SIMULTANEOUS ESTIMATION OF SPARSE SIGNALS AND SYSTEMS AT SUB-NYQUIST RATES

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ABSTRACT

The novelty of this paper is divided into two technical sections; first we propose a novel algorithm for system identification with known input sparse signal, based on the Finite Rate of Innovation sampling theory. Then we consider the problem of simultaneously estimating the input sparse signal and also the linear system and propose a novel iterative algorithm for that setup. We will show that, based on our numerical simulations, the solution to the second problem is normally convergent.

1. INTRODUCTION

In this paper we will be considering the problem of system identification based on a sparse sampling system. In many practical applications the impulse response of an unknown system is required to be estimated. This problem is usually referred to as "System Identification Problem" in literature. Unlike standard techniques for system identification which require the sampling rate to be at or above the Nyquist rate, we use sparse sampling techniques to identify the system at sub-Nyquist sampling rates. Figure 1 shows a system identification problem setup where a sparse signal $g(x)$ is fed to an unknown system with impulse response $\psi(x)$. The output signal from the unknown system then goes through a sampling process with a sampling rate T which outputs the samples s_k. The aim of system identification is to completely determine the function $\psi(x)$ from the samples s_k. Key to our formulation is the prior knowledge that the system is specified by a small number of unknown parameters.

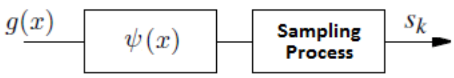

Figure 1: A system identification problem setup.

Regarding the sampling process, it has been recently shown that [15, 6, 5] it is possible to sample and perfectly estimate sparse signals. In these schemes the sparsity or parametric structure of the input signal is taken into account and perfect recovery is achieved based on a set of suitable measurements. Depending on the setup used, these sampling methods go under different names such as compressed sensing (CS), compressive sampling [6, 5] or sampling signals with finite rate of innovation (FRI) [15, 7]. Signals with the latter framework, posses a finite number of degrees of freedom per unit of time and examples of such signals include stream of Diracs and piecewise-polynomial [15, 7] signals. In this paper, we consider the finite rate of innovation sampling theory for our sampling process block where its setup is modeled as in Figure 2. In [15, 7], it is shown that perfect reconstruction of classes of FRI signals is possible by utilizing the Prony's method, which is also known as the annihilating filter method [12].

Sparse signal estimation is well known in literature and some of its applications include image registration in image super-resolution [2], multichannel sampling [1], channel estimation [10, 8], radar and ultrasound applications [13, 4] and many more. In this paper we will propose a novel algorithm for simultaneous estimation of sparse signals along with system identification using the theories of FRI sampling. Specifically, we will divide the simultaneous estimation problem into two stages where we first assume that the input sparse signal is known, so that the problem simplifies to a system identification problem only and then in the second stage, we extend the setup and modify the algorithm for the case when the input sparse signal is also unknown, that is simultaneous estimation of both the input signal and the unknown system $\psi(x)$. System identification using the sparse sampling theories has already been considered in [11] and in [9]. In [11], McCormick et al. consider a problem similar to ours but model and approximate the unknown system as a finite impulse response (FIR) filter. In [9], Hormati el al consider distributed sampling of two signals linked by an unknown sparse filter.

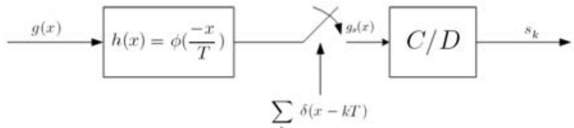

Figure 2: FRI 1-D Sampling Setup. Here, $g(x)$ is the continuous-time signal, $h(x)$ the impulse response of the acquisition device and T the sampling period. The measured samples are $s_k = \langle g(x), \phi(x/T - k)\rangle$.

The organization of the paper is as follows: In Section II we review the main results on sampling FRI signals and in particular focus on E-spline sampling kernels. In Section III, we propose our novel algorithm for system identification for the case when the input sparse signal is known and given. Then in Section IV, we extend our results and propose a novel algorithm for simultaneous estimation of both the input sparse signal and the unknown system $\psi(x)$. We finally conclude in Section V.

2. FINITE RATE OF INNOVATION SAMPLING THEORY

In Figure 2 we show the typical setup employed for sampling 1-D FRI signals where the signal $g(x)$ represents the input signal, $h(x)$ the impulse response of the sampling device, $\phi(x)$ a re-scaled and time-reversed version of $h(x)$ (also known as the sampling kernel), $g_s(x)$ the sampled version of the input signal $g(x)$, s_k the samples and T the sampling interval. The box C/D (continuous-to-discrete) reads out the sample values s_k from $g_s(x)$. From the setup shown in Figure 2, the following equations for the samples s_k can be deduced:

$$\begin{aligned} s_k &= g(x) * h(x)|_{x=kT} \\ &= \int_{-\infty}^{\infty} g(x)\ \phi(\frac{x}{T} - k)\ dx \\ &= \langle g(x), \phi(\frac{x}{T} - k)\rangle. \end{aligned}$$

Sampling kernels are characterized by the physical properties of the acquisition device which are normally specified and cannot be modified. Throughout the paper, we will focus on exponential reproducing kernels and in particular exponential splines "E-splines" [14], splines that can reproduce real or complex exponentials. In the subsequent subsection, a detailed discussion of exponential reproducing kernels is presented.

2.1 E-Spline Sampling Kernels

Any kernel $\phi(x)$ that together with its shifted versions can reproduce real or complex exponentials in the form $e^{\alpha_m x}$ with $m = 0,1,\ldots,M$ is called an exponential reproducing kernel. That is any kernel satisfying the following property (For simplicity and without loss of generality we have assumed $T = 1$):

$$\sum_{n\in\mathbb{Z}} c_n^m \phi(x-n) = e^{\alpha_m x} \qquad \text{with } \alpha_m \in \mathbb{C}, \tag{1}$$

for a proper choice of coefficients $c_n^m \in \mathbb{C}$. The coefficients c_n^m in the above equation are given by the following equation:

$$c_n^m = \int_{-\infty}^{\infty} e^{\alpha_m x}\tilde{\phi}(x-n)dt,$$

where $\tilde{\phi}(x)$ is chosen to form with $\phi(x)$ a quasi-biorthonormal set. The choice of the exponents in $e^{\alpha_m x}$ is restricted to $\alpha_m = \alpha_0 + m\lambda$ with $\alpha_0, \lambda \in \mathbb{C}$ and $m = 0,1,...,M$. This is done to allow the use of the Prony's method at the reconstruction stage [15, 7]. The theory of exponential reproducing kernels is based on the notion of E-splines [14] and this is because any exponential reproducing kernel is given by the convolution between a normal function $f(x)$ and an E-spline. A function $\beta_{\vec{\alpha}}(x)$ with Fourier transform

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \prod_{m=0}^{M} \frac{1-e^{\alpha_m - j\omega}}{j\omega - \alpha_m},$$

is called E-spline of order $M+1$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_M)$. The produced spline has compact support and can reproduce any exponential in the subspace spanned by $(e^{\alpha_0 x}, e^{\alpha_1 x}, \ldots, e^{\alpha_M x})$. In time-domain, the expression of an E-spline of order one is given by:

$$\beta_{\alpha_0}(x) = \begin{cases} e^{\alpha_0 x} & 0 \le x < 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where the higher order E-splines are obtained by successive convolutions of lower order ones with their specific α_m parameters. An interesting fact is that, if we consider the setup of Figure 2, then, when $\phi(x)$ is an exponential reproducing kernel, we can retrieve the exponential moments of the input signal from the samples s_k. To illustrate this, let us consider the following weighted sum of the samples:

$$\tau_m = \sum_k c_k^m s_k. \tag{3}$$

Substituting the equation for the samples s_k into the above equation, gives:

$$\tau_m = \langle g(x), \sum_k c_k^m \phi(x-k)\rangle.$$

The second term in the inner product can be replaced by the Equation (1) and the exponential moments of the signal are therefore obtained as follows:

$$\tau_m = \int_{-\infty}^{\infty} g(x)\, e^{\alpha_m x}\, dx.$$

When α_m is purely imaginary, the Fourier transform at α_m of the signal $g(x)$ is obtained:

$$\tau_m = G(\alpha_m).$$

Here $G(u)$ represents the Fourier transform of the signal $g(x)$. In the subsequent sections we will show how this useful feature can be employed to address the system identification problem.

3. SYSTEM IDENTIFICATION WITH KNOWN INPUT SIGNAL

Having gone through the sampling stage in the previous section, we will now show how such a sampling scheme could be employed to estimate the unknown function $\psi(x)$. First we assume, as the title of this section suggests, the input sparse signal $g(x)$ is known. Figure

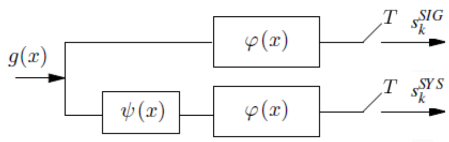

Figure 3: System identification setup with known input signal

3 shows the setup used for our proposed algorithm, which is really a combination of Figure 1 and 2 in a two-channel system. As shown in Figure 3, the two-channel system is used for sampling the input sparse signal with and without the unknown system $\psi(x)$. In the first channel, the input sparse signal is directly sampled with our pre-specified E-spline sampling kernel $\phi(x)$. The Fourier transform of the pre-specified E-spline sampling kernel $\phi(x)$ is given by:

$$\hat{\phi}(\omega) = \prod_{m=0}^{P} \left(\frac{1-e^{\alpha_m - j\omega}}{j\omega - \alpha_m}\right),$$

where P depends only on the structure of the unknown function $\psi(x)$. This will be more evident later on. The samples s_k at the output are:

$$s_k^{SIG} = \langle g(x), \phi(x-k)\rangle.$$

Given the samples, we then calculate the exponential moments of the sparse signal τ_m^{SIG} using Equation 3 as was described in Section 2:

$$\tau_m^{SIG} = G(\alpha_m)$$

Here, $G(\alpha_m)$ represents the Fourier transform of the signal $g(x)$ at α_m with $m = 0,1,\ldots,P$ and $\alpha_m = \alpha_0 + m\lambda$. This method will also work for the case of real E-splines but for simplicity we have assumed that our pre-specified E-spline sampling kernel is complex valued. In the second channel, the same input sparse signal is fed through the unknown function $\psi(x)$ and then sampled with the same sampling kernel. Therefore, its corresponding samples s_k are:

$$s_k^{SYS} = \langle g(x) * \psi(x), \phi(x-k)\rangle.$$

Same as before, given the samples we calculate the exponential moments as follows:

$$\tau_m^{SYS} = G(\alpha_m)\cdot\Psi(\alpha_m),$$

where $\Psi(\alpha_m)$ represents the Fourier transform of the function $\psi(x)$ at α_m. Moreover, the above equation is deduced from that fact that convolution in time domain corresponds to multiplication in Fourier domain. From the set of results obtained above, we can show that:

$$\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \frac{G(\alpha_m)\cdot\Psi(\alpha_m)}{G(\alpha_m)} = \Psi(\alpha_m),$$

where we have assumed that $G(\alpha_m) \neq 0$. Therefore, by dividing the exponential moments obtained from the two channels, we have shown that the Fourier transform $\Psi(\alpha_m)$ of the unknown function $\psi(x)$ can be obtained. Now, given $\Psi(\alpha_m)$ with $m = 0,1,..,P$ and $\alpha_m = \alpha_0 + m\lambda$, we will have an inverse problem to solve for the unknown parameters of the function $\psi(x)$. Once the unknown parameters are estimated, the function $\psi(x)$ will be completely determined and the system will therefore be fully identified. In the following section, we show cases where we can solve the above inverse problem (i.e. we identify the system) and highlight the applications in which the proposed system model is of interest. It should be pointed out that the above method works regardless of the structure of the input signal.

3.1 A Stream of Diracs

Consider the unknown function $\psi(x)$ to be a stream of K Diracs with unknown locations and amplitudes. Applications of such a system could be acoustic room impulse response estimation or line echo cancelation. We already know that the Fourier transform of such a function has a power-sum series form [15, 7]:

$$\Psi(\alpha_m) = \sum_{k=1}^{K} a_k u_k^m,$$

where a_k and u_k correspond to the unknown amplitudes and locations respectively. From our setup shown in Figure 3, as previously described the exponential moments of the output samples with and without the unknown function are obtained. Then, the Fourier coefficients of the signal $\Psi(\alpha_m)$ can be easily calculated as follows:

$$\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \Psi(\alpha_m), \qquad for \qquad m = 0, 1, \ldots, P.$$

As $\Psi(\alpha_m)$ has a power-sum series form, we apply the annihilating filter method to the measurements $\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \Psi(\alpha_m)$ to retrieve the unknown parameters a_k and u_k. For such a system, in order to recover the K Diracs, the E-spline sampling kernel is required to be of the order $P \geq 2K$. This set-up is similar to the one discussed in [9].

3.2 B-Spline

Let us consider $\psi(x)$ to be a B-spline $\beta^K(x)$ of unknown order $K+1$. An application of such a system could be the camera lens calibration [3]. This is because the point spread function of a camera lens is very often assumed to be a Gaussian pulse and B-splines of order $K \geq 2$ are increasingly similar to Gaussian functions. We already know that the Fourier transform function of a B-spline of order $K+1$ is given by:

$$\hat{\beta}^K(\omega) = \prod_{k=0}^{K} \frac{1-e^{-j\omega}}{j\omega} = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{K+1}.$$

Assuming the unknown function in our setup shown in Figure 3 is a B-spline of unknown order, that is $\psi(x) = \beta^K(x)$, then by calculating the exponential moments of the output samples with and without the unknown filter, we can obtain the Fourier coefficients of $\psi(x)$ as follows:

$$\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \Psi(\alpha_m) = \left(\frac{1-e^{-j\alpha_m}}{j\alpha_m}\right)^{K+1}.$$

By taking logarithms of both sides of the equation, the unknown order $K+1$ is calculated as follows:

$$\frac{log(\Psi(\alpha_m))}{log(\left(\frac{1-e^{-j\alpha_m}}{j\alpha_m}\right))} = K+1.$$

In order to estimate the unknown order of the B-spline, the E-spline sampling kernel is required to be of the order $P \geq 1$.

3.3 E-Spline

Let us consider $\psi(x)$ to be an E-spline $\beta_{\vec{\gamma}}(x)$ of known order $K+1$. An application of such a system could be the estimation of the electronic components of a finite order electronic circuit, which will be fully described in the next subsection. As stated previously, the Fourier transform function of an E-spline of order $K+1$ is:

$$\hat{\beta}_{\vec{\gamma}}(\omega) = \prod_{k=0}^{K} \frac{1-e^{\gamma_k - j\omega}}{j\omega - \gamma_k}.$$

We assume that the order of the spline is known but the parameters γ_k are unknown. From our setup we obtain the Fourier coefficients of the unknown function $\psi(x)$ as follows:

$$\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \Psi(\alpha_m) = \prod_{k=0}^{K} \frac{1-e^{\gamma_k - j\alpha_m}}{j\alpha_m - \gamma_k}.$$

Calculation of the unknown parameters of the E-splines (as the unknown function $\psi(x)$) is more involved. We first need to simplify both the numerator and denominator of the E-spline function. Simplifying the numerator gives:

$$\prod_{k=0}^{K}\left(1-e^{\gamma_k - j\alpha_m}\right) = \prod_{k=0}^{K}(1-a_k.u^m),$$

where $a_k = e^{\gamma_k}$ and $u^m = e^{j\alpha_m}$. This can be further simplified as follows:

$$\begin{aligned}\prod_{k=0}^{K}(1-a_k.u^m) &= (1-a_1u^m)(1-a_2u^m)...(1-a_Ku^m) \\ &= \sum_{k=0}^{K} q_k u^{km} \\ &= \sum_{k=0}^{K} q_k t_k^m,\end{aligned}$$

where $t_k = u^k$. Simplifying the denominator gives:

$$\prod_{k=0}^{K}(j\alpha_m - \gamma_k) = Q(m) = \sum_{k=0}^{K} r_k m^k,$$

where $Q(m)$ is a polynomial of degree $K+1$. Rearranging the above polynomial functions leads to:

$$\sum_{k=0}^{K} r_k m^k \cdot \Psi(\alpha_m) = \sum_{k=0}^{M} q_k t_k^m.$$

The above equation can be considered as a linear system consisting of $2K$ unknowns with the unknown parameters being $\theta = (r_0, r_1, \ldots, r_{K-1}, q_2, \ldots, q_{K+1})$. Here, $q_0 = 1$, $t_0 = 1$ and t_k are a known set of parameters. As we have a linear system, by constructing the following matrix equation and taking its inverse we are able to calculate the unknown parameters: Once the parameters are estimated, by taking the roots of the polynomial q_k for $k = 0, 1, \ldots, K$ and then taking the logarithm of the roots we obtain the unknown γ_k parameters, as $roots(q_k) = a_k = e^{\gamma_k}$. The E-spline sampling kernel is required to be of the order $P = 2K$ in order to estimate the parameters of the E-spline function $\psi(x)$. In the next section, we will explain how we could extend the results for the E-splines case to the case of finite order electronic circuits.

3.4 Finite Order Electronic Circuits

Any finite order electronic circuit can be thought of a modified E-spline [7]. In general any p-th order electronic circuit has a transfer function ($s = j\omega$):

$$\psi(s) = \frac{b_q s^q + b_{q-1}s^{q-1} + \ldots + b_1 q + b_0}{a_p s^p + a_{p-1}s^{p-1} + \ldots + a_1 p + a_0} = \frac{\sum_{q=0}^{Q} b_q s^q}{\sum_{p=0}^{P} a_p s^p}.$$

The equation above has a very similar structure to the E-spline case and one may suggest a similar simplifying procedure with constructing a system of linear equations with $Q+P+1$ unknowns, as discussed previously. However, the transient response of a finite order electronic circuit is infinite and this would mean that the

$$\begin{bmatrix} \psi(\alpha_0) & 0 & \dots & 0 & 1 & \dots & 1 \\ \psi(\alpha_1) & \psi(\alpha_1) & \dots & \psi(\alpha_1) & t_2 & \dots & t_K \\ \psi(\alpha_2) & 2\psi(\alpha_2) & \dots & 2^{K-1}\psi(\alpha_2) & t_2^2 & \dots & t_K^2 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ \psi(\alpha_{2K-1}) & (2K-1)\psi(\alpha_{2K-1}) & \dots & (2K-1)^{K-1}\psi(\alpha_{2K-1}) & t_2^{2K-1} & \dots & t_K^{2K-1} \end{bmatrix} \times \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{K-1} \\ q_2 \\ \vdots \\ q_{K+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1-\psi(\alpha_1)r_K \\ 1-2^K\psi(\alpha_2)r_K \\ \vdots \\ 1-(2K-1)^K\psi(\alpha_{2K-1})r_K \end{bmatrix}.$$

function $\Psi(\omega)$ will not have compact support. Assuming that the electronic circuit has a fast decay or by taking enough samples, we can approximate the above function to have finite duration and thus, similar to the E-spline case, a linear system of matrix equations with $Q+P+1$ unknown parameters could be constrcuted. As a simple example, consider the setup shown in Figure 4 where the unknown function $\psi(x)$ is a first order RC circuit (low-pass filter). We already

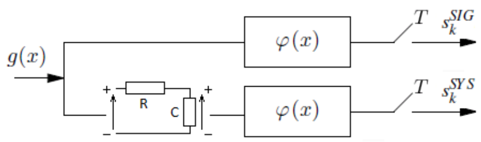

Figure 4: First order RC circuit as the unknown function $\psi(x)$

know that the transfer function of such a circuit is:

$$\Psi(\omega) = \frac{\gamma}{\gamma + j\omega},$$

where $\gamma = 1/RC$. Our goal is to estimate the parameter $\gamma = 1/RC$ and therefore identify the transfer function. Like before, we obtain the exponential moments and obtain the Fourier coefficients of the function $\psi(x)$:

$$\frac{\tau_m^{SYS}}{\tau_m^{SIG}} = \Psi(\alpha_m) = \frac{\gamma}{\gamma + j\alpha_m}.$$

The above function has one unknown parameter only and therefore the product RC can be estimated as follows:

$$\gamma = \frac{j\alpha_m \cdot \Psi(\alpha_m)}{1 - \Psi(\alpha_m)}.$$

4. SIMULTANEOUS ESTIMATION OF SPARSE SIGNAL AND SYSTEM IDENTIFICATION

When both the signal and the system are unknown, the above solution cannot be used directly and the problem is in general more involved. However, a recursive version of the previously discussed method, as shown in Figure 5(a) and 5(b), can be utilized to estimate both the input sparse signal and the unknown function $\psi(x)$.

For simplicity, let us assume that the input sparse signal is a stream of Diracs with unknown locations and amplitudes. As shown in Figure 5(a), the unknown input signal is fed to the unknown function $\psi(x)$ and then is sampled with our pre-specified E-spline sampling kernel, therefore its corresponding exponential moments are:

$$\tau_m^0 = \Psi(\alpha_m) \cdot G(\alpha_m),$$

where $\Psi(\alpha_m)$ and $G(\alpha_m)$ are not known. As our input signal is a stream of Diracs with unknown amplitudes and locations, we directly apply the annihilating filter method to the moments τ_m^0 and obtain an estimate of the input signal, denoted as $\hat{g}(x)$ (Figure 5(b)). Once an estimate of the input signal is obtained, we recursively feed the estimated signal $\hat{g}(x)$ back to our pre-specified sampling kernel and obtain its corresponding exponential moments at each recursion:

$$\tau_m^{upd} = \hat{G}(\alpha_m).$$

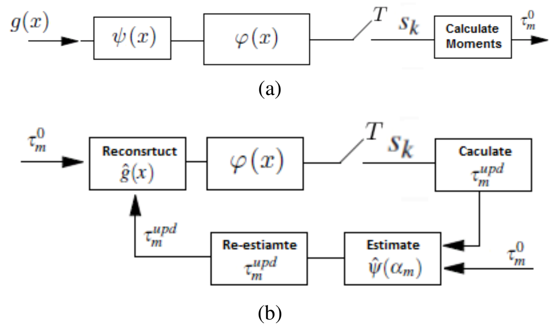

Figure 5: The setup proposed for recursive estimation

Here the superscript "upd" stands for "updated" and $\hat{G}(\alpha_m)$ is an estimate of the Fourier coefficients of the input signal. Now, we divide the updated exponential moments τ_m^{upd} and the initial measurements τ_m^0 to obtain an estimate for the unknown filter $\Psi(\alpha_m)$ as follows:

$$\frac{\tau_m^0}{\tau_m^{upd}} = \frac{G(\alpha_m) \cdot \Psi(\alpha_m)}{\hat{G}(\alpha_m)} = \hat{\psi}(\alpha_m).$$

From $\hat{\psi}(\alpha_m)$, as was shown in Section 3.2, the unknown parameters of the unknown system can be estimated. Once the parameters are estimated, from the model of the unknown system, we re-estimate the function $\psi(x)$, denoted by $\hat{\hat{\psi}}(\alpha_m)$, and from that we re-estimate the measurements τ_m^{upd} as follows:

$$\tau_m^{upd} = \frac{\tau_m^0}{\hat{\hat{\psi}}(\alpha_m)}.$$

We apply the annihilating filter method on the re-estimated τ_m^{upd} and re-estimate the unknown input signal $\hat{g}(x)$. Our empirical results show that by applying the above method recursively, the estimations converge to the actual input signal $g(x)$ and the unknown function $\psi(x)$.

As an example let us assume that our input signal consists of 2 Diracs with unknown amplitude and location. Let us also assume that the unknown system to be identified is a first order E-spline with $\gamma = 2$. Our goal is to simultaneously estimate the input signal which consist of two Diracs and also the unknown γ parameter. Figure 6 shows the results for the above example with 10 iterations. It can be seen that both the input signal and also the unknown system are estimated to a very good degree when compared to their true values.

5. CONCLUSION

In this paper we proposed our novel algorithm for simultaneously estimating input sparse signal and system identification based on the finite rate of innovation sampling theories. We described our algorithm in two stages, where in the first stage, we showed that by having access to the input signal, system identification problem could be solved at low sampling rates by employing exponential moments. Then in the second stage, we showed that a recursive method could be utilized to estimate both the input sparse signal and also the unknown system when we only have access to the output samples.

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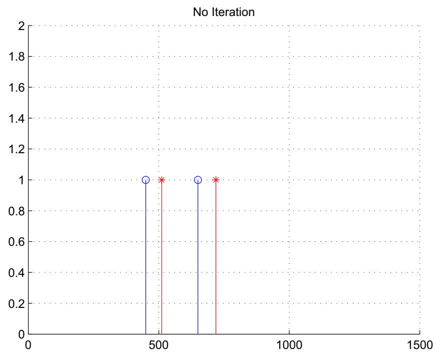
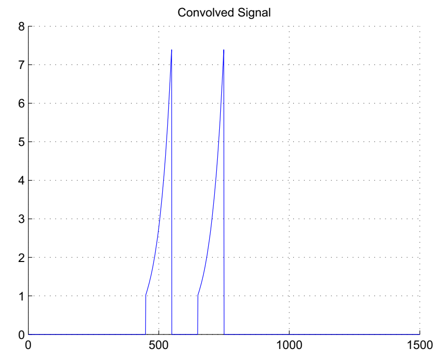


(a)



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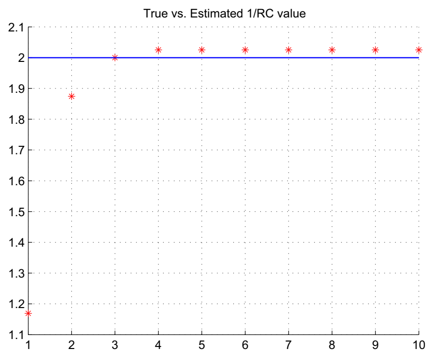
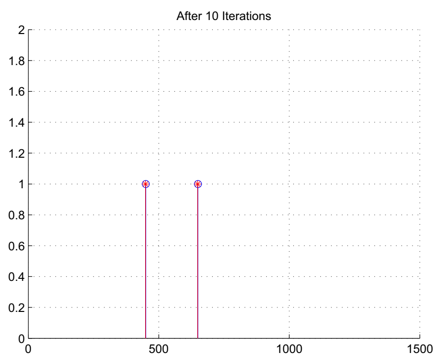


(c)



(d)

Figure 6: Simultaneous estimation of an input sparse signal with a first order E-spline as the unknown function $\psi(x)$. (a) The input signal with 2 Diracs (circle) along with the immediate estimate of the signal (star) with no iterations. (b) Input signal convolved with the function $\psi(x)$. (c) True vs. estimated values of the parameter γ after 10 iterations. (d) True vs. estimated version of the input signal after 10 iterations.