Multi-Objective Optimization[1]

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[1]Slides based on Cho, Jin-Hee, et al. "A Survey on Modeling and Optimizing Multi-Objective Systems." IEEE Communications Surveys & Tutorials (2017).

Overview

1. Multi-objective Optimization (MOO)
   - Basic Concepts

2. MOO Solution Techniques
   - Scalarization-based MOO Formulation
   - Meta-heuristics
   - Hybrid Meta-heuristics

3. Game theory-based MOO Design Techniques
   - Cooperative Game Theory

What is Multi-objective Optimization

- For m inequality constraints and p equality constraints, MOO identifies a vector $\boldsymbol{x}^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$ that optimizes a vector function

$$\bar{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})]^T \tag{1}$$

such that

$$\begin{aligned} g_i(\boldsymbol{x}) &\geq 0, \;\; i = 1, 2, \cdots, m, \\ h_i(\boldsymbol{x}) &= 0 \;\; i = 1, 2, \cdots, p, \end{aligned} \tag{2}$$

where $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^T$ is a vector of n decision variables and the feasible set is denoted by F.

Pareto Optimality

Strongly Pareto non-dominated solution

A feasible solution x is strongly Pareto non-dominated if there is no $y \in F$ such that $f_i(y) \leq f_i(x)\ \forall i = 1, 2, \cdots, k$ and $f_i(y) < f_i(x)$ for at least one i[a].

[a] Means that there is no other feasible solution that can improve some objectives without worsening at least one other objective.

Weakly Pareto non-dominated solution

A feasible solution x is weakly Pareto non-dominated if there is no $y \in F$ such that $f_i(y) < f_i(x)\ \forall i = 1, 2, \cdots, k$.

Pareto Optimality

Pareto Efficiency/Optimality

A state in which resources cannot be reallocated to make any individual gain more without hurting any other objective.

Pareto Improvement/Pareto Dominated Solution

Given an initial allocation, if we can achieve a different allocation making at least one individual function better without hurting any other, then the starting state is called *Pareto improvement*.

Pareto Optimality

Pareto Optimality

For any minimization problem, a point $\boldsymbol{x}^*$ is *Pareto Optimal* if the following holds for every $\boldsymbol{x} \in F$

$$\bar{f}(\boldsymbol{x}^*) \leq \bar{f}(\boldsymbol{x}) \quad (3)$$

where $\bar{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})]^T$ and $\bar{f}(\boldsymbol{x}^*) = [f_1(\boldsymbol{x}^*), f_2(\boldsymbol{x}^*), \cdots, f_k(\boldsymbol{x}^*)]^T$.

Strong Pareto Optimality

A feasible solution x is strongly Pareto optimal if it is strongly Pareto non-dominated.

Weak Pareto Optimality

A feasible solution x is weakly Pareto optimal if it is weakly Pareto non-dominated.

Pareto Frontier

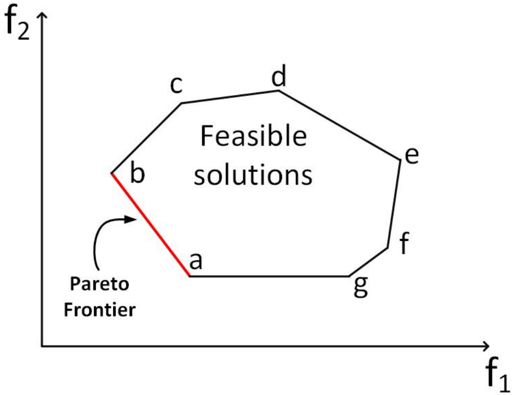

Figure: MOO with two objectives (taken from[2])

[2] Cho, Jin-Hee, et al. "A Survey on Modeling and Optimizing Multi-Objective Systems." IEEE Communications Surveys & Tutorials (2017).

Scalarization-based MOO Formulation

Weighted Sum

Weighted linear combination of the objective functions

$$\bar{f}(\boldsymbol{x}) = \sum_{i=1}^{k} r_i f_i(\boldsymbol{x}) \tag{4}$$

$$where\ 0 \leq r_i \leq 1, i = \{1, \cdots, k\}, \sum_{i=1}^{k} r_i = 1.$$

Utility Function Method

$$\begin{aligned} & max\ U(\bar{f}(\boldsymbol{x})) \\ & s.t \quad [f_1(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})] \leq \boldsymbol{z}^* \\ & \qquad [g_1(\boldsymbol{x}), \cdots, g_m(\boldsymbol{x})] \geq 0, \boldsymbol{x} \in S \end{aligned} \tag{5}$$

where $\bar{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})]^T$, and $\boldsymbol{z}^*$ is a vector of reference points

Scalarization-based MOO Formulation

Goal programming

For a target goal g_i set by the decision maker (DM), the goal is to

$$min \quad \sum_{i=0}^{k} |f_i(x) - g_i|. \tag{6}$$

Min-Max Method

$$min\ [\ max\ Z_i(x)] \quad \forall i = 1, \cdots, k \tag{7}$$

$$where \qquad Z_i(x) = \frac{|f_i(x) - g(i)|}{g(i)} \quad \forall i = 1, \cdots, k. \tag{8}$$

There are a number of different methods as well details of which can be found in [Cho et al., 2017]

Scalarization-based MOO Formulation

Table: Comparison of Scalarization-based MOO Formulation (taken from [Cho et al., 2017])

Technique	**Pros**	**Cons**
Weighted Sum	Computationally efficient particularly for strongly non-dominated solution	Weight assignment, concave trade-off curve
Utility Function	Useful with game theory for designing MOO problems for resource allocation	Hard to have a global view (for an agent) in a distributed system
Goal Programming	Computationally efficient if feasible solution space is found	Computationally inefficient if feasible solution space is not found
Min-Max	Provides the best possible optimal solution if all the objectives have equal priorities	Computationally inefficient if feasible solution space is not found

Meta-heuristics

Can't provide optimality guarantee

Evolutionary Algorithms (EA)

- An evolutionary algorithm involves the process of *recombination (crossover)*, *variation (mutation)*, and *natural selection*.

Ant Colony Optimization (ACO)

- Requires formulating the problem as the best path finding problem on a weighted graph.
- Inspired from how different ants in a colony cooperate to obtain food.

Meta-heuristics

Simulated Annealing (SA)

- Probabilistic technique for identifying the global minimum of a cost function that can multiple local minima.
- The goal is to obtain solutions by decreasing the probability of accepting worse solutions slowly.

Variable Neighborhood Search (VNS)

- Uses the distance between a current solution and its neighborhood representing local optimal leading to an improved solution.
- It is based on the idea that neighborhoods change both in descent to local optima and in escape from valleys that contain local optima.

Table: Summary of Meta-heuristics

Technique	**Pros**	**Cons**
EA	Provides heuristic, but close-to-optimal solutions	Computationally expensive, usually generate local optima
ACO	Suitable for dynamic applications (like ours)	Solution convergence time is not predictable
SA	Good approximation solution for a large size solution search space	No global optimality guarantee, can be computationally expensive
VNS	Provides efficient and good approximation solutions	For large constraint set, can be computationally expensive

Hybrid Meta-heuristics

Table: Summary of Hybrid Meta-heuristics

Technique	**Pros**	**Cons**
EA + Dynamic Programming	Produces efficient feasible solution	May reduce solution diversity
ACO + Constraint Programming	Generates efficient but good quality solutions by leveraging the benefits of using CP	Update of global constraints incurs extra overhead
SA + Tabu Search	Controls worsening solutions using SA's temperature parameter	Computationally expensive for problems with few local optima
VNS + Large Neighborhood Search	Provides good quality neighborhood search region	Finding LNS using exact algorithms is NP-hard

Cooperative Game (CG) Theory

- A group of players/users known as *Coalitions* cooperate to enhance their utilities/benefits by joining a grand coalition.
- The game is played by the coalition of players rather than the players in each coalition.
- CG is denoted by by the pair (N, v) where $N = \{1, 2, \cdots, n\}$ is the set of players and v computes the value obtained from subset S of N
  - $v(S)$ is the value of forming a coalition consisting of all the players in S
  - v captures the objective of the system.

Non-transferable Utility (NTU)

- CG's are mostly referred to as NTU games.
- Denoted by a pair $(\mathcal{N}, \mathcal{V})$
  - $\mathcal{N}$ is the set of players and $\mathcal{V}$ is the function assigning payoff to each coalition $S \subset \mathcal{N}$

[3]

Hedonic Game

- Special case of NTU in which no externalities are considered.
  - The members of the coalition are only affected by other members of the same coalition

[3] NTU means that the agents do not have a common scale to measure the payoff for a coalition

Nash Bargaining Solution (NBS)

In CGT, NBS is applied when two or more players are required to select one of possible outcomes from any joint collaboration.

- Particularly when two parties negotiate something associated with each party's interest, a bargaining game may result in a disagreement outcome, i.e., a payoff each player receives when a negotiation is not successful.

Shapley Value

- Indicates how valuable a player is to the overall cooperation and the payoff player can expect from joining the coalition.

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)}{n!}(v(S \cup \{i\}) - v(S)) \tag{9}$$

Table: Summary of CGT

Technique	**Pros**	**Cons**
NTU	Generic	Generally no guarantee of a unique solution
Hedonic Games	Generic	Requires additional conditions to ensure stable partitioning for different presentations
Shapley Value	Simple to measure utility for a coalition	High communication overhead, no guarantee in hostile environments
NBS	Generic with less complexity	Not straightforward for a cooperative concept

References

Cho, Jin-Hee, et al. "A Survey on Modeling and Optimizing Multi-Objective Systems." IEEE Communications Surveys & Tutorials (2017).