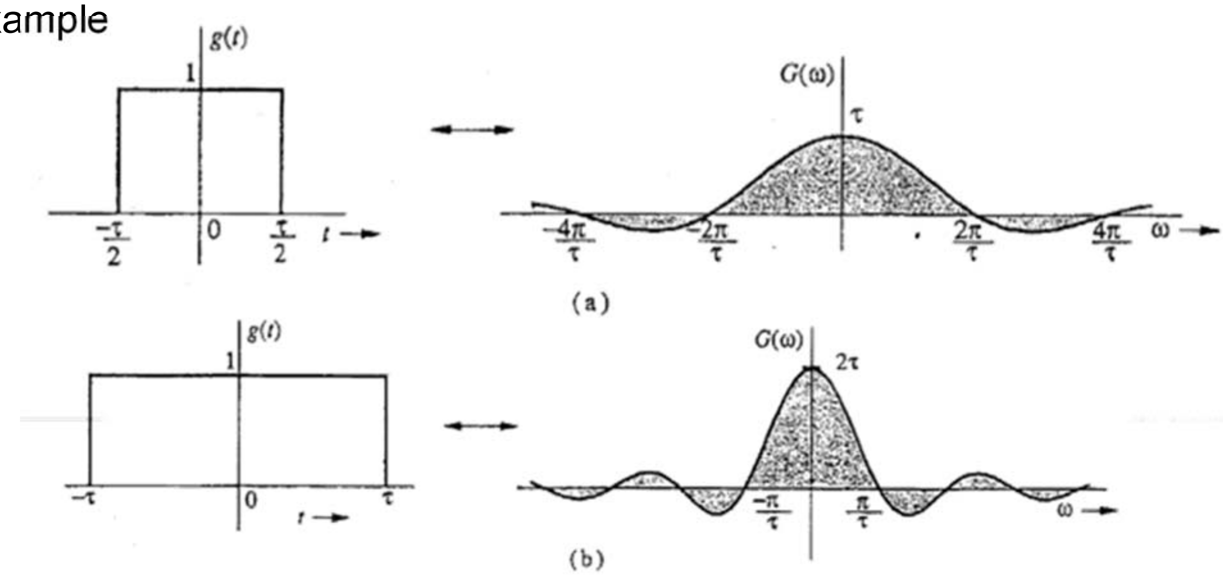
Scaling Property

- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(\omega)$$

- Then $$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

- Example



Time-Shifting Property

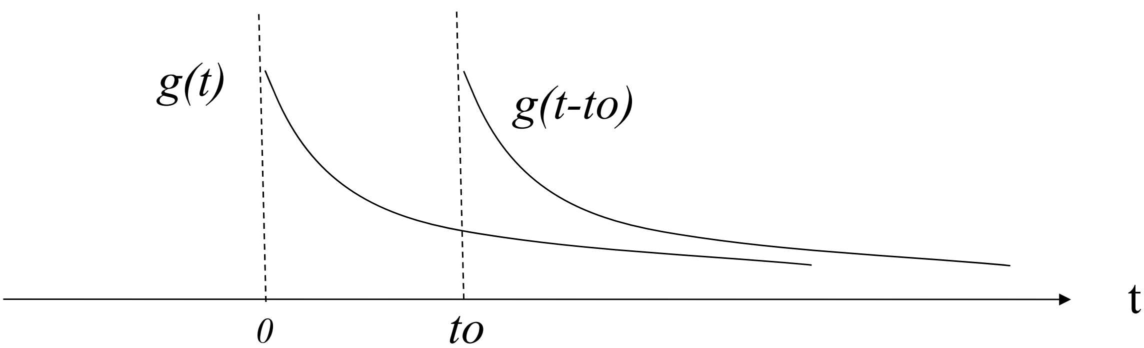
- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(\omega)$$

- Time shifting introduces phase shift

$$g(t - t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$$

- Example



Frequency-Shifting **Property**

- Consider the Fourier transform pair

$$g(t) \Leftrightarrow G(\omega)$$

- Exponential multiplication introduces frequency shift

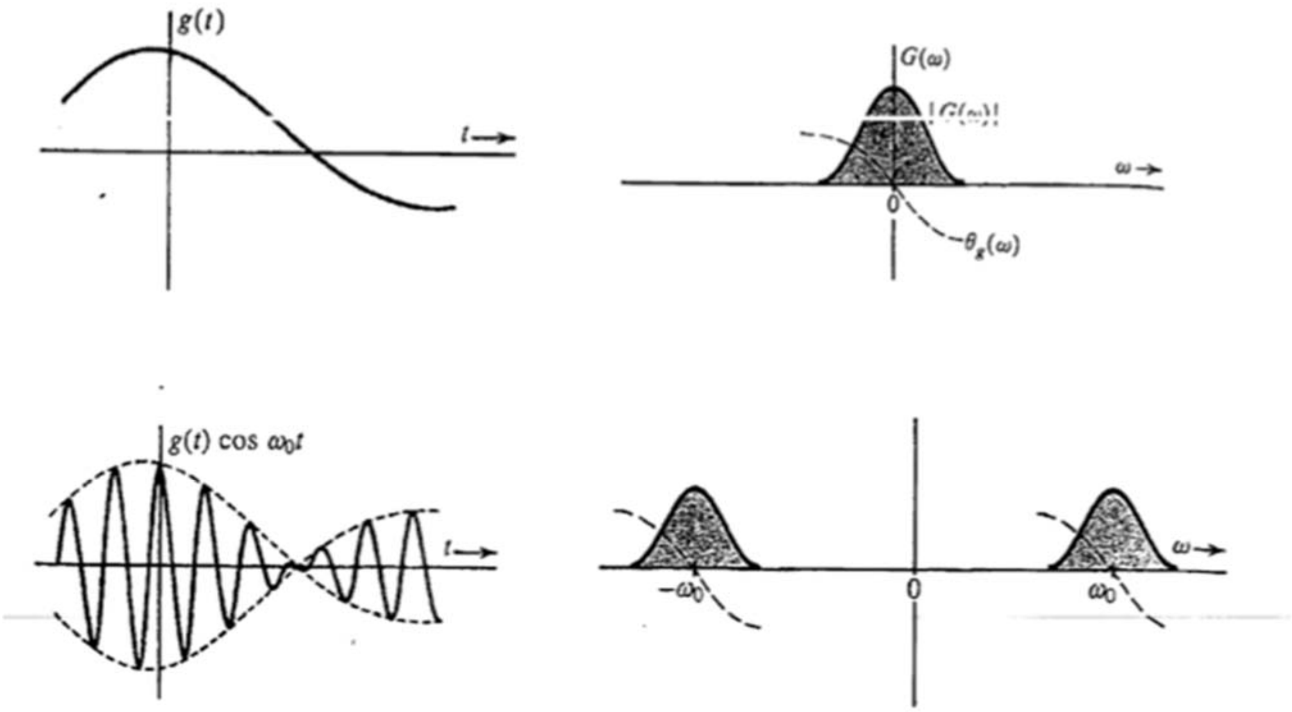
$$g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0) \qquad g(t)e^{-j\omega_0 t} \Leftrightarrow G(\omega+\omega_0)$$

- Cosine multiplication leads to

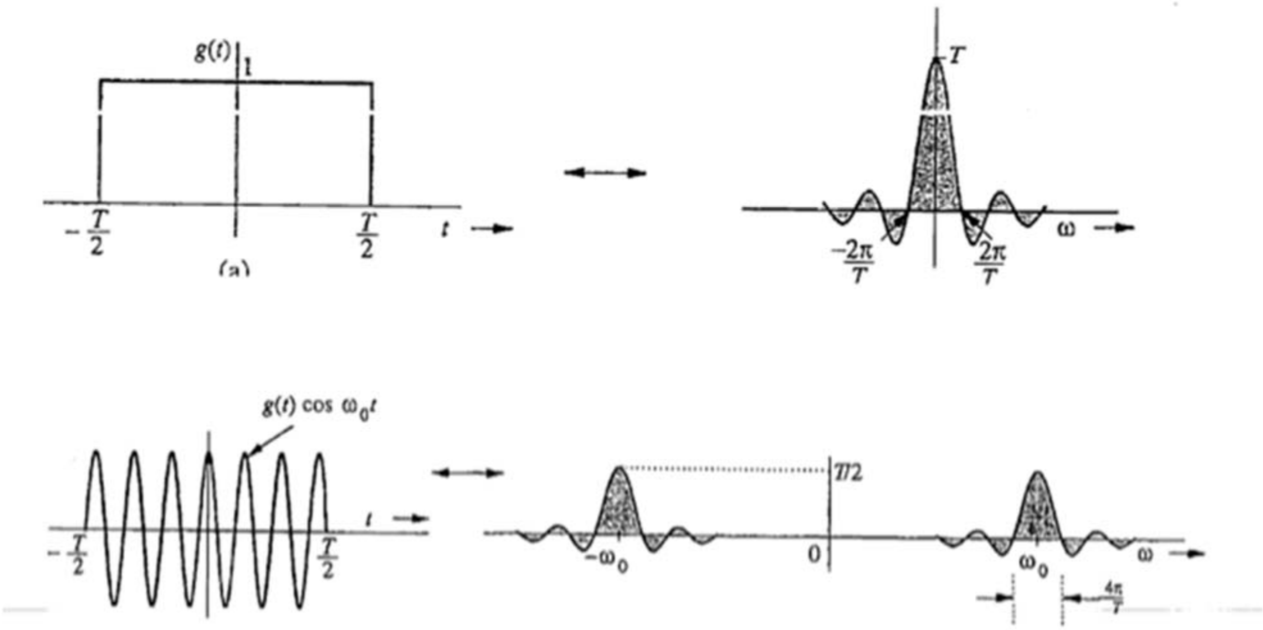
$$g(t)\cos\omega_0 t = \frac{1}{2}\left[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t} \right]$$

$$g(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}\left[G(\omega-\omega_0) + G(\omega+\omega_0) \right]$$

Frequency-Shifting **Property**



Frequency-Shifting Property



Fourier transform of periodic functions

- Find the Fourier transform of a general periodic signal $g(t)$ of period T_0
- A periodic signal $g(t)$ can be expressed as an exponential Fourier series as

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$g(t) \Leftrightarrow \sum_{n=-\infty}^{\infty} F\left[D_n e^{jn\omega_0 t}\right]$$

$$g(t) \Leftrightarrow 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\omega - n\omega_0\right)$$

Fourier transform of periodic functions

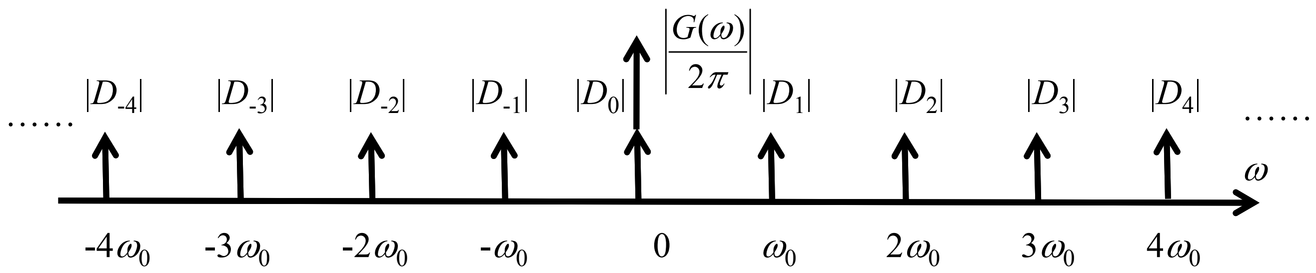
- Consider a periodic waveform given by

$$g(t) = \sum_{n=-\infty}^{n} w(t - nT_0) \qquad w(t) = \begin{cases} \text{non-zero} & T_0/2 \le |t| \\ 0 & \text{otherwise} \end{cases}$$

- where

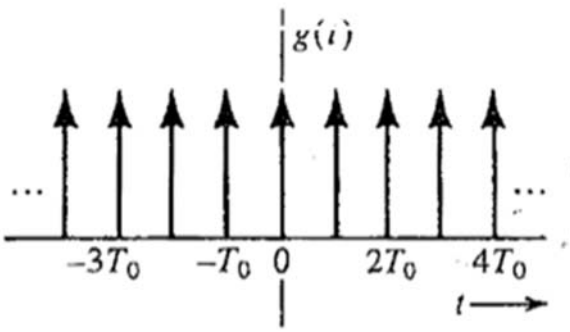
$$g(t) = \sum_{n=-\infty}^{n} D_n \, e^{jn\omega_0 t} \qquad w(t) \Leftrightarrow W(\omega)$$

$$g(t) \Leftrightarrow G(\omega) = 2\pi \sum_{n=-\infty}^{n} D_n \delta(\omega - n\omega_0) \qquad D_n = \frac{1}{T_0}\int_{T_0} g(t) e^{-jn\omega_0 t}\, dt = \frac{W(n\omega_0)}{T_0}$$



Fourier transform of periodic functions

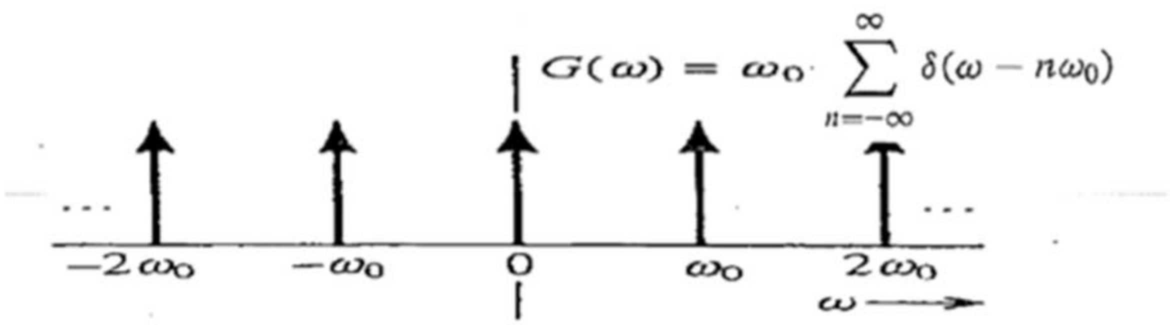
- Find the Fourier transform of a unit impulse train $\delta(t)$ of period T_0



$$w(t) = \delta(t) \quad \Leftrightarrow \quad W(\omega) = F(\delta(t)) = 1$$

$$D_n = \frac{W(n\omega_0)}{T_0} = \frac{1}{T_0}$$

$$g(t) \Leftrightarrow \frac{2\pi}{T_0} \sum_{n=-\infty}^{n} \delta(\omega - n\omega_0)$$



Convolution

The convolution of two functions $g(t)$ and $w(t)$,

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau) w(t-\tau) d\tau$$

- Consider two waveforms

$$g_1(t) \Leftrightarrow G_1(\omega) \qquad g_2(t) \Leftrightarrow G_2(\omega)$$

- Convolution in time domain

$$g_1(t) * g_2(t) \Leftrightarrow G_1(\omega) G_2(\omega)$$

- Convolution in the frequency domain

$$g_1(t) g_2(t) \Leftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$$

Time Differentiation and Time Integration

- Consider the Fourier transform relationship

$$g(t) \Leftrightarrow G(\omega)$$

- The following relationship exists for integration

$$\int_{-\infty}^{t} g(\tau) d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0) \delta(\omega)$$

- The following relationship exists differentiation

$$\frac{dg(t)}{dt} \Leftrightarrow j\omega G(\omega) \qquad \frac{d^n g(t)}{dt^n} \Leftrightarrow (j\omega)^n G(\omega)$$

Important Fourier Transform Operations

Fourier Transform Operations

Operation	$g(t)$	$G(\omega)$
Addition	$g_1(t) + g_2(t)$	$G_1(\omega) + G_2(\omega)$
Scalar multiplication	$kg(t)$	$kG(\omega)$
Symmetry	$G(t)$	$2\pi g(-\omega)$
Scaling	$g(at)$	$\frac{1}{\|a\|}G\left(\frac{\omega}{a}\right)$
Time shift	$g(t-t_0)$	$G(\omega)e^{-j\omega t_0}$
Frequency shift	$g(t)e^{j\omega_0 t}$	$G(\omega-\omega_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(\omega)G_2(\omega)$
Frequency convolution	$g_1(t)g_2(t)$	$\frac{1}{2\pi}G_1(\omega) * G_2(\omega)$
Time differentiation	$\frac{d^n g}{dt^n}$	$(j\omega)^n G(\omega)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$

An Example of Fourier Transform Properties

Given the Fourier Transform pair of *g(t)* and *G(ω)*,

find the Fourier Transform *F(ω)* of $\frac{dg(t)}{dt}$.

$$F(\omega) = \int_{-\infty}^{\infty} \frac{dg(t)}{dt} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dg(t)$$

$$= e^{-j\omega t} g(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(t) de^{-j\omega t}$$

$$= 0 + j\omega \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$= j\omega G(\omega)$$

Conclusions

- Examined some properties of Fourier transforms
  - Scaling property
  - Time shifting property
  - Frequency shifting property
- Examined Fourier Transform of periodic functions
  - General case
  - Unit Impulse function
- Examined Convolution
- Examined Fourier transforms for
  - Integration
  - Differentiation