Solutions to problem sheet three

1. $$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} g(t)\cos\omega t dt - j\int_{-\infty}^{\infty} g(t)\sin\omega t dt.$$

   If $g(t)$ is an even function of t, $g(t)\sin\omega t$ is odd and $g(t)\cos\omega t$ is even. Therefore, the second integral in the equation above vanishes, whereas the first integral is twice the integral over the interval 0 to ∞. Thus when $g(t)$ is even

   $$G(\omega) = 2\int_{0}^{\infty} g(t)\cos\omega t dt. \tag{1}$$

   Similarly, when $g(t)$ is odd one can show that

   $$G(\omega) = -2j\int_{0}^{\infty} g(t) sin\omega t dt \tag{2}$$

   If $g(t)$ is real and even, the integral in (1) is real and

   $$G(-\omega) = 2\int_{0}^{\infty} g(t)\cos\omega t dt = G(\omega).$$

   Hence, $G(\omega)$ is a real and even function. With the same arguments, one can show that if $g(t)$ is real and odd, $G(\omega)$ is imaginary and odd.

2. $g(t)$ is real. Therefore $G^*(\omega)$ is given by

   $$G^*(\omega) = \int_{-\infty}^{\infty} g^*(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} g(t)e^{j\omega t}dt = G(-\omega).$$

3. $$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t-5)e^{-j\omega t}dt = \int_{4.5}^{5.5} e^{-j\omega t}dt = -\tfrac{1}{j\omega}[e^{-j\omega t}]_{4.5}^{5.5} \\ &= \tfrac{e^{-j5\omega}}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}] = \mathrm{sinc}(\omega/2)e^{-j5\omega} \end{aligned}$$

4. Time shift property:

   $$g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$$

   In our case

   $$g(t) = \mathrm{rect}(t) \Leftrightarrow G(\omega) = \mathrm{sinc}(\omega/2)$$

   Therefore, the Fourier transform of $g(t) = \mathrm{rect}(t-5)$ is

   $$\mathrm{sinc}(\omega/2)e^{-j5\omega}$$